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The hyperbolic sets with the fitting shadowing property

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Abstract

We explain around C^1-stable fitting shadowing property of a diffeomorphism f of C^∞(closed manifold M) denote by SFSP of invariant closed set $\mathfrak{B}$ in M; we want to prove the main Theorem that f_(| B) satisfies a C^1- stable fitting shadowing property if and only if B is a hyperbolic set with the same index; if partially hyperbolic diffeomorphisms map contains q, r are types of saddle points with different indices, then f does not investigate the fitting shadowing property.

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1. Introduction

The shadowing property is one of the effective concepts in the qualitative theory of dynamical systems, asserting that there is a true orbit of the system closed to compute orbit. Where the δ -pseudo-orbit shadowing property (POSP) is important in investigating stability and ergodic theories [1]. By shadowing theorem that is a diffeomorphism has the shadowing property (SP) in the neighbourhood of a hyperbolic set [2],[3], and a structurally stable diffeomorphism has the shadowing property (SP) on the whole manifold [4 - 6]. A metric space $\mathfrak{J}$ and $f\colon \mathfrak{J} \to \mathfrak{J}$ be a map; we say that a sequence $\{y_\kappa\} \in \mathfrak{J}$ is a δ –fitting pseudo-orbit (FPO) of a map f on $\mathbb{Z}$ if there is $\delta > 0$ and $\kappa \in \mathbb{N}$ we have

$$d(f(y_k), y_{k+1}) \le \delta$$

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It has the fitting shadowing property, denote (FSP), if for every sequence $\{y_\kappa\}_{\kappa=0}^{\infty}$ in $\mathfrak{J}$ there is positive integer $\aleph = \aleph(\delta)$, there is $\delta > 0$, for every $\varepsilon > 0$, for all $m \geq \aleph$ and $\kappa \in \mathbb{N}$, there is $z \in \mathfrak{J}$ we have[7]

$$\lim_{m\to\infty} sup \sum_{\kappa=0}^{m-1} d(f^{\kappa}(z), y_\kappa) < \varepsilon$$

Let $Per(f)$be the set of all periodic points of f, symbolize for $O_f(p)$, if the periodic f-orbit of $r \in Per(f)$ type of hyperbolic saddle with period $\alpha(r) > 0$, then there are the local stable (unstable) manifolds $W_\epsilon^s(r)$,$W_\epsilon^u(r)$of r respectively for some $\epsilon = \epsilon(r) > 0$.It is easy to shaw that if $d(f^m(q), f^m(r)) \leq \epsilon$ for any $m \geq 0$, then $q \in W_\epsilon^s(r)$(similarity $W_\epsilon^u(r)$to f^{-1}).The stable and unstable manifolds $W^s(r)$, $W^u(r)$respectively of r are always known. In this work, we symbolize the dimension of the stability of M $W^s(r)$ by index(r). Let $\mathfrak{B} \subset M$ be a closed f-invariant set and $f_{|\mathfrak{B}}$ the restriction of f to this set. Let $U \subset M$ be a compact neighbourhood of $\mathfrak{B}$, such that

$$\mathfrak{B}_f(U) = \cap_{n\in\mathbb{Z}} f^n(U) .$$

$\exists U$ of $\mathfrak{B}$ previously defined such that if $\mathfrak{B} = \mathfrak{B}_f(U)$, then $\mathfrak{B}$ is called locally maximal and denote (L.M). We call that $f|_{\mathfrak{B}f(U)}$ satisfies the C^1-stable fitting shadowing property (C^1 $-$SFSP) if there is a compact neighbourhood $\mho$ of $\mathfrak{B}$ and a C^1-neighbourhood $\mho(f)$, $g|_{\mathfrak{B}g(\mho)}$ satisfies the FSP.

Here $\mathfrak{B}_g(\mho) = \cap_{n\in\mathbb{Z}} g^n(\mho)$ is the continuation of $\mathfrak{B}_f(\mho) = \mathfrak{B}$ .In this work to prove the Main Theorem (2.1).

2. Some of the Basic results

Let $\mathfrak{B}$ be a closed f $-$invariant set. Then $f|_{\mathfrak{B}f(\mho)}$ satisfies the C^1 $-$SFSP if and only if $\mathfrak{B}$ is a hyperbolic set.

Lemma 2.1 [8]: *Suppose $f \in diff(M)$ and let $\mho(f)$ be in above known , every $\varepsilon > 0$ there is $\delta > 0$ such that for a finite set $\{y_i\}$ where $i = 1, \dots, N$ a neighborhood $\mho$ of $\{y_i\}$ and $\xi: T_{y_i}M \to T_{f(y_i)}M$ is a linear map satisfying $d(\xi, D_{y_i}f) < \delta$, then $\exists$ a diffeomorphism β from a manifold to it self in the C^1-topology, $D_{y_i}\beta = \xi$, $\exists$ $\epsilon_\circ > 0$ and $\beta \in \mho(f)$ then $f(y) = \beta(y)$ for all $y \in M \setminus \mho$ and*

$$\beta(y) = exp_{f(y_i)} \circ \xi \circ exp_{(y_i)}{}^{-1}(y) \text{ if } y \in B_{\epsilon_\circ}(y_i), \text{ for all } 1 \leq i \leq N.$$

Theorem 2.1: *If $f|_{\mathfrak{B}f(U)}$ satisfies the C^1 $-$SFSP, then there is a C^1 $-$neighbourhood $\mho(f)$ of f, for any $g \in \mho(f)$, and any periodic point of $\mathfrak{B}_g(U)$ is the hyperbolic point and has the same index.*

If $\mathfrak{B} = M$ we call that f verification the C^1 $-$SFSP, for any transitive Anosov diffeomorphisms, denoted by(TAD) of M is connected and topologically mixing [9], then we get:

Proposition 2.1: Let $r, q \in \mathfrak{B} \cap \mathrm{Per}(f)$ are hyperbolic saddles points. If $f|_{\mathfrak{B}}$ has the SFSP, then $W^s(O_f(r)) \cap W^u(O_f(q)) \neq \emptyset$.

Proof: Let $r, q \in \mathfrak{B} \cap \mathrm{Per}(f)$ are hyperbolic saddles points, and let $\epsilon(r)$ and $\epsilon(q) > 0$. Put $\epsilon = \min\{\epsilon(r), \epsilon(q)\}$. And let $\aleph = \aleph(\epsilon) \ > 0$ be the number of the SFSP of $f|_{\mathfrak{B}}$, there is $r_n \in \mathfrak{B}$ such that $d(f^k(r_n), f^k(f^{-n}(r))) \leq \epsilon \; for \; 0 \leq k \leq n$, $d(f^k(r_n), f^k(f^{-\aleph-n}(q))) \leq \epsilon \; for \; \aleph + n \leq k \leq \aleph + 2n$ implies that

$$d(f^k(f^{\aleph}(f^n(r_n))), f^k(q)) \leq \epsilon \; for \; 0 \leq k \leq n$$

put $z_n = f^n(r_n)$ and let $z = \lim_{n\to\infty} z_n$ if necessary we take a subsequence. Then

$$d(f^{-k}(z_n), f^{-k}(r)) \leq \epsilon \leq \epsilon(r) \quad \text{and} \quad d(f^k(f^{\aleph}(z_n), f^k(q)) \leq \epsilon \leq \epsilon(q)$$

for $0 \leq k \leq n$ we have that $z \in W^u_{\epsilon(r)}(r) \subset W^u(r)$ and $f^{\aleph}(z) \in W^s_{\epsilon(q)}(q)$, that is, $z \in W^s\left(f^{-\aleph}(q)\right)$. Hence

$$z \in W^u(r) \cap W^s\left(f^{-\aleph}(q)\right) \subset W^u\left(O_f(r)\right) \cap W^s\left(O_f(q)\right) \neq \emptyset$$

Suppose that $q \in Per(f)$hyperbolic point, then for every $g \in Diff(M) C^1$-nearby f, $\exists q_g \in Per(g)$ is a unique hyperbolic periodic point nearby q such that $\alpha(q) = \alpha(q_g)$and index (q_g) =index(q). Such a q_g is the continuation of q.

f is called Kupka- Smale if the periodic points of f are hyperbolic points, and if $q, r \in Per(f)$, then $W^s(q)$ is transversal to $W^u(r)$. The set of all Kupka-Smale diffeomorphisms be C^1-residual of $Diff(M)$[10].

Proposition 2.2: Suppose $f|_{\mathfrak{B}f(U)}$ is verification the $C^1 -$ SFSP, let $\mho(f)$ be as previously known. Every hyperbolic saddles points $r, q \in \mathfrak{B}_g(U) \cap Per(g)$ $(g \in \mho(f))$, index (r) =index(q).

Proof: For any $g \in \mho(f)$, and let $r, q \in \mathfrak{B}_g(U) \cap \mathrm{Per}(g)$ be hyperbolic saddles. Then there is a C^1 −neighborhood $V(g) \subset \mho(f)$ of g such that for any $Q \in V(g)$, there are the continuation $r_Q$and q_Q(of r and q)in $\mathfrak{B}_Q(U)$, respectively (recall that since $\mathfrak{B}_f(U) = \mathfrak{B} \subset int\, U$ for any $g \in \mho(f)$reducing $\mho(f)$. To prove the Proposition by contradiction, Let index(r) be less than index(q), and thus dim $W^s(r,g)$ +dim $W^u(q,g)$ are less than dim M (also dim $W^u(r,g)$ +dim $W^s(q,g)$ <dim M). Now $W^s(r,g)$ and $W^u(q,g)$, by Kupka -Smale diffeomorphism $Q \in V(g)$. Then $W^s\left(r_Q, Q\right) \cap W^u\left(q_Q, Q\right) = \emptyset$.

Since dim$W^s(r,g)$ =dim$W^s\left(r_Q, Q\right)$ and dim $W^u(q,g)$ =dim $W^u\left(q_Q, Q\right)$.

On the other hand, since $Q \in \mho(f)$, so $Q|_{\mathfrak{B}_Q(U)}$satisfies the FSP thus that $W^s\left(r_Q, Q\right) \cap W^u\left(q_Q, Q\right) \neq \emptyset$ by Proposition (2.1), it is not possible.

A Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that $E^s$and E^u are uniformly contracting (expanding) respectively, and at least one of them is(resp.

both of them are)not dimension of zero and $E^c$is the central direction of the splitting,is called partial hyperbolic diffeomorphisms was introduced in[5].

Proposition 2.3 [11]: Let $\mathfrak{B}$ be L.M in U, and let $\mho(\mathrm{f})$ be as previously known. Let $\mathrm{w} \in \mathfrak{B}_{\mathrm{g}}(\mathrm{U}) \cap \mathrm{Per}(\mathrm{g})$ where $(\mathrm{g} \in \mho(\mathrm{f}))$ be not hyperbolic set, then $\exists \mathrm{Q} \in \mho(\mathrm{f})$ take possession of w_1 and w_2 in $\mathfrak{B}_{\mathrm{Q}}(\mathrm{U})$are hyperbolic periodic points with index $(\mathrm{w}_1) \neq$ index (w_2).

Proof Theorem 2.1: Let $\mathrm{f}|_{\mathfrak{B}\mathrm{f}(\mathrm{U})}$ be satisfies the C^1 $-$SFSP, and let $\mho(\mathrm{f})$ be as previously known, it is enough to show that every $\mathrm{w} \in \mathfrak{B}_g(\mathrm{U}) \cap \mathrm{Per}(g)$ $(g \in \mho(\mathrm{f}))$ is non-hyperbolic. Then by Proposition (2.2), we prove this Theorem by contradiction. Let $\mathrm{w} \in \mathfrak{B}_g(\mathrm{U}) \cap \mathrm{Per}(g)$ $(g \in \mho(\mathrm{f}))$ be non-hyperbolic, by Proposition (2.3), $\exists \mathrm{Q} \in \mho(\mathrm{f})$, we get hyperbolic periodic points w_1 and w_2 in $\mathfrak{B}_{\mathrm{Q}}(\mathrm{U})$ that is index $(\mathrm{w}_1) \neq$ index (w_2). This is impossible again by Proposition (2.2) because $\mathrm{f}|_{\mathfrak{B}\mathrm{f}(\mathrm{U})}$ is C^1 $-$SFSP.

Corollary 2.1: *The set of diffeomorphism of M satisfying the C^1 $-$SFSP is characterized as the set of transitive Anosov diffeomorphisms map TAD.*

3. Conclusion

In general, the stable fitting shadowing property is not fulfilled in all cases of the hyperbolic dynamical systems, it satisfying of a diffeomorphism f of C^∞(closed manifold M) invariant closed set $\mathfrak{B}$ of M, that $f_{|\mathfrak{B}}$ satisfies a C^1-stable fitting shadowing property if and only if $\mathfrak{B}$ is hyperbolic set, also has the same index. In this paper, several concepts are introduced. Future studies on hyperbolic dynamical systems can be carried out using the concept of eventual fitting shadowing property. These concepts can be re-examined in used in solving some mathematical problems.

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