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Stability analysis of robust control problems in linear descriptor systems with matching conditions

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Abstract

We investigate the optimal control equivalent and state feedback approach to linear time-invariant descriptor systems with norm-bounded uncertainty in matrix A(q). The goal is to find a controller that can stabilize the system even in the presence of random perturbations and to study the effect of initial conditions on the system's stability. The systems with matching conditions and also linear algebraic equations with the full rank of the algebraic coefficient matrix are investigated. The solution to the optimal control problem (OCP) for the linear descriptor system (LDS) is the solution to the robust linear descriptor control problem provided. We will show that with theorems and examples.

Subject Classification: *03B47, 05C50.*

Keywords: *Descriptor system, Optimal control, Robust control, Uncertainty, Matching condition.*

1. Introduction

The topic of robust control problems (RCP) of linear descriptor systems (LDS) with matching conditions deals with the presence of uncertainty in the state matrix, where uncertainty theory is a branch of mathematics [1].

Consider a descriptor system(DS) described by the following set of differential-algebraic equations DAE $\dot{z}_1 = f_1(z_1, z_2, u)$

$$0 = f_2(z_1, z_2, u)$$

or in a more compact form

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$$E\dot{z} = A(q)z + Bu \tag{1}$$

where E = $\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$ is a matrix (Σ is an $m_1 \times m_1$identity matrix) and z = (z_1, z_2) are local coordinates for an $(m_1 + m_2)$-dimensional state space manifold. In the state space z, the state variables are divided into two parts: the dynamic state varied is $z_1 \in R^{m_1,}$ and instantaneous state variables $z_2 \in R^{m_2}$ (algebraic variables).

LDS theory is a significant aspect of the broader subject of control system theory[2]. A mix of differential equations regulates unique systems dynamics and algebraic equations. The question of how beginning values affect the stability gained in this situation is raised. In this sense, the algebraic equations indicate the solutions to the limitations of the differential part. It is also known as singular generalized or differential-algebraic semi-state systems)[3]. It can be defined as a more complete dynamical model than state-space systems(Zang &Zang,2021)[4]. To obtain (RCP), implemented (OCP) approach, as proposed by (Petersen,1987)[5]. The well-known Lyapunov direct technique is the most frequent way to derive the stability requirements for an ODE's equilibrium point Khalaf & Khalaf, 2020) [6]. We also investigated the Lyapunov approach (Trigeassou, Maamri, Sabatier, & Oustaloup, 2011)[7,8]. The uncertainty in the. Emerge in various applications, including aircraft dynamics, electrical, networks, chemical neutral delay systems, optimization, economics, and feedback systems. Also, study the controllability of linear descriptor systems using direct methods. The performance index (cost function) and state-space dynamic system equations are the two main parts of the general formula for the OCP dynamic system.

This research required transforming the DS into an ordinary system to get the solution of the differential part. Also, the type of the condition was identified as matching or unmatched condition, and splitting the system was into a nominal portion and an objective function part with suitable perturbation restrictions to aid in the convergence stability. The problem of robust linear descriptor control was solved via the solution of OCP. Using theorems and some assumptions, we investigated how initial values affect the degree of the system's stability.

The paper is ordered as follows in Section 2, transform the system into parabolic form after building and analyzing an uncertainty mathematical model, including the effect of uncertainty on the system's stability. Section 3 analyses broad dynamics in search of control to ensure that the closed-loop is asymptotically stable, and the matrix K must be determined. Section 4 tries to control that minimizes the goal function and presents an analysis of a more complicated model that presupposes the presence of the required stability condition. Lastly, in section 5, we discuss and conclude.

2. Problem Formulation

Consider the linear descriptor system as follows:

$$E\dot{z} = A(q)z + Bu \tag{1}$$

where, $E, A(q) \in R^{m\times m}$, $q \in [a,b]$, $rankE = m_1|0 < m_1 \leq m$, $z \in R^m, u \in R^r$, $B \in R^{m\times r}$ are the system coefficients q unknown uncertainty. $A(q) - A(q_0) = BB^{\mathrm{T}}\Delta A(q) + (1 - BB^{\mathrm{T}})\Delta A(q)$. In matching condition $(1 - BB^{\mathrm{T}})\Delta A(q) = 0$. Define h as the following uncertainty upper limits. For some $q_0 \in [a,b]$, and for all $q \in [a,b]$. Used the following condition in the ordinary system.

$$G(q)G^T(q) \leq \mathrm{h} \tag{2}$$

Lemma [9] The pencil of $E\dot{z} = A(q) + Bu$. is regular $|SE - A(q)| \neq 0$ for some $s \in C$ where C is the field of a complex number and it is equivalent to the system. Premultiplying $E\dot{z}$=A(q)z+Bu by matrix $Q \in R^{m\times m}$, $QE\dot{z} = QA(q)z + QBu$, and defining the new state vector $\bar{z} = P^{-1}z = \begin{pmatrix}\bar{z}_1\\ \bar{z}_2\end{pmatrix}, \bar{z}_1 \in R^{m_1}, \bar{z}_2 \in R^{m_2}$. We obtain $QEPP^{-1}\dot{z} = QA(q)PP^{-1}z + QBu$

$$QEP = U^{-1}\left(U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^t\right) = \begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}\bar{A}_1(q) & \bar{A}_2(q)\\ A_3(q) & A_4(q)\end{pmatrix} = QA(q)P,$$

$$QB = \begin{pmatrix}\bar{B}_1\\ B2\end{pmatrix}, \Sigma\dot{z}_1 = \bar{A}_1(q)z_1 + \bar{A}_2(q)z_2 + \bar{B}_1u$$

$$0 = A_3(q)z_1 + A_4(q)z_2 + B_2u$$

$A_1(q) \in R^{m_0\times m_0}, A_4(q) \in R^{m_2\times m_2}, B_2 \in R^{(m-m_0)\times r}$, $B_1 \in R^{m_o\times r}$.

System (1) is equivalent to the system.

$$\dot{z}_1 = A_1(q)z_1 + A_2(q)z_2 + B_1u \tag{3. a}$$

$$0 = A_3(q)z_1 + A_4(q)z_2 + B_2u \tag{3. b}$$

Take B_2=0 in (3. b) the system above becomes

$$\dot{z}_1 = A_1(q)z_1 + A_2(q)z_2 + B_1u \tag{4. a}$$

$$0 = A_3(q)z_1 + A_4(q)z_2 \tag{4. b}$$

Assumption 1: If A_4 is a Full rank matrix in eq(4) then we can write the system as

$$\dot{z}_1 = (A_1(q)z_1 - (A_2(q)L)z_1 + B_1u_1 \tag{5. a}$$

$$z_2 = Lz_1, L = A_4^{-1}(q)A_3(q) \tag{5. b}$$

Assumption 2: [10] Assume there wants on open set $\hbar_z \subset D$ such that for all $z_1 \in \hbar_z$ it is possible to solve $A_3(q)z_1 + A_4(q)z_2 = 0$, for z_2, we define the corresponding solution manifold as

$\hbar = \{z_1 \in \hbar_z, z_2 \in R^{m-m_1}|A_3(q)z_1 + A_4(q)z_2\} = 0$

$P_k = \{(z_1(0), z_2(0))|z_1(0)\hbar_z, z_2(0) = L(z_1(0)\}$, and $U[0,t] = \{u(0) \ni u(t)\ is\ differentiable\ on\ [0,t]\}$, the solution to the system eq (4) is $z_1(t) \in \hbar_z$ and $z_2(t) = L(z_1(t))$.

3. Robust Descriptor Control Problem

Consider the system $\dot{z}_1 = (A_1(q) - A_2(q)L - B_1k)z_1$

$$\dot{z}_1 = (A_1(q_0) - A_2(q_0)L - B_1 k)z_1 + BG(q) \tag{6}$$

Find a feedback control (FBC) u =- kz such that the closed loop (CLS) of the system (6) is asymptotically stable for all $q \in [a, b]$. In general, this linear RCP is difficult to solve. In order to be a stable system, self-values for this matrix $((A_1(q_0) - A_2(q_0)L - B_1 k))$ are negative the real part. Our approach is to translate (6) into the following problem.

4. Optimal Control Equivalent Problem

Now, the RCP eq(1) which is equivalent to the system eq(6), can be put in the equivalent quadratic OCP. For the nominal systems,

$$\dot{z}_1 = (A_1(q_0) - A_2(q_0)L)z_1 + B_1 u \tag{7}$$

Find minimizes FBC low $u = -kz$, which minimizes the following cost function, where

$$J(u(.)) = \int_o^\infty \left(z_1^T H z_1 + u^T R u + z_1^T Q_1 \ z_1\right) dt \, , Q_1 = I = 1 \tag{8}$$

Theorem: *Global asymptotic stability of linear control systems if we get positive definite state matrix: An optimal control approach.*

Proof: The OCP finds the minimum of the cost function

$$J(z_1) = \min_u \int_o^\infty (z_1^T H z_1 + u^T R u + z_1^T Q_1 \ z_1) dt$$

Which is subject to the systems $\dot{z}_1 = (A_1(q_0) - A_2(q_0)L)z_1 + B_1 u$, from $z_0 = 0$. Then the necessary condition for the existence of $OCP$is that $J(z_1)$ must satisfy $Hamilton - Jacobi - Bellman(HJB)\ equation.$

$$0 = \min_{u \in \Delta}[z_1^T Q_1 z_1 + z_1^T H z_1 + u^T R u + J_{z_1}^T\big((A_1(p_0) - A_2(p_0)L)z_1 + B_1 u\big) \tag{9}$$

Now if $\Delta \neq \emptyset$ and $\exists\ u \in \Delta$ Now if $\Delta \neq 0,$ and where ($\Delta$the set of all successful controllers $\ni eq(9)$ are satisfied, and if $u = -kz$ then there exists a solution to the OCP and $\partial J / \partial u = 0$

$$\left(z_1{}^T Q_1 z_1 + z_1{}^T H z_1 + (kz_1)^T R(Kz_1) + J_{z_1}^T (A_1(q_0) - A_2(q_0)L)z_1 + B_1 u\right) = 0 \tag{10}$$

$$\partial J / \partial u = 2u^T R + J_{z_1}^T B_1 \tag{11}$$

Now we show that $u = -kz$ is the solution to the RCP, i.e, $z_1 = 0$ of RCP is globally asymptotically stable for all admissible uncertainties q, and we can show that $J(z_1)$ is $alypunov\ function$ for the system.

$\dot{z}_1$=$(A_1(q_0) - A_2(q_0)A_4^{-1}(q_0)A_3(q_0))z_1 + B_1 u + B_1 G(q)z$, the system achieves the following goals: $J(z_1) > 0$, $z_1 \neq 0$, $J(z_1) = 0$, $z_1 = 0$,

$\dot{J}(z_1) < 0$ for all $z_1 \neq 0$, we will prove it

$$\dot{J}(z_1) = J_{z_1}^T \dot{z}_1$$
$$= J_{z_1}^T (A_1(q_0) - A_2(q_0)A_4^{-1}(q_0)A_3(q_0))z_1 + B_1 u + B_1 G(q)z_1$$

From eq(10) and $eq(11)$ we get

$$\dot{J}(z_1) = J^T z_1 \left(\left(A_1(q_0) - A_2(q_0)A_4^{-1}(q_0)A_3(q_0)\right)z_1 + B_1 u\right) + J^T z_1 (B_1 G(q) z_1)$$

$\mathrm{j}(z_1) = -{z_1}^T z_1 - {z_1}^T h z_1 - u^T u + J^T z_1(B_1 G(q) z_1)$
$\dot{J}(z_1) = -{z_1}^T z_1 - {z_1}^T h z_1 - u^T u - 2u^T G(q)$
$\dot{J}(z_1) = -{z_1}^T z_1 - {z_1}^T (h - G(q)G(q)^T) z_1 - {z_1}^T (k + G(q)(k + G(q)^T) z$
$\dot{J}(z_1) \leq -{z_1}^T z_1$

Thus, the conditions of the Lyapunov local stability are satisfied. Consequently, there exists a neighbourhood $I = \{z\colon \|z(t)\| < C\}$ for some $\mathrm{C} > 0$. Thus, if $z(t)$ enters I then $lim_{t\to\infty} z(t) = 0$

But z(t) cannot remain forever outside I; otherwise,

$z^T z > \mathrm{C}$ For all $\mathrm{t} > 0$, therefore

$$J(z(t)) - J(z(0)) = \int_0^t \dot{J}(z(s))ds$$
$$J(z(t)) \leq J(z(0)) - \mathrm{C}^2 \mathrm{t}$$

Illustration example

Consider the robust descriptor system $E\dot{z} = A(q)z + Bu$ where z^T=(z_1, z_2, z_3, z_4) and

$$E = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & -2 \\ 1+q & q & q & 0 \\ 1 & 0 & 0 & 1 \\ 0 & q & q & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

since rank $E = 2 = m_1$, and $q \in [-10,10]$, unknown uncertainty, then E is a singular matrix. calculate such that from the lemma, the system above became afterwards defining the new state vector $P^{-1}z = \begin{pmatrix} \mathrm{y}_1 \\ \mathrm{y}_2 \end{pmatrix}$ we get:

$$\dot{y}_1 = A_1(q)y_1 + A_2(q)y_2 + B_1 u$$
$$0 = A_3(q)y_1 + A_4(q)y_2 \text{ , where}$$

$$A_1 = \begin{pmatrix} 1/2 & 0 \\ q & 1+q \end{pmatrix}, A_2 = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From assumption (1) and assumption (2)

$$|A_4| = -q \; then \; |A_4^{-1}| = \begin{pmatrix} 0 & \frac{1}{q} \\ 1 & 0 \end{pmatrix}, y_2 = -A_4^{-1} A_3 y_1 \text{ get, } y_2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y_1$$

From assumption **2** the initial condition is given :

$$P_k = \{(y_{1,0} y_{2,0} y_{3,0}, y_{4,0}) | y_{3,0} = -y_{1,0}, y_{4,0} = -y_{2,0}\}$$

The nominal system will be where $q_0 = 2$ is the nominal uncertainties in the [-10,10]. $The\ system\ redues\ to\ the\ \dot{y}_1 = \begin{pmatrix} 1/2 & 1 \\ 0 & 3 \end{pmatrix} y_1 + B_1 u$

From condition (2), H=$\begin{pmatrix} (q-2)^2 & 0 \\ 0 & (q-2)^2 \end{pmatrix}$. Find the FBC Low u=-kz that minimizes the cost function for all$(y_1, y_2) \in \hbar$ and $Q_1 = \mathrm{I} = 1$, $H + I$=Q_2

$J(z_1) = \min_{u \in \Delta} \int_0^\infty \left(\left(Y_1^T \begin{pmatrix} 65 & 0 \\ 0 & 65 \end{pmatrix} Y_1 \right) + (u_1^T u_1) \right) dt$. By solving the linear quadratic regulator (LQR) problem and $algebraic\ Riccati\ Equation\ A^T S +$

$SA - SBR^{-1}B^TS + Q$=0 in $matlab$ we get S=$\begin{pmatrix} 136.5647 & 14.1973 \\ 14.1973 & 13.1190 \end{pmatrix}$, and the eigenvalues of S are (11.50072,138.1765). Take $u = -ky$, where, $k = R^{-1}B^TS\ get\ k = (14.1973 \quad 13.1190)$. Then, $u = -14.1973y_1 - 13.1190y_2$. Stability is being checked by using

The eigenvalues of Bk are(-1.0687,-8.5503) therefore the system is stable

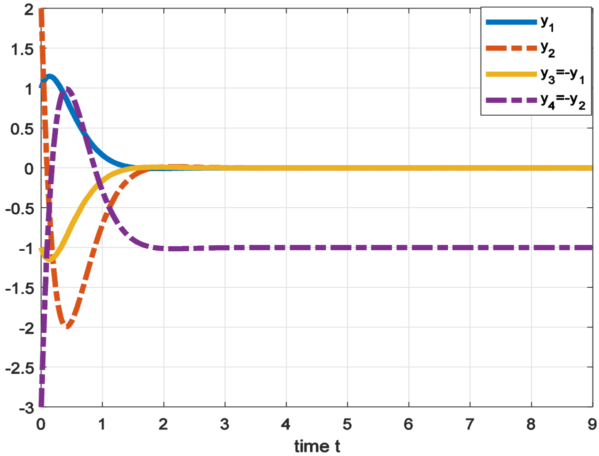

Figure 1

(not asymmetrically stable) Robust solution, q=6 $\in [-\mathbf{10}, \mathbf{10}] = (\boldsymbol{y_{1,0}}, \boldsymbol{y_{2,0}}, \boldsymbol{y_{3,0}}, \boldsymbol{y_{4,0}})$
$(\boldsymbol{y_1}, \boldsymbol{y_2}, \boldsymbol{y_3}, \boldsymbol{y_4}) = (\mathbf{1}, \mathbf{2}, -\mathbf{1}, -\mathbf{3}) \notin \boldsymbol{P_k}$

5. Conclusions

In state-space ODE systems, the system is stable regardless of the beginning circumstances if the nominal system's eigenvalues have a negative real portion. There were two components to the initial condition in descriptor systems: the dynamic ODE and the algebraic equation. Initial circumstances impacted a symmetrically stable system even when the dynamic system's spectrum is shown in the left half of Figure 1. The solution to the provided descriptor with the matched condition was the solution of an analogous OCP to the uncertain linear descriptor system. We can use this approach in many systems, and it improves and reduces the original problem for point-of-application applications. In future work, we plan to improve the problem in unmatched conditions in two cases, the full rank matrix, and the deficient rank matrix.

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