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Some generalized numerical methods for solving higher-order of fractional partial differential equations with application

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Abstract

This article recommends a computer method for solving some classes of higher-order fractional partial differential equations (FrPDEs). The updated techniques were made by combining the method of lines (MOL) and the numerical RK-type method. For solving certain types of FrPDEs that aren't in the order, you might expect. Some problems with second-, third-, and fourth-order FrPDEs are solved with the methods made, and the numerical solutions are compared with the analytical answers. Also, a new type of application in classical physics called a wave FrPDE has been made for mechanical waves like water, earthquake, sound, and electromagnetic waves. Many examples are given to show how flexible and useful the method is. The numerical results are seen as the same as the exact solutions of the implementations to show how accurate and useful the suggested methods are. Also, the numerical implementations compare the suggested methods with exact solutions to show that they work and are accurate. The suggested methods used RKM analytical solutions to come up with numerical solutions to the test problems. Some examples are given to show how flexible and useful the method is. Also, the numerical implementations are compared

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with exact results to show that the proposed methods work and are accurate. R.K. techniques were used to find numerical answers to the suggested test problems.

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1. Introduction

All types of differential equations(D.E.s) are essential tools in the branches of applied mathematics and its applications, especially in engineering and science. Nowadays, much mathematical modelling in science or engineering involves different types of D.E.s, especially fractional ones. Many phenomena in chemistry, engineering, physics and other sciences have recently been discovered to be extremely well characterized by models based on the theory of fractional derivatives(FrDs) and integrals. For instance, the FrDs can be used to describe nonlinear seismic oscillations, and the FrDs can be used to alleviate the inadequacy caused by the assumption of continuous traffic flow in a fluid-dynamic traffic model. FrPDEs are proposed based on experimental data for seepage flow in porous media. In various applications, FrPDEs are often used to describe applied mathematical models. For example, the fluids in the lungs, physics, engineering, and geosciences all contain a vast spectrum of dynamic processes and physical phenomena. Also, the physical fluxes include ice production, the body's transverse vibration, the homogeneous elastic bands, and the plate deflection hypothesis [1]. There are some approaches to finding analytical or approximated solutions to the linear FrPDEs. The majority of FrPDEs do not have analytical solutions. However, the numerical and approximated approaches should be utilized for solving FrPDEs since most do not have exact analytical solutions. For instance, the Adomian decomposition method(ADM) and the variational iteration method(VIM) are two relatively new approaches for approximating the solutions to different types of these problems. The decomposition method has been used in the literature to approximate the solutions to a wide range of D.E.s, especially FrDEs. Many modern or classical analytical methods for solving D.E.s have been studied for a long time [3-4,15]. For nonlinear FrPDEs, a variety of approaches have been utilized to determine the analytical solution, including the homotopy perturbation method(HPM), variation iteration method(VIM), and homotopy analysis method(HAM) [10]. Engineers and scientists are very interested in FrPDE solutions. However, there is a need to develop effective numerical methods for dealing with them. Numerous classes of FrPDE problems have emerged in various areas of mathematics in recent years, leading to analytical solutions for FrPDEs and the development of some methods, such as the variational iteration method, numerical solutions for FrPDEs, such as the HPM proposed by J.- H. for solving linear FrPDEs [2,17]. [14] defined fractional derivatives in the Caputo sense and contrasted the results of the HPM with those of the VIM. [5-6] used the local fractional Laplace decomposition method to produce a rough analytical solution

of the telegraph and Laplace equations on Cantor sets. To review the literature in the scope of this study, many authors studied PDEs and FrPDEs; for example, [7] has derived a direct numerical method for solving initial-boundary value problems, which include PDEs, [12] has solved the nonlinear PDEs using improved Kudryashov method. In contrast, [16] solved FrPDEs using Tau approximate, [1] used a finite element method for solving time FrPDEs, [11] proposed an analytical approach for solving linear FrPDEs arising in fluid mechanics, [10] has compared the HPM verse the VIM for solving linear FrPDEs, [17] studied the numerical solution of the FrPDEs by using the two-dimensional fractional-order Legendre functions, and [18] derived generalized R.K. integrators for solving a class of sixth-order ODEs. Lastly, [19] derived embedded explicit RK-type methods for directly solving a class of seventh-order ODEs. This work compares the numerical solutions of FrPDEs with the exact solutions by combining the RKD and RKM approaches with the method of lines. As a result, we applied the proposed numerical methods to various problems of FrPDEs together with the application of wave FrDEs in physics.

2. Preliminary

We have introduced the basic background that is related to this study.

2.1 *Basic Definitions of the Fractional Derivatives*

The FrDs and fractional integral are old as calculus. In 1695, the topic of what it means was raised. L'Hospital asks what's the value of $\frac{d^n f(t)}{dt}$? if $n = \frac{1}{2}$. Many researchers have attempted to define FrDs since then. For the FrD, the majority of them employed an integral form. Two of which are the most popular ones. Two famous of FrDs are known as follows:

(a) The definition of Riemann-Liouville.

Definition 2.1. The left and right Riemann-Liouville fractional integral operator of order $\alpha > 0$ or $\alpha \in [n-1; n)$,

$$D_a^{\alpha} f(t) = \frac{1}{\mathbb{\Gamma}(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(t)}{(t-x)^{\alpha-n-1}} dx \quad , x > 0 \tag{1}$$

$$^{a}D_t^{\alpha} y(t) = \frac{1}{\mathbb{\Gamma}(n-1)} \frac{d^n}{dt^n} \int_a^{\tau} (t-s)^{n-\alpha-1}\, y(s) ds \quad , x > 0 \tag{2}$$

and

(b) Caputo definition. For $\alpha \in [n-1, n)$, where n is an integer, and the derivative of f is

Definition 2.2.

$$D_a^{\alpha} f(t) = \frac{1}{\mathbb{\Gamma}(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n-1}} dx \quad , x > 0 \tag{3}$$

2.2 *Classical Partial Fractional Integrals and Derivatives*

This section gives basic definitions, preliminaries and some properties of partial fractional calculus: [16]

2.2.1 Basic Definitions of the Partial Fractional Calculus

We state the definitions and preliminaries of partial fractional calculus.

Definition 2.3: For m to be the smallest integer that exceeds α, Caputo's space-fractional derivative operator of order $\alpha > 0$ is defined as follows:

$$\mathrm{D}_{\boldsymbol{\chi}}^{\boldsymbol{\alpha}}\mathrm{u}(\chi,\tau) = \frac{\partial^{\alpha}u(\chi,\tau)}{\partial\chi^{\alpha}} == \begin{cases} I_{\chi}^{m-\alpha}D_{\chi}^{m}u(\chi,\tau), & if\ m-1<\alpha<m, \\ D_{\chi}^{m}u(\chi,\tau), & if\ \alpha=m, m\epsilon\mathbb{N} \end{cases} \quad (4)$$

Where $\boldsymbol{I}_{\chi}^{\boldsymbol{\alpha}}\mathrm{u}(\chi,\tau) = \frac{1}{\mathbb{\Gamma}(\alpha)}\int_0^{\chi}(\chi-\sigma)^{\alpha-1}u(\sigma,\tau)d\sigma, \alpha>0, \chi>0$ (6)

and the time-fractional derivative operator of order $\alpha > 0$ is defined as follows:

$$\mathrm{D}_{\boldsymbol{\tau}}^{\boldsymbol{\alpha}}\mathrm{u}(\chi,\tau) = \frac{\partial^{\alpha}u(\chi,\tau)}{\partial\tau^{\alpha}} = \begin{cases} I_{\tau}^{m-\alpha}D_{\tau}^{m}u(\chi,\tau), & if\ m-1<\alpha<m, \\ D_{\tau}^{m}u(\chi,\tau), & if\ \alpha=m, m\epsilon\mathbb{N} \end{cases} \quad (8)$$

Where $I_{\tau}^{\alpha}\mathrm{u}(\chi,\tau) = \frac{1}{\mathbb{\Gamma}(\alpha)}\int_0^{\chi}(\chi-\sigma)^{\alpha-1}u(\sigma,\tau)d\sigma, \alpha>0, \chi>0$ (9)

and the time-fractional derivative operator of order $\alpha > 0$ is defined as

2.2.2 The Properties of the Operators J_{χ}^{α} and J_{τ}^{α}

Some of the most important properties of the operators J_{χ}^{α} and J_{τ}^{α} are as given follows:

I. $J_{\chi}^{\alpha}(\chi^{\gamma}) = \frac{\mathbb{\Gamma}(\gamma+1)}{\mathbb{\Gamma}(\alpha+\gamma+1)}\chi^{\alpha+\gamma}$, II. $J_{\tau}^{\alpha}(\tau^{\gamma}) = \frac{\mathbb{\Gamma}(\gamma+1)}{\mathbb{\Gamma}(\alpha+\gamma+1)}\tau^{\alpha+\gamma}$,

II. $J_{\chi}^{\alpha}\left(\chi^{\gamma}\tau^{\beta}\right) = \tau^{\beta}J_{\chi}^{\alpha}(\chi^{\gamma}), IV. J_{\tau}^{\alpha}\left(\chi^{\gamma}\tau^{\beta}\right) = \chi^{\gamma}J_{\tau}^{\alpha}\left(\tau^{\beta}\right),$

Caputo's fractional differentiations are the linear operators concerning and. Hence, the properties of this operator are given as follows:

i. $D_{\tau}^{\alpha}\big(w(\chi,\tau)\big) = \lambda D_{\tau}^{\alpha}u(\chi,\tau) + \eta D_{\tau}^{\alpha}v(\chi,\tau), where\ w(\chi,\tau) = \lambda u(\chi,\tau) + \eta v(\chi,\tau)$

ii. $D_{\chi}^{\alpha}\big(w(\chi,\tau)\big) = \lambda D_{\chi}^{\alpha}u(\chi,\tau) + \eta D_{\chi}^{\alpha}v(\chi,\tau)$, where λ and η are constants.

2.3 *The Quasi-Linear Fractional Partial Differential Equations(FrPDEs)*

The general quasi-linear fractional FrPDE can be written as follows:

$$D_{\tau(n-times)}^{n\alpha}(\chi,\tau) = f\left(\chi,\tau,u(\chi,\tau),u_{\chi}(\chi,\tau),..,u_{\chi(n-times)}(\chi,\tau)\right), \chi_1 \le \chi \le \chi_2, 0<\tau\le T(10)$$

with initial conditions(ICs):

$$u_{\chi(n-times)}(\chi,0) = f_i(\chi), i = 0,1,2,\dots,n-1, \quad (11)$$

and the boundary conditions (BCs):

$$u(\chi_1,\tau) = \varphi_1(\tau), \quad u(\chi_2,\tau) = \varphi_2(\tau), \quad (12)$$

2.4 *The Finite Difference Method (FDM)*

The finite difference method(FDM) is a traditional approach to solving ODEs and PDEs. To solve ODE or PDE, the researchers employed the forward, backwards, or central finite difference(F.D.) formula [3]. To explain the numerical constructed method for solving the following FrPDE of the issues quasi-linear PDE of n^{th}-order, Firstly, in this study, we will employ FDM to solve the issues quasi-linear FrPDE in Equation (10) with I.C.s in Equation (11) and the B.C.s in Equation (12). We divide the intervals of definitions of the variables χ and τ which are named as $[\chi_1, \chi_2]$ and $[0;T]$ to N and M subintervals, respectively, where, $\chi_i = \chi_1 + ih$, $\tau_j = jk$ and $\mathrm{h} = \frac{\chi_2 - \chi_1}{N}_1$, $k = \frac{T}{M}$. At the position (χ, τ), we put (χ_i, τ_j) for i = 1,2,…,N - 1 and j = 1,2,…,M. This will be transformed FrPDE into the fractional finite difference system as bellows.

$$\frac{\partial^{n\alpha} u(x,\tau)}{\partial \tau^{n\alpha}}|_{(\chi,\tau)=(\chi_i,\tau_j)} = f\left(\chi, \tau, \mathrm{u}(\chi,\tau), \frac{\partial u(\chi,\tau)}{\partial \chi}, \frac{\partial^{2\alpha} u(x,\tau)}{\partial \chi^{2\alpha}}, \ldots \frac{\partial^{n\alpha} u(x,\tau)}{\partial \chi^{n\alpha}}\right)|_{(\chi,\tau)=(\chi_i,\tau_j)} \tag{13}$$

By using the following central finite difference formulas of the orders one, two or three etc. as follows

$$\frac{\partial \mathrm{u}(\chi,\tau)}{\partial \chi}|_{(\chi,\tau)=(\chi_i,\tau_j)} = \frac{u_{i+1,j} - u_{i-1,j}}{h}, \quad \frac{\partial^2 u(x,\tau)}{\partial \chi^2}|_{(\chi,\tau)=(\chi_i,\tau_j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\mathrm{h}^2} \tag{14}$$

$$\frac{\partial^3 u(x,\tau)}{\partial \chi^3}|_{(\chi,\tau)=(\chi_i,\tau_j)} = \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} + u_{i-2,j}}{2\mathrm{h}^3} \tag{16}$$

$$\frac{\partial^4 u(x,\tau)}{\partial \chi^4}|_{(\chi,\tau)=(\chi_i,\tau_j)} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{\mathrm{h}^4} \tag{17}$$

3. The Proposed Numerical Method

In this paper, combining the MOL technique with the R.K., RKD and RKM approaches, numerical methods for solving FrPDEs have been developed. Let the numerical solution's $u(\chi_i, \tau_j)$ where $(\chi_i, \tau_j) \in [\chi_i, \tau_j] \times [0;T]$ where $\tau_j = jk$ and $\chi_i = \chi_1 + ih$ with $k = \frac{T}{M}$ and $\mathrm{h} = \frac{\chi_2 - \chi_1}{N}_1$, where N, and M are the numbers of points in the directions of χ and τ respectively for i = 1,2,…,N -1; j =1,*2,…,M*. However, after some adjustments, we obtain the following set of explicit fractional difference equations:

$$u^{n\alpha}_{\tau(n-times)}(\tau)|_{\chi=\chi_i} = \mathrm{f}(\chi_i, \tau_i, \mathrm{u}_{i-2,j}, \mathrm{u}_{i,j}, \mathrm{u}_{i+1,j}, \mathrm{u}_{i+2,j}, \mathrm{u}_{i,j+1}, \mathrm{u}_{i,j-1}, \mathrm{u}_{i,j-2})$$

with the I.C.s (11) and the B.C.s (12). We can combine the method of lines (MOL)with RKD or RKM method to solve the problem with the I.C.s and the B.C.s using the steps of the proposed algorithm in the following subsection.

3.1 *Proposed Algorithm*

1. Do steps 2-6 while $1 \le j \le m$
2. Fix $\chi = \chi_i$ at the point (χ, τ) of the FrPDE in Equation (10) converting to the following fractional differential equations

$$u^{n\alpha}_{\tau(n-times)}(\tau) = f\left(\chi, \tau, u(\tau), u_{\chi}(\tau), u_{\chi\chi}(\tau), \dots., u_{\chi(n-times)}(\tau)\right)|_{\chi=\chi_i}, \quad (20)$$

where, $u_i^{n\alpha}(\tau) = \frac{d^n u(\tau)}{d\tau^n}|_{\chi=\chi_i}$, for i = 1,2,...,n−1.

3. Substitute suitable finite difference formula, which is defined in equations (14)-(19), into the derivatives of the right side of FODE in Equation (13); then, a system of finite difference FrODEs has been obtained.

$$u_i^{n\alpha} = f\left(\chi_i, \tau, u_{i-n+1}(\tau), u_{i-n}(\tau), \dots, u_{i+n-2}, u_{i+n-1}(\tau)\right), \quad (21)$$

For i=1, 2, …, n-1. when $j = 1$ then the I.C.s are, $u_i^{(k)}(0) = f_k(\chi_i)$, for k = 0,1,…,n - 1 and $2 \le j \le m$; the ICs are,

$$u_i(\tau_{j-1}) = u(\chi_i, \tau_{j-1}),\ u_i^{(k)}(\tau_{j-1}) = \frac{du^{(k)}(\chi, \tau_{j-1})}{d\chi^{(k)}}|_{\chi=\chi_i}, \quad (22)$$

4. Put the B.C.s,

$$u_{0j} = u(\chi_1, \tau_j) = \varphi_1(\tau_j),\ u_{Nj} = u(\chi_2, \tau) = \varphi_2(\tau_j), \quad (23)$$

5. Solve the system of *nth-order* FrODEs in Equation (21) at $\tau = \tau j$ with I.C.s in Equation (22) and B.C.s in Equation (23) using the RK-type methods. In general, this algorithm is for solving a FrPDE of *n^{th}-order* in Equation (21) with I.C.s in Equation (22) and B.C.s in Equation (23).

3.2 The Quasi-Linear *n^{th} -order* ODEs

In general, the special quasi-linear ODE of the *n^{th}-order* has the following form:

$$y^{(n)}(x) = f\left(x, y(x)\right), x \ge x_0, 0 < \alpha < 1, \tau \ge \tau_0 \quad (24)$$

With the I.C.s $y^{(j)}(x_0) = \alpha^j$, and, $(\tau) = [y_1(\tau), y_2(\tau), \dots, y_N(\tau)]$, $f(x, y) = [f_1(x, y), f_2(x, y), \dots, f_N(x, y)]$, and $\alpha^j = \left[\alpha_1^j, \alpha_2^j, \dots, \alpha_N^j\right]$, where j = 0, 1, 2, ..., n.

3.3 *RKD and RKM Numerical Methods*

In this paper, the problem of interest is the special value problems of second-, third- and fourth-order ODEs, that mean n=2, 3, 4 in Equation (24). [8-9,13] derived the general form of RKD and RKM methods with *s*-stage for solving special third- and fourth-order ODEs in Equation (24) with the I.C.s.

4. Application of the Fractional Wave Equation

The fractional wave equation is a second-order linear FrPDE that explains mechanical waves in classical physics, such as seismic, sound, water, and electromagnetic waves (including light waves) [14]. It is defined as follows:

$$u_{\tau\tau}^{2\alpha}(\chi,\tau) = \alpha^2 u_{\chi\chi}(\chi,\tau), \quad \chi_1 \le \chi \le \chi_2, 0 < \tau \le T \tag{37}$$

with the I.C.s: $u_{\chi(i-times)}(\chi,0) = f_i(\chi), \quad i = 0,1.$ (38)

and the B.C.s: $u(\chi_i,\tau) = \varphi_i(\tau), \quad i = 1,2, \quad 0 < \alpha \le 1$ (39)

Using the proposed algorithm in section 3, the fractional wave equation in Equation (37) with I.C.s in Equation (38) and B.C.s in Equation (39) has been solved using modified RKN method, n=2, and the numerical solution of this problem has plotted in Figure 1-(a) for N=100, M=10, $T = \alpha = 1$, with I.Cs $f_1(\chi) = e^{-\chi}$ and $f_2(\chi) = -e^{-\chi}$, the B.Cs $u(\chi_i,\tau) = 0, i = 1,2$. The numerical solution on the section line $\tau = \frac{Mh}{2}$ has been compared with the exact solution defined by $u(\chi,\tau) = -e^{-\chi-\tau}$.

5. Numerical Examples

We implemented the proposed algorithm using the RKD method, n=3, and the RKM method, n=4, in section 3 for solving three different examples of FrPDEs using MATLAB software with M=10, N=100 and T=1. We tested the proposed methods in this study by comparing the numerical solutions versus exact solutions for three examples in which the numerical solutions on the section line $\tau = \frac{Mh}{2}$ have been compared with the exact solutions and plotted in Figure 1 (b)-(c).

Example 5.1: (Homogenous)

Consider $u_\tau^{3\alpha}(\chi,\tau) = (-1)^{3\alpha}u(\chi,\tau)$; $\chi_1 \le x \le \chi_2, \tau > 0$ with ICs: $u(\chi,0) = \chi^3, u_\chi(\chi,0) = 3\chi^2, u_{\chi\chi}(\chi,0) = 6\chi$, and BCs : $u(\chi_1,\tau) = \chi_1^3 e^{-\tau}, u(\chi_2,\tau) = \chi_2^3 e^{-\tau}$,

Example 5.2: Consider $u_\tau^{3\alpha} = (2)^{3\alpha-1}\left(u(\chi,\tau) - u_{\chi\chi}(\chi,\tau)\right)$, $\chi_1 \le \chi \le \chi_2, \tau > 0$. with ICs: $u(\chi,0) = \cos(\chi), u_\chi(\chi,0) = u_\chi(\chi,0) = -\sin(\chi), u_{\chi\chi} = -\cos(\chi)$, with B.C.s: $u(\chi_1,\tau) = e^{2\tau}\cos(\chi_1), u(\chi_2,\tau) = e^{2\tau}\cos(\chi_2)$, Exact solution: $u(\chi,\tau) = e^{2\tau}\cos(\chi)$, with $\chi_1 = 0$ and $\chi_2 = \pi$

Example 5.3. (Homogenous)

Consider $u_\tau^{4\alpha}(\chi,\tau) = u(\chi,\tau)$; $\chi_1 \le x \le \chi_2$, with ICs: $u(\chi,0) = -u_\chi(\chi,0) = u_{\chi\chi}(\chi,0) = e^{-\chi}$, with B.C.s: $u(\chi_1,\tau) = e^{\chi_1}e^{\tau}, u(\chi_2,\tau) = e^{\chi_2}e^{\tau}$, Exact solution: $u(\chi,\tau) = e^{\tau}e^{-\chi}$ with $\chi_1 = 0$ and $\chi_2 = 4$.

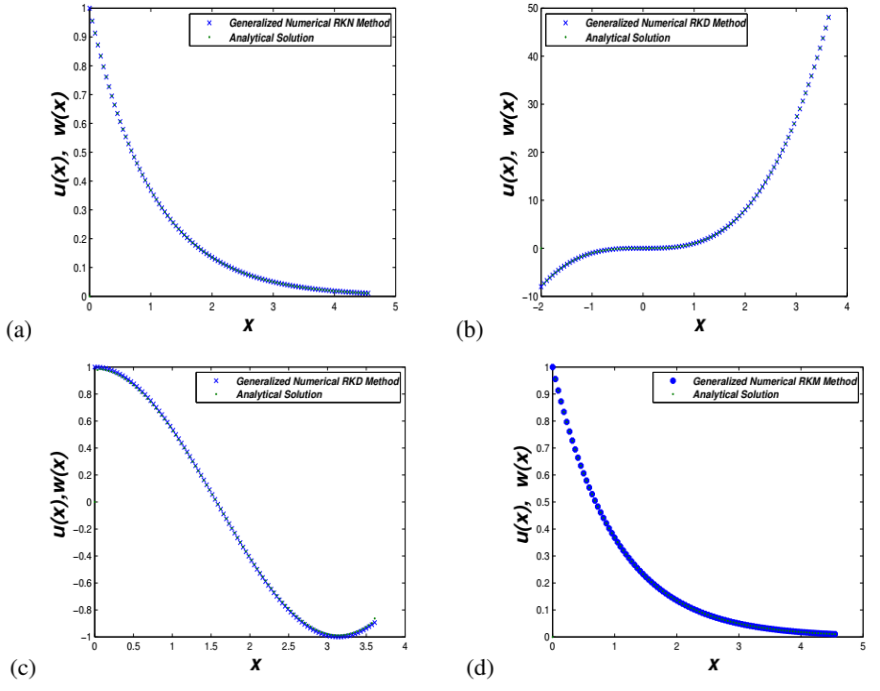


Figure 1

A Comparison between the Numerical Solutions evaluated by Generalized Numerical Methods versus the Analytical Solutions for the Application in Section 4 in (a) and the Examples 5.1, 5.2 and 5.3 in (b)-(d) Using Generalized RKN, RKD, and RKM Methods.

6. Discussion and Conclusion

The main goal of this study is to provide numerical methods for solving FrPDEs of different orders. The objective of this article has been proved by generalizing some numerical methods of RK-types to be efficient with the numerical solving of FrPDEs. The fractional wave equation has been introduced, and hence, it is solved using the proposed method. We compared the numerical findings with the exact solution because the method is a new contribution and is reliable in finding solutions to problems that arise in applied sciences. The proposed method can be used in the fractional R.K. methods for solving FrPDEs of the *nth*-order. Finally, recent advances in the field of FrPDEs of higher order in applied mathematics necessitate using such equations to obtain numerical solutions. We can conclude from the numerical results for the four implementations in Figure 1 (a)-(b) that the proposed methods are efficient and suitable for evaluating the numerical solutions of higher-order quasi-linear FrPDEs.

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