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Solutions of nonlinear fractional integro - differential equations by combined sumudu transform with adomian decomposition approach

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Abstract

The Caputo formulation is used to handle fractional derivatives in this study. The well-known method form employs Adomian's polynomials as the proper mathematical tool for dealing with nonlinear variables. The integral equation may be solved by applying the Sumudu transformation to both sides and then the terms can be written out. The solution is given as a convergent infinite series. Nonlinear Volterra integro-differential equations with fractional derivatives are amenable to the proposed method. The results strongly support the validity, applicability, and efficacy of the approach.

Subject Classification: *05C72, 11J54.*

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1. Introduction

In order to achieve an approximation solution for certain fractional integro-differential equations, the Sumudu decomposition method is introduced in this study. Using the Caputo definition for fractional derivatives, we show that the results of our solution analysis may be trusted. Due to the presence of nonlinear variables, the integral equation must be transformed into a system of Adomian polynomials [1-4]. We obtain a convergent infinite series with computable terms

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by solving the two-sided problem using the Sumudu transformation. This method is used to effectively resolve a wide variety of nonlinear integral equations, including fractional derivatives [4-7]. The authors also go through the salient features of the Sumudu transformation and how it can be used to create and solve mathematical models. Methods that have proven to be successful are investigated [8-14]. The implications of this study's findings could be huge.

2. The Sumudu Decomposition Algorithm in Practice

By Consider the Fractional nonlinear integro-differential equation in the following form:

$$\sum_{K=0}^{m} a_k(\xi)\,\Upsilon^{(k)}(\xi) = g(\xi) + \lambda_1 \int_a^{\xi} \sum_{i=0}^{S} A_i(\xi,\tau)\, R_i(\Upsilon(\tau))d\tau + \lambda_2 \int_a^{b} \sum_{j=0}^{r} Bj(\xi,\tau)\, F_j(\Upsilon(\tau))\, d\tau \tag{1}$$

with the initial conditions $\Upsilon^{(n)}(0) = a_n$, $n = 0,1,2,\dots.k-1$

Where $a_k(\xi)$, $k = 0,1 \dots\dots m$, $A_i(\xi,\tau)$, $(i = 0,1,\dots s)$, $Bj(\xi,\tau)$, $(j = 0,1,\dots r)$ and $g(\xi)$) are given functions, $\Upsilon^{(n)}$ indicates the n-th derivative of $\Upsilon(\tau)$, $R_i(\Upsilon(\tau))$ and $F_j(\Upsilon(\tau))$ are nonlinear functions and $\lambda_1, \lambda_2, a_n$ are constants. Now, apply the Sumudu transform to solve Eq. (1) to find an approximate solution for the fractional integro-differential equation as follows:

$$\Upsilon^{(\alpha)}(\xi) = \frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_m(\xi)\Upsilon^{(k)}(\xi) + g(\xi) + \lambda_1 \int_a^{\xi} \sum_{i=0}^{S} A_i(\xi,\tau)\, R_i(\Upsilon(\tau))d\tau + \lambda_2 \int_a^{b} \sum_{j=0}^{r} Bj(\xi,\tau)\, F_j(\Upsilon(\tau))\, d\tau \quad m-1 < \alpha \le m \tag{2}$$

We assume that $\Upsilon(\xi)$ is sufficiently distinct. Put both sides of (2) via a Sumudu transformation. is sufficiently differentiable.

$$S[\Upsilon^{(\alpha)}(\xi)] = S\left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_m\, \Upsilon^{(k)}(\xi)\right] + S[g(\xi)] + S\left[\lambda_1 \int_a^{\xi} \sum_{i=0}^{S} A_i(\xi,\tau)\, R_i(\Upsilon(\tau))d\tau\right] + S\left[\lambda_2 \int_a^{b} \sum_{j=0}^{r} Bj(\xi,\tau)\, F_j(\Upsilon(\tau))\, d\tau\right], m-1 < \alpha \le m$$

Then

$$u^{-\alpha} S[\Upsilon(\xi)] - \sum_{k=0}^{n-1} u^{-(\alpha-k)}\, \Upsilon^{(k)}(0) - S\left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_m\, \Upsilon^{(k)}(\xi)\right] + S[g(\xi)] + S\left[\lambda_1 \int_a^{\xi} \sum_{i=0}^{S} A_i(\xi,\tau)\, R_i(\Upsilon(\tau))d\tau\right] + S\left[\lambda_2 \int_a^{b} \sum_{j=0}^{r} Bj(\xi,\tau)\, F_j(\Upsilon(\tau))\, d\tau\right]$$

Then

$$S[\Upsilon(\xi)] = u^{\alpha} \sum_{k=0}^{n-1} u^{-(\alpha-k)}\, \Upsilon^{(k)}(0) + \left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_m\, \Upsilon^{(k)}(\xi)\right] + u^{w} S[g(\xi)] + u^{w} S\left[\lambda_1 \int_a^{\xi} \sum_{i=0}^{S} A_i(\xi,\tau)\, R_i(\Upsilon(\tau))d\tau\right] + u^{w} S\left[\lambda_2 \int_a^{b} \sum_{j=0}^{r} Bj(\xi,\tau)\, F_j(\Upsilon(\tau))\, d\tau\right]$$

Now, the Sumudu decomposition method represents the solution in the form of infinite series given as the following expressions

$$\Upsilon(\xi) = \sum_{n=0}^{\infty} \Upsilon_n(\xi) \tag{3}$$

and let $$R_i(\Upsilon(\xi)) = \sum_{n=0}^{\infty} (C_i)_n$$

$$F_j(\Upsilon(\xi)) = \sum_{n=0}^{\infty} (D_j)_n \text{ and } \Upsilon^{(k)}(\xi) = \sum_{n=0}^{\infty} E_n \tag{4}$$

Where $(C_i)_n$, $(D_j)_n$, E_n are the Adomain polynomials. Substituting Eqs (3) and (4) yields:

$$S[\sum_{n=0}^{\infty} \Upsilon_n(\xi)] = u^\alpha \sum_{k=0}^{n-1} u^{-(\alpha-k)} \Upsilon^{(k)}(0) + u^\alpha S\left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_k(\xi) \sum_{n=0}^{\infty} E_n\right]$$
$$+u^\alpha S[g(\xi)] + u^\alpha S\left[\lambda_1 \int_a^x \sum_{j=0}^{s} A_i(\xi,\tau) \sum_{n=0}^{\infty} (C_i)_n \, d\tau\right] +$$
$$u^\alpha S\left[\lambda_2 \int_a^b \sum_{j=0}^{r} B_j(\xi,\tau) \sum_{n=0}^{\infty} (D_j)_n \, d\tau\right]$$

The recursive relation is built on the principle that the zeroth component is the starting point $S[y_0]$ is defined by all terms that arise from

$$S[\Upsilon_0] = u^\alpha \sum_{k=0}^{n-1} u^{-(\alpha-k)} \Upsilon^{(k)}(0) + +u^\alpha S[g(\xi)]$$

And $$S[\Upsilon_{n+1}(\xi)] = u^\alpha S\left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_k(\xi) \sum_{n=0}^{\infty} E_n\right]$$
$$+u^\alpha S\left[\lambda_1 \int_a^x \sum_{j=0}^{s} A_i(\xi,\tau) \sum_{n=0}^{\infty} (C_i)_n \, d\tau\right] +$$
$$u^\alpha S\left[\lambda_2 \int_a^b \sum_{j=0}^{r} B_j(\xi,\tau) \sum_{n=0}^{\infty} (D_j)_n \, d\tau\right], k \geq 0$$

When the preceding equation is transformed using the inverse Sumudu operator, we get the following:

$$\Upsilon_0(x) = S^{-1}\left[u^\alpha \sum_{k=0}^{n-1} u^{-(\alpha-k)} \Upsilon^{(k)}(0)\right] + S^{-1}\left[u^\alpha S[g(\xi)]\right] \tag{5}$$

And

$$\Upsilon_{n+1}(\xi) = S^{-1}\left[u^\alpha S\left[\frac{-1}{a_m(\xi)} \sum_{k=0}^{m-1} a_k(\xi) \sum_{n=0}^{\infty} E_n\right]\right] +$$
$$S^{-1}\left[u^\alpha S\left[\lambda_1 \int_a^x \sum_{j=0}^{s} A_i(\xi,\tau) \sum_{n=0}^{\infty} (C_i)_n \, d\tau\right]\right]$$
$$+S^{-1}\left[u^\alpha S\left[\lambda_2 \int_a^b \sum_{j=0}^{r} B_j(\xi,\tau) \sum_{n=0}^{\infty} (D_j)_n \, d\tau\right]\right], k \geq 0$$

Thus, the series solutions are determined. The n^{th} term approximation $\phi_n = \sum_{k=0}^{n-1} \Upsilon_k$ can be used to approximate the solution.

3. Results

In the above example, Consider the nonlinear Volterra Integro- Differential Equation with Fractional order as follows:

$$D^\alpha y(t) = 3\int_0^t y^2 \, (r) dr + 1 - t^2 y(t), 0 < \alpha \leq 1 \tag{6}$$

And $y(0) = 0$

Solution: To Solve Eq. (1) by the proposed method, we apply Sumuda transform Adomain method on both Sides and using the inverse Sumudu transform, we have

$$\left.\begin{aligned} g(t) &= S^{-1}[u^{\alpha}S(1)] \\ R\big(y(t)\big) &= -S^{-1}[u^{\alpha}S(t^2y)] \\ N\big(y(t)\big) &= 3S^{-1}[u^{\alpha+1}S(y^2)] \end{aligned}\right\} \tag{7}$$

and, by using the recursive relation by Eq. (3.2), we get

$$\left.\begin{aligned} y_0 &= \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ y_1 &= -S^{-1}[u^{\alpha}\, S\,(t^2y_0)] + 3S^{-1}[u^{\alpha+1}S(y_0^2)] \\ y_{n+1} &= -S^{-1}[u^{\alpha}\, S\,(t^2y_n)] + 3S^{-1}[u^{\alpha+1}S(y_n^2)] \end{aligned}\right\}$$

By using the inverse Sumudu transform of Eq.(3.2), we get

$$y_1 = -\frac{(\alpha+2)\, t^{2\alpha+2}}{2\Gamma(2\alpha+2)} + \frac{6\Gamma(2\alpha)t^{3\alpha+1}}{\alpha\, \Gamma^2(\alpha)\, \Gamma(3\alpha+2)}$$

$$y_2 = -S^{-1}[u^{\alpha}\, S(t^2y_1)] + 3S^{-1}[u^{\alpha+1}\, S(2y_0\, y_1 + y_1^2)]$$

$$= \frac{(\alpha+2)\, \Gamma(2\alpha+5)}{2\, \Gamma(2\alpha+2)\, \Gamma(3\alpha+5)} t^{3\alpha+4} - \frac{6\, \Gamma(2\alpha)\, \Gamma\,(3\alpha+4)}{\alpha\, \Gamma^2(\alpha)\, \Gamma(3\alpha+2)\, \Gamma(4\alpha+4)} t^{4\alpha+3}$$
$$- \frac{3\,(\alpha+2)\, \Gamma(3\alpha+3)}{\alpha\, \Gamma(\alpha)\, \Gamma(2\alpha+2)\, \Gamma(4\alpha+4)} t^{4\alpha+3} + \frac{36\, \Gamma\,(2\alpha)\, \Gamma(4\alpha+2)}{\alpha^2\, \Gamma^3(\alpha)\, \Gamma(3\alpha+2)\, \Gamma(5\alpha+3)} t^{5\alpha+5}$$
$$+ \cdots$$

For $\alpha = 1$, then $y\,(t) = t$, which is the exact solution.

Example: Nonlinear Fractional Volterra integro-differential equation as follows:

$$D^{\alpha}\, y(t) = 1 + \int_0^t e^{-r}\, y^2(r)\, dr\, , 3 < \alpha \le 4\ 0 \le t < 1 \tag{8}$$

$$y(0) = 0\, , y'(0) = 1, y''(0) = 1, y'''(0) = 1 \tag{9}$$

When $\alpha = 4, y(t) = e^t$ and can be considered as the optimal solution.

Solution: To solve this problem by the proposed method, We apply STADM of Eq. (4) and using the inverse ST, we get:

$$\left.\begin{aligned} g(t) &= S^{-}\left[\textstyle\sum_{k=0}^{3} u^k\, y^{(k)}(0)\right] + S^{-1}[u^{\alpha}\, S\,(1)] \\ N\big(y(t)\big) &= S^{-1}\, [u^{\alpha+1}\, S(e^{-t}\, y^2)] \end{aligned}\right\} \tag{10}$$

Thus,

$$y_0 = 1, y_1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + S^{-1}\left[u^{\alpha+1}\, S\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\right) y_0^2\right]$$

Then, by using the recursive relation (6) , we get : $y_0 = 1$

$$y_1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$

Hence, the 2-term approximate solution of problems (3) and (4) is

$$y(\tau) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)}$$

When $\alpha = 4$, the exact solution is $y(t) = e^t$

4. Conclusion

The primary objective of this study is to propose a viable approach for solving specific types of fractional Volterra integro-differential equations. We employ Adomian's polynomials and the Sumudu Transform, a domain decomposition technique to address the challenge posed by nonlinear terms. Several examples of nonlinear fractional Volterra integro-differential equations are tested to validate the effectiveness and benefits of the proposed method. The outcomes are carefully evaluated to ensure the efficacy of the procedure. This approach holds great potential for various engineering and scientific applications.

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