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# Soft *P*–field with some of their properties

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**Abstract**

In this paper, we introduce the notion of soft *P*–field. Characterization and different properties of this concept are studied as well as the restriction concept of soft *P*–field is introduced.

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## 1. Introduction

$\sigma -$ field is of a significant topic in measure theory. Wang [1] and Ahmed et al. [2-6] have studied many types of classes of sets, which are generalizations of $\sigma$–field as $\beta -$ field, $\sigma$–ring and $\alpha - \sigma -$ field as well as explained some important results of these notions in measure space. Endou [7] has studied the $\sigma$–ring concept as a family $\mathfrak{M}$ such

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that it is closed under countable union and the difference between any two sets. In 2021 Ahmed et al. [8] have been studied the $\sigma-$field concept to define an outer measure. The idea of $\sigma-$field and field are studied by [9]. The notion of the soft set was studied by [10], where $\mathcal{W}$ is a universal set, and the set of parameters is denoted by $\mathcal{E}$ with respect to $\mathcal{W}$. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{E}$. Then a set of $\mathcal{F}_{\mathcal{A}} = \{(a, \mathfrak{f}_{\mathcal{A}}(a)) : a \in \mathcal{E}, \mathfrak{f}_{\mathcal{A}}(x) \in \mathrm{P}(\mathcal{W})\}$ is called a soft set on $\mathcal{W}$, where $\mathfrak{f}_{\mathcal{A}}: \mathcal{E} \longrightarrow \mathrm{P}(\mathcal{W})$ such that $\mathfrak{f}_{\mathcal{A}}(a) = \Phi$ if $a$ not belong to $\mathcal{A}$. The null soft set is $\mathcal{F}_{\Phi}$ *or* $\widetilde{\Phi}$. $\mathcal{F}_{\tilde{\mathcal{A}}}$ or $\tilde{\mathcal{A}}$ is denotes to $\mathcal{A}$ −universal soft set . If $\mathcal{A} = \mathcal{E}$, then the $\mathcal{A}$ −universal soft set is called a universal soft set and denoted by $\mathcal{F}_{\tilde{\mathcal{E}}}$ or $\tilde{\mathcal{E}}$. $\mathcal{F}_{\mathcal{A}} \widetilde{\bigcup} \mathcal{F}_{\mathcal{B}}$ is symbolized to soft union for $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}$ and defined as the approximate function $\mathfrak{f}_{\mathcal{A}}(a) \bigcup \mathfrak{f}_{\mathcal{B}}(a)$ $\forall a \in \mathcal{E}$. $\mathcal{F}_{\mathcal{A}} \widetilde{\bigcap} \mathcal{F}_{\mathcal{B}}$ is symbolized to soft intersection for $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}$ which is define by the approximate function $\mathfrak{f}_{\mathcal{A}}(a) \bigcap \mathfrak{f}_{\mathcal{B}}(a)$ $\forall a \in \mathcal{E}$.

## 2. The main results

We will involve the definition of soft set in the field, so we introduced the definition of $soft\,\mathcal{P} - field$ and proof many propositions and gave important examples.

**Definition 2.1:** A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$ and let $\widetilde{\mathfrak{M}}$ be a class of soft subsets of $\mathcal{F}_{\tilde{\mathcal{E}}}$ ($\widetilde{\mathfrak{M}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$). Then $\widetilde{\mathfrak{M}}$ is called $soft\,\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ for short ($soft\,\mathcal{P} - field$ on $\mathcal{F}_{\tilde{\mathcal{E}}}$) if the following statement holds:

1- $\mathcal{F}_{\Phi}, \mathcal{F}_{\tilde{\mathcal{E}}} \in \widetilde{\mathfrak{M}}$.

2- If $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \in \widetilde{\mathfrak{M}}$, then $\widetilde{\bigcap_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}} \in \widetilde{\mathfrak{M}}$.

3- If $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \in \widetilde{\mathfrak{M}}$, then $\widetilde{\bigcup_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}} \in \widetilde{\mathfrak{M}}$.

**Definition 2.2:** $(\mathcal{F}_{\tilde{\mathcal{E}}}, \widetilde{\mathfrak{M}})$ is called soft measurable space over soft $\mathcal{P} -$ field, and each member in $\widetilde{\mathfrak{M}}$ is said soft measurable set, where $\mathcal{F}_{\tilde{\mathcal{E}}}$ is universal soft set and $\widetilde{\mathfrak{M}}$ is soft $\mathcal{P} -$ field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

**Example 2.3:** Let $\mathcal{W} = \{\hbar_1, \hbar_2,, \hbar_3\}$ and $\mathcal{E} = \{c_1, c_2, c_3\}$. Define $\widetilde{\mathfrak{M}}$ as follows :

$$\widetilde{\mathfrak{M}} = \left\{\begin{matrix} \mathcal{F}_{\Phi},\ \{(c_1,\{\hbar_1,\hbar_2\}),(c_2,\{\hbar_1,\hbar_3\}),(c_3,\{\hbar_1\})\}, \\ \{(c_1,\{\hbar_1,\hbar_3\}),(c_2,\{\hbar_1,\hbar_2\}),(c_3,\{\hbar_1\})\}, \\ \{(c_1,\{\hbar_1\}),(c_2,\{\hbar_1\}),(c_3,\{\hbar_1\})\}, \\ \{(c_1,\mathcal{W}),(c_2,\mathcal{W}),(c_3,\{\hbar_1\})\},\ \mathcal{F}_{\tilde{\mathcal{E}}} \end{matrix}\right\}.$$

Then, $\widetilde{\mathfrak{M}}$ is a soft $\mathcal{P} -$ field on $\mathcal{F}_{\tilde{\mathcal{E}}}$.

**Proposition 2.4:** Let $\{\widetilde{\mathfrak{M}}_{\alpha}\}_{\alpha \in \mathrm{I}}$ be a collection of soft $\mathcal{P} -$ fields on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\widetilde{\bigcap_{\alpha \in \mathrm{I}}} \widetilde{\mathfrak{M}}_{\alpha}$ also, is the $soft\,\mathcal{P} - field$ on $\mathcal{F}_{\tilde{\mathcal{E}}}$

**Definition 2.5**: A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$. If $\tilde{\mathcal{J}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$. Then $\widetilde{\mathcal{P}(\tilde{\mathcal{J}})}$ is the $soft\,\mathcal{P} - field$ generated by $\tilde{\mathcal{J}}$ which is the soft intersection of all $soft\,\mathcal{P} - fields$ on $\mathcal{F}_{\tilde{\mathcal{E}}}$, that contains $\tilde{\mathcal{J}}$.

**Proposition 2.6:** $\widetilde{\mathcal{P}(\tilde{\mathcal{J}})}$ is smallest $soft$ $\mathcal{P}$-*field* $\mathcal{F}_{\tilde{\mathcal{E}}}$, that contains $\tilde{\mathcal{J}}$.

**Definition 2.7**: Assume $\widetilde{\mathfrak{M}}$ is a $soft\,\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}}$, then a restriction of $\widetilde{\mathfrak{M}}$ over $\mathcal{F}_{\mathcal{B}}$ is:

$$\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \{ \mathcal{F}_{\mathrm{N}} : \mathcal{F}_{\mathrm{N}} = \mathcal{F}_{\mathrm{M}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}, \text{ where } \mathcal{F}_{\mathrm{M}} \in \widetilde{\mathfrak{M}}\}.$$

**Proposition 2.8**: Let $\widetilde{\mathfrak{M}}$ is a $soft\, \mathcal{P}-field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and

$\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$, then $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ is a $soft\, \mathcal{P}-field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\mathcal{B}}$.

***Proof:*** Since $\mathcal{F}_{\Phi}, \mathcal{F}_{\tilde{\mathcal{E}}} \in \widetilde{\mathfrak{M}}$ and $\mathcal{F}_{\Phi} = \mathcal{F}_{\Phi} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}$, $\mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\tilde{\mathcal{E}}} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}$, Then $\mathcal{F}_{\Phi}, \mathcal{F}_{\tilde{\mathcal{E}}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$. Let $\mathcal{F}_{\mathrm{N}_1}, \mathcal{F}_{\mathrm{N}_2}, \ldots \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$, then $\mathcal{F}_{\mathrm{N}_\mathrm{k}} = \mathcal{F}_{\mathrm{M}_\mathrm{k}} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}$ where $\mathcal{F}_{\mathrm{M}_\mathrm{k}} \in \widetilde{\mathfrak{M}}\ \forall k = 1,2, \ldots$. then, we get

$$\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{N}_\mathrm{k}} = \widetilde{\bigcap_{\mathrm{k}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{k}} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}) = (\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{k}}) \widetilde{\cap} \mathcal{F}_{\mathcal{B}} \text{ and}$$

$$\widetilde{\bigcup_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{N}_\mathrm{k}} = \widetilde{\bigcup_{\mathrm{k}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{k}} \widetilde{\cap} \mathcal{F}_{\mathcal{B}}) = (\widetilde{\bigcup_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{k}}) \widetilde{\cap} \mathcal{F}_{\mathcal{B}}. \text{ But}$$

$\widetilde{\mathfrak{M}}$ is a $soft\, \mathcal{P}-field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{\mathrm{M}_\mathrm{k}} \in \widetilde{\mathfrak{M}}\ \forall k = 1,2, \ldots$

Then $\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{k}}, \widetilde{\bigcup_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{k}} \in \widetilde{\mathfrak{M}}$. So, we have $\widetilde{\bigcap_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{N}_\mathrm{k}}, \widetilde{\bigcup_{\mathrm{k}=1}^{\infty}} \mathcal{F}_{\mathrm{N}_\mathrm{k}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$. Therefore, $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ is a $soft\, \mathcal{P}-field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\mathcal{B}}$.

**Remark 2.9:** A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$. If $\tilde{\mathfrak{J}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$, then $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$ is $soft\, \mathcal{P}-field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\mathcal{B}}$.

**Lemma 2.10:** *A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$. If $\tilde{\mathfrak{J}} \subseteq \tilde{P}(\mathcal{F}_{\tilde{\mathcal{E}}})$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. Define a class $\mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})} = \widetilde{\cap}\{\widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}} : \widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}} is\ a\ soft\, \mathcal{P}-field$ on $\mathcal{F}_{\mathcal{B}}$ and $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}}\ \forall \alpha \in I\}$, then $\mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ is smallest $soft$ $\mathcal{P}$-field on $\mathcal{F}_{\mathcal{B}}$ which contain $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$.*

***Proof:*** Since $\forall \alpha \in \mathrm{I}$, $\widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}}$is a soft $\mathcal{P}-$ field on $\mathcal{F}_{\mathcal{B}}$, then from the definition of soft $\mathcal{P}$-field directly $\mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ is soft $\mathcal{P}$-field on $\mathcal{F}_{\mathcal{B}}$.

Assume $\widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}}$is a soft $\mathcal{P}-$ field on $\mathcal{F}_{\mathcal{B}}$ and $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}}\ \forall \alpha \in \mathrm{I}$, then $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \bigcap_{\alpha \in \mathrm{I}} \widetilde{\mathfrak{M}}_{\alpha}|_{\mathcal{F}_{\mathcal{B}}}$, hence $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$.

Let $\widetilde{\mathfrak{M}}\,|_{\mathcal{F}_{\mathcal{B}}}$is a soft $\mathcal{P}-$ field on $\mathcal{F}_{\mathcal{B}}$ and $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathfrak{M}}\,|_{\mathcal{F}_{\mathcal{B}}}$. Then $\widetilde{\mathfrak{M}}\,|_{\mathcal{F}_{\mathcal{B}}} \supseteq \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$. Therefore, the prove is complete.

**Proposition 2.11>** A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$. If $\tilde{\mathfrak{J}} \subseteq \widetilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. Define a class $\widetilde{\mathfrak{M}} = \{\mathcal{F}_{\mathrm{M}} \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_{\mathrm{M}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}} \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}\}$, then $\widetilde{\mathfrak{M}}$ also is a $soft$ *$\mathcal{P}$-field* on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$.

***Proof:*** $\mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ is a $soft\, \mathcal{P}-field$ on $\mathcal{F}_{\mathcal{B}}$ from the previous lemma.

So, $\mathcal{F}_{\Phi}, \mathcal{F}_{\mathcal{B}} \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$. Since $\mathcal{F}_{\Phi} = \mathcal{F}_{\Phi} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\tilde{\mathcal{E}}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}$, then we get $\mathcal{F}_{\Phi}, \mathcal{F}_{\tilde{\mathcal{E}}} \in \widetilde{\mathfrak{M}}$. Assume that $\mathcal{F}_{\mathrm{M}_1}, \mathcal{F}_{\mathrm{M}_2}, \ldots \in \widetilde{\mathfrak{M}}$. Then $(\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}) \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ $\forall$i=1,2,... Now, $(\widetilde{\bigcap_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}}) \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}} = \widetilde{\bigcap_{\mathrm{i}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}})$ and

$(\widetilde{\bigcup_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}}) \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}} = \widetilde{\bigcup_{\mathrm{i}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}})$. Since $\mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ is a soft $\mathcal{P}-$ field on $\mathcal{F}_{\mathcal{B}}$ and $(\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}) \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ $\forall$i=1,2,... Then $\widetilde{\bigcap_{\mathrm{i}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}) \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ and $\widetilde{\bigcup_{\mathrm{i}=1}^{\infty}} (\mathcal{F}_{\mathrm{M}_\mathrm{i}} \widetilde{\cap}\, \mathcal{F}_{\mathcal{B}}) \in \mathcal{P}\widetilde{(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$.Thus $\widetilde{\bigcap_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}} \in \widetilde{\mathfrak{M}}$ and $\widetilde{\bigcup_{\mathrm{i}=1}^{\infty}} \mathcal{F}_{\mathrm{M}_\mathrm{i}} \in \widetilde{\mathfrak{M}}$.

Therefore, $\widetilde{\mathfrak{M}}$ is a $soft$ $\mathcal{P}$*-field* on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$.

**Theorem 2.12**: *A universal set* $\mathcal{W}$ *and a set of parameters* $\mathcal{E}$ *with respect to* $\mathcal{W}$. *If* $\tilde{\mathfrak{J}} \subseteq \tilde{P}(\mathcal{F}_{\tilde{\mathcal{E}}})$ *and* $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. *Then* $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})} = \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$.

***Proof:*** From Proposition2.6: we will get $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}$ is a soft $\mathcal{P}$-field on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$. Hence from Proposition2.8: we will have $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$ is a soft $\mathcal{P}$-field on $\mathcal{F}_{\mathcal{B}}$. Assume $\mathcal{F}_{\mathrm{N}} \in \tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$. Then $\mathcal{F}_{\mathrm{N}} = \mathcal{F}_{\mathrm{M}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$ where $\mathcal{F}_{\mathrm{M}} \in \tilde{\mathfrak{J}}$. But $\tilde{\mathfrak{J}} \subseteq \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}$, so we have $\mathcal{F}_{\mathrm{M}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}$ and thus $\mathcal{F}_{\mathrm{N}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$.

Hence $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$. Therefore, $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$ is $soft\ \mathcal{P} - field$ on $\mathcal{F}_{\mathcal{B}}$ that contains $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$. But by Lemma2.10: $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ is smallest is $soft$ $\mathcal{P}$*-field* on $\mathcal{F}_{\mathcal{B}}$that contains $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$, which implies that $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})} \subseteq \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$.

Now, if we define $\widetilde{\mathfrak{M}}$ as below :

$\widetilde{\mathfrak{M}} = \{\mathcal{F}_{\mathrm{M}} \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}} : \mathcal{F}_{\mathrm{M}} \tilde{\cap} \mathcal{F}_{\mathcal{B}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}\}$, then by Proposition 2.11:

We will get $\widetilde{\mathfrak{M}}$ is a soft $\mathcal{P} -$ field on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$.

Let $\mathcal{F}_{\mathrm{C}} \in \tilde{\mathfrak{J}}$, then $(\mathcal{F}_{\mathrm{C}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}) \in \tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$, but $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$ implies that $(\mathcal{F}_{\mathrm{C}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}) \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$, hence $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}$ and $\tilde{\mathfrak{J}} \subseteq \widetilde{\mathfrak{M}}$.

Now, if $\mathcal{F}_{\mathrm{N}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$, then $\mathcal{F}_{\mathrm{N}} = \mathcal{F}_{\mathrm{M}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$, where $\mathcal{F}_{\mathrm{M}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}$.

But $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})} \subseteq \widetilde{\mathfrak{M}}$, leads to $\mathcal{F}_{\mathrm{M}} \in \widetilde{\mathfrak{M}}$, hence $\mathcal{F}_{\mathrm{N}} \in \widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$. Consequentially, $\widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \widetilde{\mathcal{P}(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}})}$. This completes the proof.

**Proposition 2.13:** Suppose $\widetilde{\mathfrak{M}}$ is a $soft\ \mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and

$\mathcal{F}_{\mathrm{C}} \tilde{\subset} \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$ such that $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}$, then $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$.

***Proof:*** Since $\mathcal{F}_{\mathrm{C}} \tilde{\subset} \mathcal{F}_{\mathcal{B}}$, then $\mathcal{F}_{\mathrm{C}} = \mathcal{F}_{\mathrm{C}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$ , but $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}$, hence $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ by the definition of $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$.

**Note 2.14:** Suppose $\widetilde{\mathfrak{M}}$ is $soft\ \mathcal{P} - field\ o\hbar$ $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ &

$\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$.Then not necessarily that $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \subset \widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{M}} \subset \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$.

The following examples explain these facts.

**Example 2.15**:Let $\mathcal{W} = \{\hbar_1, \hbar_2, \hbar_3\}$ and $\mathcal{E} = \{c_1, c_2, c_3\}$. Define $\widetilde{\mathfrak{M}}$ as follows :

$$\widetilde{\mathfrak{M}} = \begin{Bmatrix} \mathcal{F}_{\Phi}, \{(c_1, \{\hbar_1, \hbar_2\}), (c_2, \{\hbar_1, \hbar_3\}), (c_3, \{\hbar_1\})\}, \\ \{(c_1, \{\hbar_1, \hbar_3\}), (c_2, \{\hbar_1, \hbar_2\}), (c_3, \{\hbar_1\})\}, \\ \{(c_1, \{\hbar_1\}), (c_2, \{\hbar_1\}), (c_3, \{\hbar_1\})\}, \\ \{(c_1, \mathcal{W}), (c_2, \mathcal{W}), (c_3, \{\hbar_1\})\},\ \mathcal{F}_{\tilde{\mathcal{E}}} \end{Bmatrix}.$$

Then $\widetilde{\mathfrak{M}}$ is $soft\ \mathcal{P} - field$ on $\mathcal{F}_{\tilde{\mathcal{E}}}$. Put $\mathcal{F}_{\mathcal{B}} = \{(c_1, \{\hbar_1\}), (c_2, \{\hbar_2\}), (c_3, \{\hbar_3\})\}$, then $\{\hbar_k\} \subseteq \mathcal{W}\ \forall k = 1,2,3$. Hence $\mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. So, we have $\mathcal{F}_{\Phi} = \mathcal{F}_{\Phi} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$, $\mathcal{F}_{\tilde{\mathcal{E}}} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\mathcal{B}}$.

$\{(c_1, \{\hbar_1, \hbar_2\}), (c_2, \{\hbar_1, \hbar_3\}), (c_3, \{\hbar_1\})\} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \{(c_1, \{\hbar_1\})\}$ ,

$\{(c_1, \{\hbar_1, \hbar_3\}), (c_2, \{\hbar_1, \hbar_2\}), (c_3, \{\hbar_1\})\} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \{(c_1, \{\hbar_1\}), (c_2, \{\hbar_2\})\}$ ,

$\{(c_1, \{\hbar_1\}), (c_2, \{\hbar_1\}), (c_3, \{\hbar_1\})\} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \{(c_1, \{\hbar_1\})\}$,

$\{(c_1, \mathcal{W}), (c_2, \mathcal{W}), (c_3, \{\hbar_1\})\} \tilde{\cap} \mathcal{F}_{\mathcal{B}} = \{(c_1, \{\hbar_1\}), (c_2, \{\hbar_2\})\}$.

Thus $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \{\mathcal{F}_{\Phi}, \{(c_1, \{\hbar_1\})\}, \{(c_1, \{\hbar_1\}), (c_2, \{\hbar_2\})\}, \mathcal{F}_{\mathcal{B}}\}$.

Now, $\{(c_1, \{\hbar_1, \hbar_2\}), (c_2, \{\hbar_1, \hbar_3\}), (c_3, \{\hbar_1\})\}$ belong to $\widetilde{\mathfrak{M}}$ but it does not belong to $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ and $\{(c_1, \{\hbar_1\}), (c_2, \{\hbar_2\})\}$ belong to $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ but it does not belong to $\widetilde{\mathfrak{M}}$.

In the following proposition, we put a necessary and sufficient condition for us to obtain $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \subset \widetilde{\mathfrak{M}}$.

**Proposition 2.16:** Assume $\widetilde{\mathfrak{M}}$ is $soft\,\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. If $\mathcal{F}_{\mathcal{B}} \in \widetilde{\mathfrak{M}}$, then $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$ and $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \subset \widetilde{\mathfrak{M}}$.

***Proof:*** Assume $\mathcal{F}_{\mathrm{N}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$, then $\mathcal{F}_{\mathrm{N}} = \mathcal{F}_{\mathrm{M}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$, where $\mathcal{F}_{\mathrm{M}} \in \widetilde{\mathfrak{M}}$.

Since $\widetilde{\mathfrak{M}}$ is $soft\,\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{\mathrm{M}}, \mathcal{F}_{\mathcal{B}} \in \widetilde{\mathfrak{M}}$, then $\mathcal{F}_{\mathrm{N}} = \mathcal{F}_{\mathrm{M}} \tilde{\cap} \mathcal{F}_{\mathcal{B}} \in \widetilde{\mathfrak{M}}$, hence $\mathcal{F}_{\mathrm{N}} \in \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$.

Therefore, $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \subseteq \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$. Let $\mathcal{F}_{\mathrm{D}} \in \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$. Then $\mathcal{F}_{\mathrm{D}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathrm{D}} \in \widetilde{\mathfrak{M}}$. Hence $\mathcal{F}_{\mathrm{D}} = \mathcal{F}_{\mathrm{D}} \tilde{\cap} \mathcal{F}_{\mathcal{B}}$, thus $\mathcal{F}_{\mathrm{D}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$ because $\mathcal{F}_{\mathrm{D}} \in \widetilde{\mathfrak{M}}$. So, $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \supseteq \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$. Then, $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$.To claim $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} \subset \widetilde{\mathfrak{M}}$, assume $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$, then $\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}$.

**Remark 2.17:** Suppose $\widetilde{\mathfrak{M}}$ is soft $\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ &

$$\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}},\ \text{if}\ \mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\tilde{\mathcal{E}}},\ \text{then}\ \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \widetilde{\mathfrak{M}}.$$

***Proof:*** Since $\mathcal{F}_{\mathcal{B}} = \mathcal{F}_{\tilde{\mathcal{E}}}$ and $\mathcal{F}_{\tilde{\mathcal{E}}} \in \widetilde{\mathfrak{M}}$, then $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\}$.

Hence for $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$, we get $\mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}$.For any $\mathcal{F}_{\mathrm{M}} \in \widetilde{\mathfrak{M}}$, then $\mathcal{F}_{\mathrm{M}} \tilde{\subseteq} \mathcal{F}_{\tilde{\mathcal{E}}} = \mathcal{F}_{\mathcal{B}}$. Hence $\mathcal{F}_{\mathrm{M}} \in \{\mathcal{F}_{\mathrm{C}} \tilde{\subseteq} \mathcal{F}_{\mathcal{B}} : \mathcal{F}_{\mathrm{C}} \in \widetilde{\mathfrak{M}}\} = \widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$. Thus $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}} = \widetilde{\mathfrak{M}}$.

**Conclusion**

The main results of this work are if $\widetilde{\mathfrak{M}}$ is a soft $\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\tilde{\mathcal{E}}}$ & $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$, then $\widetilde{\mathfrak{M}}|_{\mathcal{F}_{\mathcal{B}}}$is a $soft\,\mathcal{P} - field$ on $\mathcal{W}$ relative to $\mathcal{F}_{\mathcal{B}}$. $\mathcal{P}\widetilde{\left(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}\right)}$ is smallest soft $\mathcal{P} - field$ on $\mathcal{F}_{\mathcal{B}}$ which contain $\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}$ as well as where A universal set $\mathcal{W}$ and a set of parameters $\mathcal{E}$ with respect to $\mathcal{W}$. If $\tilde{\mathfrak{J}} \subseteq \tilde{\mathrm{P}}(\mathcal{F}_{\tilde{\mathcal{E}}})$ and $\mathcal{F}_{\Phi} \neq \mathcal{F}_{\mathcal{B}} \tilde{\subset} \mathcal{F}_{\tilde{\mathcal{E}}}$. Then $\mathcal{P}\widetilde{\left(\tilde{\mathfrak{J}}|_{\mathcal{F}_{\mathcal{B}}}\right)} = \widetilde{\mathcal{P}(\tilde{\mathfrak{J}})}|_{\mathcal{F}_{\mathcal{B}}}$.

## References


[1] Z. Wang and G. Klir, Generalized Measure Theory, 1st ed. Springer Science and Business Media, LLC, New York (2009).

[2] I. S. Ahmed and H. H. Ebrahim, "Generalizations of σ-field and new collections of sets noted by δ-field," AIP Conf. Proc., vol. 2096, no. 20019, p. 020019-1 - 020019-6 (2019), doi: 10.1063/1.5097816.

[3] I. S. Ahmed and H. H. Ebrahim, "On $\alpha$-field and $\beta$-field," *J. Phys. Conf. Ser.*, vol. 1294, no. 3 (2019), doi: 10.1088/1742-6596/1294/3/032015.

[4] S. H. Asaad, I. S. Ahmed, and H. H. Ebrahim, "On Finitely Null-additive and Finitely Weakly Null-additive Relative to the $\sigma$–ring," *Baghdad Sci. J.*, vol. 19, no. 5, pp. 1148–1154 (2022).

[5] I. S. Ahmed and H. H. Ebrahim, "On measure relatively to fuzzy $\sigma$-algebra," J. Phys. Conf. Ser., vol. 2322, no. 1 (2022), doi: 10.1088/1742-6596/2322/1/012020.

[6] I. S. Ahmed, H. H. Ebrahim, and A. Al-Fayadh, "$\gamma$– algebra of Sets and Some of its Properties," *Iraqi J. Sci.*, vol. 63, no. 11, pp. 4918–4927 (2022), doi: 10.24996/ijs.2022.63.11.28.

[7] N. Endou, K. Nakasho, and Y. Shidama, "$\sigma$-ring and $\sigma$-algebra of Sets," Formaliz. Math., vol. 23, no. 1, pp. 51–57 (2015), doi: 10.2478/forma-2015-0004.

[8] I. S. Ahmed, S. H. Asaad, and H. H. Ebrahim, "Some new properties of an outer measure on a $\sigma$–field," *J. Interdiscip. Math.*, vol. 24, no. 4, pp. 947–952 (2021), doi: 10.1080/09720502.2021.1884386.

[9] R. B. ASH, Real Analysis and Probability, 1st ed. Academic Press (1972).

[10] N. Çağman, S. Karata, and S. Enginoglu, "Soft topology," *Comput. Math. with Appl.*, vol. 62, no. 1, pp. 351–358 (2011), doi: 10.1016/j.camwa.2011.05.016.