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Robust and optimal control of linear control systems and their equivalency

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Abstract

In the field of control theory, there has been an increase in interest in studies on the equivalence of robust and optimum control of linear control systems. To provide appropriate constraints for increased stability and control performance, we study in this work the link between linear systems under robust and optimum control as well as the equivalence between the two. We will present in this research significant changes to the way constraints are designed, which may also improve system control and stability. We will show that with theorems and examples.

Subject Classification: *14C20, 47A48.*

Keywords: *Optimal control problem, Robust control problem, Ordinary differential equations, Uncertainty, Matched condition.*

1. Introduction

In the field of control theory, studies on the equivalence of robust and optimum control of linear control systems have been of increasing interest [1]. Robust and optimal control has been extensively studied to design controllers that can provide satisfactory performance even in the presence of system uncertainties or to find the best control. Previous research investigated the robust control of nonlinear systems by Lin, Brandt, and Sun in 1992 [2], and the optimal control approach to robust control of robot manipulators by Lin and Brandt in 1998 [3]. Other studies have focused on improving the stability of

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linear systems; for example, Cao and Lin 2004 studied continuous-time singular systems with exogenous disturbance control [4], while Tan, Shu, and Lin 2009 developed an OCP approach to RCP tracking of linear systems [5]. Khalaf and Flayyih 2021 analyzed the stability of a fractional SIR model regarding delays in state and control variables [6].

In this research, we aim to improve the results obtained in a study by Lin in 2007 on robust control transformation to an optimal control approach by putting appropriate restrictions on the perturbed state matrix A(q) of linear control systems [7]. We demonstrate that a cost function that captures uncertainty's constraint, regulation, and control of the OCP solution will solve the RCP. Our technique provides fresh insights into building resilient and optimal control strategies for linear control systems, which we hope will improve this topic.

The remaining parts of this paper are structured as described below.: In Section 2, we build and analyze a mathematical uncertainty model, including the effect of uncertainty on the system's stability. Section 3 describes our approach to developing appropriate restrictions on the state matrix A(q) of linear control systems. Section 4 presents an example of our simulation results and compares them to those obtained in previous studies. Finally, in Section 5, we conclude our study and discuss the implications of our findings for future research in this area.

2. Problem Formulation

Take into account that the linear system is

$$\dot{z} = A(q)z + Bu \tag{1}$$

Where $E, A(q) \in R^{m\times m}, z \in R^{m}$, $u \in R^{r}$, $B \in R^{m\times r}$ are the system coefficients and for all $q \in [a, b]$. To solve the RCP uncertainty problem, the first step is to write A(q) in equivalent form. Where our write to decomposition sum of matched and unmatched parts, i.e. A(q)-A(q_0) $= BB^T\Delta A(q) + (I - BB^T)\Delta A(q)$. If their uncertainties are in the range of B, then the form of A(q) became, A(q)=A(q_0) $+ BB^T\Delta A(q)$. In the case of a match, then$(I - BB^T)\Delta A(q)$=0. The treatment of the issue lies in conditions placed on the constraints of matrix A(q) using one of the most important properties of matrices, which is the conversion of the constant into a matrix. Let G(q)= $B^T\Delta A(q)$.

3. Robust Control Problem

Find a feedback control (AFC) low u=-kz Such that the closed-loop system (CLS) of

$$\dot{z} = A(q_0)z + Bu + BG(q)z \tag{2}$$

$$\dot{z} = (A(q_0)z - Bk)z + BG(q)z \tag{3}$$

Is asymptotically stable for some $q_0 \in [a, b]$. If the eigenvalues of the matrixes $(A(q_0)z - Bk)$ are negative, then the system (1) is stable.

In case matched for all q∈[a,b], the upper bounded h on the uncertainty $G(q)\ G(q)^T$ is (constant), such that

$$G(q)\ G(q)^T < h \tag{4}$$

To transform an RCP into an OCP by constraining the perturbation and adding the constraint to the objective function, we have obtained the nominal system, then by applying the Hamilton − Jacob − Bellman (HJB)necessary condition, we have obtained the optimal control. This linear RCP is difficult to overcome in general. The difficulty is to find the matrix k in this problem. Our approach is to transform (2) into the following problem.

4. Optimal Control Equivalent Problem

The OCP becomes the following linear quadratic regulator problem (LQR). For the nominal system:

$$\dot{z} = A(q_0)z + Bu \tag{5}$$

Find AFC low u = −kz that minimizes the following cost function

$$\mathrm{J(z)} = \int_0^\infty (z^T Hz + z^T z + u^T\, u)dt \tag{6}$$

To solve the LQR, the first step solves the algebraic Riccati equation (ARE)

$$\mathrm{SA} + A^T\mathrm{S} - \mathrm{SB}R^{-1}B^T\mathrm{S} + \mathrm{Q} = 0 \tag{7}$$

then the workaround to the LQR is given by u = −kz(t) where (k = $-R^{-1}B^T$S). S is the only symmetrical positive definite workaround of the ARE using Matlab to get it.

Lemma: *The necessary condition for the existence of OCP is that J(z) would have to be satisfied the HJB equation.*

Theorem 1: [2] *Assume the solution to the OCP exists, then it is a solution to the RCP.*

Theorem 2: *Global asymptotic stability of linear control systems if we get positive definite state matrix: An optimal control approach.*

Proof: Take into account the system (1):

$$\dot{z} = A(q)z + Bu$$

Define the OCP, and finds the minimum of the cost function

$$\mathrm{J(z)} = \min_u\ \int_0^\infty (z^T Hz + z^T z + u^T\, u)dt.$$

subject to the system.

$\dot{z} = A(q_0)z + Bu$. From lemma get

$$\min_u (z^T Hz + z^T z + u^T u + J_z^T (A(q_0)z + Bu) = 0 \tag{8}$$

Now if $\Delta \neq 0$and $\exists u \in \Delta$, where ($\Delta$the set of all successful controllers)∋ Eq (8) satisfied and if u = −kz then there exists a solution to the OCP and $\partial J/\partial u\ =\ 0$

$$\min_u\ \left(z^T Hz\ + z^T Q\, z\ + \mu^T R\mu +\ + J_z^T((A\,(q_0)z + +B\,u)\right) = 0 \tag{9}$$

$$2\mu^T R + J_z^T B = 0 \tag{10}$$

Now we show that u = −kz is the solution to the RCP, i.e., z = 0 of Eq (1) is globally asymptotically stable for all passable uncertainty A(q), and we can show that J(z) is a Lyapunov function for system (2), the system achieves the following goals

- $J(z) > 0, z \neq 0$
- $J(z) = 0, z = 0$
- $\dot{J}(z) < 0$ for all $z \neq 0$ we will prove it

$$\dot{J}(z) = J_z^T \dot{z}$$
$$= J_z^T (A\,(q_0)\mathrm{z} + \mathrm{Bu} + B\,G(q)z)$$

From Eq (9),(10) get:

$$\mathrm{j(z)} = -z^T z - z^T hz - u^T u + J_z^T (B\,G(q)z)$$
$$\dot{J}(z) = -z^T z - z^T hz - u^T u - 2u^T G(q)$$
$$\dot{j}(z) = -z^T z - z^T (h - G(q)G(q)^T)z - z^T (k + G(q)(k + G(q)^T)z$$
$$\dot{J}(z) \le -z^T z$$

Thus, the conditions of the Lyapunov local stability are satisfied. Consequently, there is a neighbourhood $N = \{z{:}\ \|z(t)\| < C\}$ for some $\mathrm{C} > 0$. Such that if $z(t)$ enters N then $lim_{t\to\infty}\, z(t) = 0$

But z(t) cannot remain forever outside N, otherwise

$z^T z > \mathrm{C}$ For all $\mathrm{t} > 0$, therefore

$$J(z(t)) - J(z(0)) = \int_0^t \dot{J}(z(s))ds$$
$$J(z(t)) \le J(z(0)) - \mathrm{C}^2\mathrm{t}$$

Illustration example

Take into account the RCP system

$$\dot{z} = A(q)z + Bu$$
$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+\ q & q \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Where q $\in$ $[-10,10]$ is the uncertainty. We want to create a reliable CLS by designing a robust control u=-kz(t). Assuming q_0=0, and cheek the controllability of the matrix (B: A(q_0)B) is C=(B: A(q_0)B) = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The nominal system may be controlled since C has a complete rank. The equation of state may be expressed as,

$$\begin{pmatrix} \dot{z_1} \\ \dot{z_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (q \quad q) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

From Eq(4) get $(q)\ G(q)^T = (q \quad q) \begin{pmatrix} q \\ q \end{pmatrix} = 2q^2 < h$, where h is constant.

As a result, the RCP of determining the AFC rule that stabilizes the system, including all q$\in$ $[a, b]$ may be transformed into the OCP system shown below in the case of the nominal system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Find u = kz, AFC rule that leads to the lowest functional cost.

$$\int_0^\infty (z^T Hz + z^T z + u^T\, u)dt = \int_0^\infty (z^T (H + I)z + u^T\, u)dt$$

$$= \int_0^\infty (z^T Qz + u^T\, u)dt$$

Where H = hI, I is the identity matrix and Q = H +I is positive and seems definite, for the (LQR), Q = $\begin{pmatrix} 201 & 0 \\ 0 & 201 \end{pmatrix}$

We can use MATLAB to solve this LQR, and then the solution of the ARE Eq (7) gives us S=$\begin{pmatrix} 216.2127 & 15.2127 \\ 15.2127 & 15.2127 \end{pmatrix}$, the eigenvalues of S are (14.0678, 217.3576), therefore S is positive definite symmetric take u = −kz, where(k = $-R^{-1}B^T$S) then u = −15.2127z1− 15.2127z2 see Figure (1). Stability is being checked by using the eigenvalues of −Bk is (−1, −14.2127) therefore the system $\dot{z} = (A(q_0)z - Bk\,)z$. Is asymptotically stable see Figure (3).

But in[6] used the condition

$G(q)^T G(q) = \begin{pmatrix} q \\ q \end{pmatrix} (q \quad q) \leq h = \begin{pmatrix} 100 & 10\,0 \\ 100 & 100 \end{pmatrix}$, where h is the matrix, Q = $\begin{pmatrix} 101 & 10\,0 \\ 100 & 101 \end{pmatrix}$, and S=$\begin{pmatrix} 12.0995 & 11.0995 \\ 11.0995 & 11.0995 \end{pmatrix}$, and the eigenvalues of S are (0.447, 22.7103), then u=−11.0995z1−11.0995z2 see Figure (2).

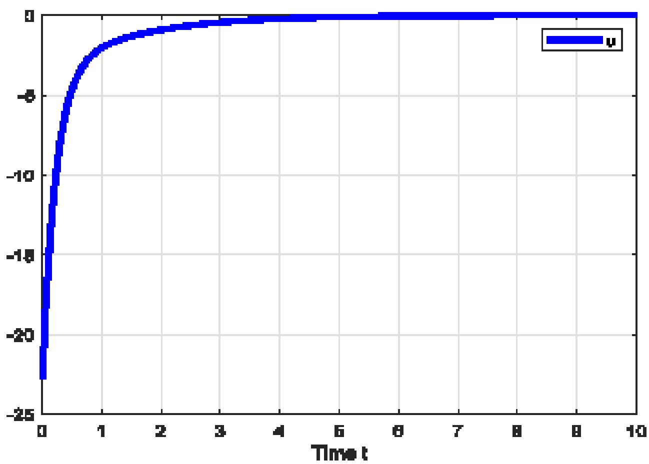

Figure 1

q=10 robust control

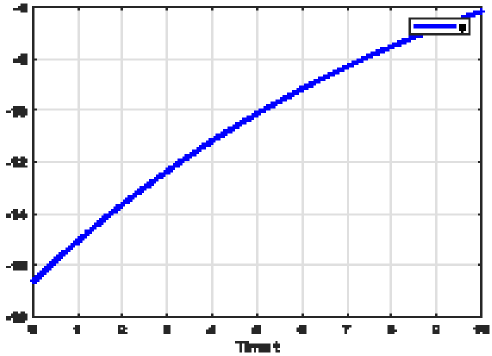

Figure 2
q = 10 robust control

Figure (1) represents the fast response to asymptotically stability due to the restrictions imposed on the perturbation, while Figure (2) represents the opposite.

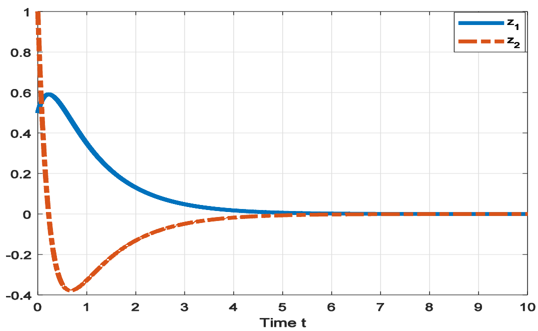

Figure 3
q=10 robust solution

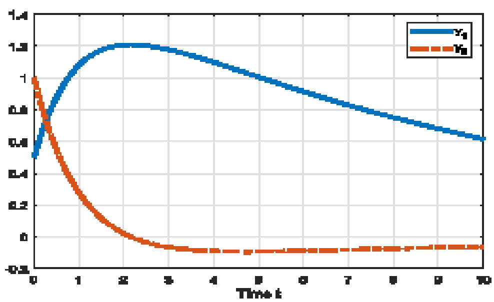

Figure 4
q = 10 robust solutions

We can see Figure (3) when q=10 and the time is [0,10] is asymptotically stable, but the Figure (4) when q=10 and the time is [0,10] is not asymptotically stable this is due to the limitation used on matrix A(q).

5. Conclusion

In this paper, we have investigated the relationship between robust and optimal control of linear control systems through the use of appropriate constraints for increased stability and control performance. Our results show that the two approaches are equivalent when dealing with matched uncertainties in the system. Our approach offers new insights into the development of robust and optimal control strategies for linear control systems and suggests that constraining the state matrix A(q) can lead to improved stability and control performance. Future research could explore the extension of this approach to nonlinear systems and the effectiveness of the approach under more general classes of uncertainties. These findings have significant potential applications in real-world systems and control engineering design and could result in significant improvements in system control and stability.

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