

Rational map has a Cantor Circle Julia set and a Sierpinski Carpet Julia set of degree m

Iftichar M. T. Al-Shara'a *

Department of Mathematics

College of Education of Pure Sciences

University of Babylon

Babylon

Iraq

Hassanein Q. Al-Salami [†]

Department of Biology

College of Science

University of Babylon

Babylon

Iraq

Abstract

This research aims to study complex dynamics, and one of the best examples of this concept is the Julia set, for this we will present new concepts specific to the Julia set of the rational map $J(G_\mu)$. We introduce the family of complex rational maps as:

$$G_\mu(z) = 2\mu^{m+1}z^{-m} + z^{2m} - \mu^{3m+1} / z^m(z^{2m} - \mu^{m-1}),$$

with $m \geq 2$ and $\mu \in \mathbb{C} \setminus \{0\}$ ($\mathbb{C}$ complex numbers) such that $\mu^{m+1} \neq 1$ and $\mu^{2m+2} \neq 1$. We prove $J(G_\mu)$ is a degenerate Sierpinski carpet or a Sierpinski carpet or a Cantor circle by condition one of the free critical points has an image for G_μ is diverge on ∞ or 0.

Subject Classification: Primary 37F50, Secondary 37P40.

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Introduction

In Eighty of the past century, McMullen found the concept of a Julia set to be cantor circles of the rational map $R_\alpha(z) = \frac{\alpha}{z^3} + z^2$, given that $0 \neq \alpha$ is small,

* E-mail: Pure.iftichar.talb@uobabylon.edu.iq (Corresponding Author)

[†] E-mail: haszno732@gmail.com

as in [7]. After that, the idea of a generalization of McMullen maps $R_\alpha(z) = \frac{\alpha}{z^n} + z^d$, came by some scientists by studying some dynamic phenomena, see [2], [4], [8]. In [4] Devaney also some authors introduces a theorem about escape for the iterations of critical points. In [6] had studied the maps $f_\alpha(z) = \frac{z^d(z^{2d}-\alpha^{d+1})}{z^{2d}-\alpha^{3d-1}}$, and They found the following: Julia sets are either Sierpinski carpet or quasi circles or cantor circles for the orbit of the critical points. Look also [1],[6]. Milnor and Tan introduced the rational map with a Sierpinski carpet Julia set. Lyubich, Bonk, and Merenkov studied Julia sets of quasi-symmetric geometry for post-critically-finite maps see [5].

The paper is organized as follows. In section 2, we give the symmetric dynamical behaviors for G_b and some lemmas and propositions. Section 3 discusses a proof of some of the theorem. Section 4 concludes and considers future work.

1. Statement of the basic results

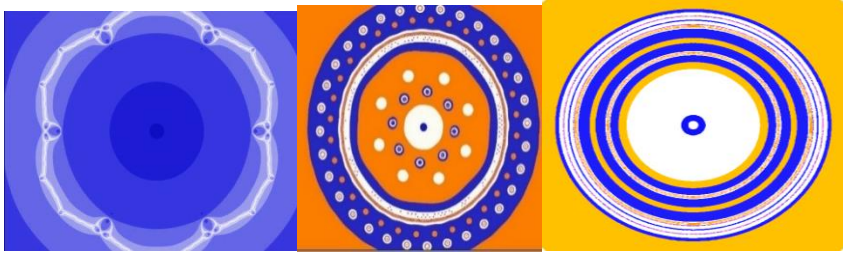
If $\gamma \subset \mathbb{C}_\infty$ genesis from many simple closed curves uncountably that are homeomorphic to Cantor middle third $\times S^1$. Then γ is a Cantor circle. A $\Gamma \subset \mathbb{C}_\infty$ is a Sierpinski carpet iff $\text{int}(\Gamma) = \emptyset$, also $\Gamma = \mathbb{C}_\infty \setminus \bigcup_{i \in \mathbb{N}} V_i$, wherever Jordan disks $V_i \subseteq \mathbb{C}$ such that $\partial V_i \cap \partial V_\ell = \emptyset$ for $i \neq \ell$, also as $i \rightarrow \infty$, $\text{diam} V_i \rightarrow 0$. We say a compact set Γ is a degenerate Sierpinski carpet if $\partial V_i \cap \partial V_\ell \neq \emptyset$. In [3], we offer the subject (quasi circle) for Julia set when l_∞ or l_0 contains one of the free critical points.

Now, we offer the basic results:

Theorem 1: *If the iteration of $G_\mu(c_\mu)$ goes to 0 or ∞ . Then*

- (1) *$J(G_\mu)$ is a Cantor circle. If $G_\mu(c_\mu) \in l_\infty$ or $G_\mu(c_\mu) \in l_0$ but $c_\mu \notin l_\infty$ or $c_\mu \notin l_0$.*
- (2) *Suppose that $G_\mu^i(c_\mu) \in l_\infty$ or $G_\mu^i(c_\mu) \in l_0$ where $i \geq 2$ also $G_\mu^\ell(c_\mu) \notin l_\infty$ or $G_\mu^\ell(c_\mu) \notin l_0$ where $0 \leq \ell < i$ also,*
- (2') *$J(G_\mu)$ is a Sierpinski carpet if $\partial l_\infty \cap \partial l_0 = \emptyset$.*
- (2'') *$J(G_\mu)$ is a degenerated Sierpinski carpet if $\partial l_\infty \cap \partial l_0 \neq \emptyset$.*

Theorem 2: *Each McMullen map is not topologically conjugated to G_μ analogical to Julia sets if $J(G_\mu)$ is a Cantor circle as shown in Figure 1.*

**Figure 1**

When $m = 4$. Right: $\mu = 0.0055492149i$ and $J(G_\mu)$ is a Cantor circles;
 median: $\mu = 0.3398 + 0.266i$, so $J(G_\mu)$ is a Sierpinski carpet; left: $\mu = 1.25$ and
 $J(G_\mu)$ is a degenerated carpet.

2. Preliminaries

The following Lemma is helping to our work.

Lemma 2.1: For any map $G_\mu : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ such that

$$G_\mu(z) = 2\mu^{m+1}z^{-m} + \frac{z^{2m} - \mu^{3m+1}}{z^m(z^{2m} - \mu^{m-1})},$$

1. Let $\omega \in \mathbb{C}$ satisfy $\omega^{2m} = 1$. Then $G_\mu^k(\omega z) = \omega^{-mk} G_\mu^k(z)$ for some $k \geq 1$.
2. For any $T(z) = \frac{\mu^2}{z}$, then G_μ holds the conjugate
 $G_\mu \circ T(z) = T \circ G_\mu(z)$, for any $z \in \mathbb{C}_\infty$.
3. Assume that $\mathbb{U}$ is a Fatou component of G_μ , then $\mathbb{U} = T(\mathbb{U})$. In particular, $l_\infty = T(l_0)$ and $l_0 = T(l_\infty)$.
4. Suppose that $z \in l_\infty$ or l_0 , then $\omega z \in l_\infty$ or l_0 respectively, such that $\omega^{2m} = 1$ satisfy for ω . So both l_∞ and l_0 have $2m$ -fold symmetry.
5. If G_μ contain for all Fatou components V . If $z_0, \omega^k z_0 \in V$, where $\omega^k \neq 1$ with for integer $k \neq 0 \pmod{2n}$. Then $\omega^i z_0 \in V$ for all i . In particular, V has $2m$ -fold symmetry around 0.

For the proof Lemma 2.1, the method is similar to the proof technique for [1, Corollary 2.2].

If $G'_\mu(z) =$

$$-mz^{-m-1} \left[\frac{-(3\mu^{3m+1} + 4\mu^{2m} - \mu^{m-1})z^{2m} + z^{4m}(1 + 2\mu^{m+1}) + \mu^{4m} + 2\mu^{3m-1}}{(z^{2m} - \mu^{m-1})^2} \right], \quad (2.1)$$

then $c_\mu^{2m} = \frac{(3\mu^{3m+1} + 4\mu^{2m} - \mu^{m-1}) \pm \sqrt{9\mu^{6m+2} + 16\mu^{5m+1} - 10\mu^{4m} - 16\mu^{3m-1} + \mu^{2m-2}}}{2 + 4\mu^{m+1}}$, or
 $z = \infty$ with multiplicity $m - 1$, by Lemma (2.1), 0 is a critical point of G_μ with multiplicity $m - 1$.

Also, we have $C_\mu(G_\mu) = \{\omega_0^k c_b, \omega_0^d \frac{\mu^2}{c_\mu} : 0 \leq d \leq 2m-1\}$, where $\omega_0 = e^{\frac{\pi i}{m}}$, thus G_μ contains only one free critical orbit.

Theorem 2.2: $J(G_\mu)$ is quasi circle if one of c_μ belong to l_∞ or l_0 .

Proposition 2.3: Each of the preimages of the boundaries of l_∞, l_0 also the $\partial l_\infty, \partial l_0$ are quasi circles surrounding 0.

For the proof Theorem 2.2 and Proposition 2.3, the method is similar to the proof technique for [3]

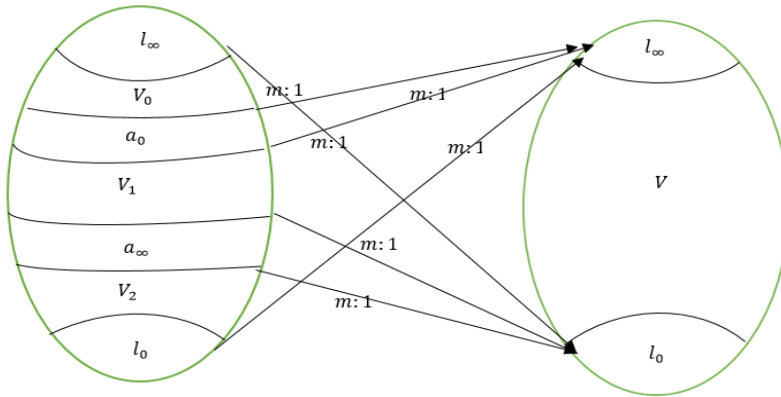


Figure 2
A description of the map G_μ on Riemann's Sphere.

3. Proof Theorems 1 and 2

The same propositions are given to show Theorem 1 and 2, note that the l_0 and l_∞ have no critical points.

By Theorem 2.2, if one of c_μ lies in l_0 or l_∞ , we have $J(G_\mu)$ is quasi circle. Suppose that $c_\mu \in a_0$, where $a_0 = G_\mu^{-1}(l_\infty) \setminus l_0$. Thus $a_0 \neq \emptyset$ and by Lemma 2.1 so that $\frac{\mu^2}{c_\mu} \in a_\infty$, where $a_\infty = G_\mu^{-1}(l_0) \setminus l_\infty$. Hence $a_\infty \neq \emptyset$. G_μ are both m to 1 on l_∞ and l_0 .

Proposition 3.1: Each l_∞ and l_0 are simply connected if $c_\mu \in a_\infty$ or a_0 also, the rings a_∞ and a_0 around the origin is $2m$ -fold symmetry.

Proof: There is the super-attracting periodic orbit of 0, and ∞ period 2 includes $0 \in l_0$. Let $\mathbb{D}_0 = \{z : |z| < r\}$ be a small disk such that the intersection of the boundary of the small disk and the orbits of each free critical point is empty. In simplest, for $i = 0, 1$ and $z = G_\mu^i(0) \in G_\mu^i(l_0)$. For any of $G_\mu^{-n}(\mathbb{D}_0)$ the connected component $\mathbb{D}_n$ includes z such that $i = n \bmod 2$. Note that $\mathbb{D}_0 \subset l_0$

also $\mathbb{D}_1 \subset l_\infty$. So that $\mathbb{D}_{2j+i} \subset \mathbb{D}_{(2j+1)+i} \subset \cdots \subset l_i$ ($l_1 = l_\infty$), thus $l_\infty = \bigcup_{j \geq 0} \mathbb{D}_{2j+1}$ also $l_0 = \bigcup_{j \geq 0} \mathbb{D}_{2j}$. So $\mathbb{D}_0$ is simply connected, also a covering map with (m^{2j}) degree $G_\mu^{2j}: \mathbb{D}_{2j} \setminus \{0\} \rightarrow \mathbb{D}_0 \setminus \{0\}$. From the Riemann–Hurwitz’s formula [1], $\chi(\mathbb{D}_{2j} \setminus \{0\}) + \delta(\mathbb{D}_{2j} \setminus \{0\}) = m^{2j} \chi(\mathbb{D}_0 \setminus \{0\})$. Because G_μ^{2j} contain no critical points in $\mathbb{D}_{2j} \setminus \{0\}$ also $\chi(\mathbb{D}_0 \setminus \{0\}) = 0$, so that $\chi(\mathbb{D}_{2j} \setminus \{0\}) = 0$. Therefore $\mathbb{D}_{2j} \setminus \{0\}$ form an annulus, so $\mathbb{D}_{2j}$ is connected. Therefore l_∞ is simply connected, also similar l_0 is connected. From Lemma 2.1, hence a_∞ contains either 1 component or $2m$. Suppose a_0 has $2m$ components, thus, 0 has $5m$ preimages. Since the map has degree two from each component of a_∞ to l_0 , also contains $2m$ components. This is impossible since the degree of G_μ is $3m$. So that a_∞ contains one component only also is connected. From Riemann–Hurwitz’s formula for $G_\mu: a_\infty \rightarrow l_0$, therefore $\chi(a_\infty) + 2m = 2m\chi(l_0)$, since l_0 is connected. Therefore $\chi(a_\infty) = 0$, so a_∞ is an annulus surround 0 with $2m$ – fold symmetry. Similarly, we prove that l_∞ is the simply connected and a_0 is an annulus surrounding the origin with $2m$ – fold symmetry.

Proposition 3.2: If a_0, a_∞ are two annuli, then $a_\infty \cap a_0 = \emptyset$. Therefore $\overline{a_\infty}, \overline{l_0}, \overline{l_\infty}$ and $\overline{a_0}$ are disjoint to any other.

Proof: From Proposition 3.1, hence the component $\mathbb{C}_\infty \setminus a_0$ contains a_∞ and 0, we claim $\overline{a_\infty}, \overline{a_0}, \overline{l_0}$ and $\overline{l_\infty}$ are reciprocally disjoint. Because the image of a_0 is a_∞ for G_μ such that $l_\infty \cap l_0 = \emptyset$. So that $a_\infty \cap a_0 = \emptyset$. Also a_0, a_∞ are two annuli separate ∞ and 0. Hence $\overline{l_\infty} \cap \overline{l_0} = \emptyset$. We have $G_\mu(\overline{a_\infty}) = G_\mu(\overline{l_\infty}) = \overline{l_0}$ also $G_\mu(\overline{a_0}) = G_\mu(\overline{l_0}) = \overline{l_\infty}$. So that $\overline{a_\infty} \cap \overline{a_0} = \emptyset, \overline{l_\infty} \cap \overline{a_0} = \emptyset$ also $\overline{a_\infty} \cap \overline{l_0} = \emptyset$. Therefore $\overline{a_0} \cap \overline{l_0} = \emptyset$ also $\overline{l_\infty} \cap \overline{a_\infty} = \emptyset$.

Theorem 3.3: $J(G_\mu)$ is a Cantor circle if $G_\mu(c_\mu) \in l_\infty$ or I_0 , but $c_\mu \notin l_\infty$ or I_0 .

Proof: By Figure (2), we have $V := \mathbb{C}_\infty \setminus l_0 \cup l_\infty$ is closed set amongst l_∞ and l_0 divided into closed sets V_0, V_1, V_2 amongst l_∞ and a_0, a_0 and a_∞, a_∞ and l_0 . All the map $G_\mu: V_k \rightarrow V$ is covering at m to 1, for $0 \leq k \leq 2$. So that $J(G_\mu) = \bigcap_{i \geq 0} G_\mu^{-i}(V)$. For each inverse branch $h: V \rightarrow V_k$ of G_μ where $0 \leq k \leq 2$. Then any component of the Julia set has $j_{k_0, k_1, \dots, k_i, \dots} = \bigcap_{i=0}^\infty h_{k_i} \circ \dots \circ h_{k_1} \circ h_{k_0}(V)$, where $i \in \mathbb{N}$ and $k_i \in \{0, 1, 2\}$. Separate ∞ and 0 by the compact set, which is, all $j_{k_0, k_1, \dots, k_i, \dots}$. Hence $j_{k_0, k_1, \dots, k_i, \dots}$ is locally connected from [2, Corollary 1.3], and because G_μ is hyperbolic. For each $\mathcal{L} = j \cup \mathbb{J}$, $j = j_{2,0,2,0,\dots} = \partial l_0$ and $\mathbb{J} = j_{0,2,0,2,\dots} = \partial l_\infty$. Because $V_k \subset V$, also $\text{id}: V_k \hookrightarrow V$ is not homotopic to a constant map. Therefore $j_{k_0, k_1, \dots, k_i, \dots}$ is Jordan curve from [2, Lemma 2.4 and Proposition Case 2]. Hence $j_{k_0, k_1, \dots, k_i, \dots}$ is a quasi circle from Theorem 2.2 and because G_μ is hyperbolic. So $\Sigma_3 = \{s = (s_0 s_1 s_2, \dots); s_k \in \{0, 1, 2\}\}$ is a one-

sided shift on the space of 3 and is isomorphic to the dynamics on $j_{k_0, k_1, \dots, k_i, \dots}$. In particular, $J(G_\mu)$ is homeomorphic to $\Sigma_3 \times S^1$, so that $J(G_\mu)$ is a Cantor circle.

Theorem 3.4: *Each McMullen map is not topologically conjugated to G_μ analogical to Julia sets if $c_\mu \notin l_0$ or l_∞ , for some $G_\mu(c_\mu) \in l_0$ or l_∞ .*

Proof: From theorem 3.3 above, there is an isomorphism amidst $\Sigma_3 = \{s = (s_0 s_1 s_2, \dots); s_m \in \{0, 1, 2\}\}$ and the $j_{k_0, k_1, \dots, k_i, \dots}$. So that there is an isomorphism amidst the set of Julia components of each McMullen map and $\Sigma_2 = \{s = (s_0 s_1, \dots); s_k \in \{0, 1\}\}$. Then, G_μ is not topologically conjugate to the McMullen map corresponding to Julia sets, look Figure 3.

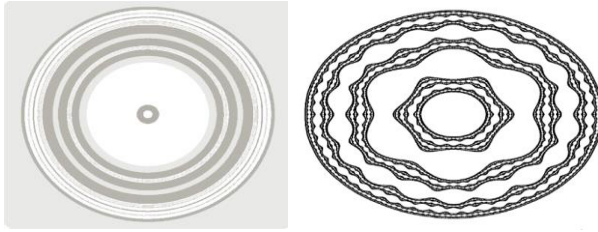


Figure 3

$J(G_\mu)$ if $d = 4, \mu = 0.0055492149i$ and $R(z) = z^3 + \frac{0.01}{z^3}$.
So are both of them Cantor circles.

The Julia set is given as either a Sierpinski carpet or degenerate Sierpinski carpet.

Proposition 3.5: $J(G_\mu)$ is nowhere dense, compact, connected and locally connected, also any Fatou components of G_μ are simply connected if c_μ belong to W_∞^k or W_0^k for a few $k \geq 2$.

Proof: By Proposition 3.1, we get l_0 and l_∞ are connected. So $W_0^1 = G_\mu^{-1}(l_0) \setminus l_\infty$ of l_0 contain $2m$ - symmetry Fatou components. Because G_μ maps l_0 onto any one of them is conformal also $c_\mu \in W_0^k$ where $k \geq 2$, so that each component of W_0^j is simply connected for $j, 1 \leq j \leq k-1$, so at least there is $2m$ symmetry components in W_0^k from Proposition 2.3 around 0. For each simply connected component $\mathcal{U}$ in W_0^{k-1} . If the critical orbits do not belong to $\mathcal{U}$, then any components of $G_\mu^{-1}(\mathcal{U})$ are simply connected. The critical value belong to $\mathcal{U}$, so $\exists \mathcal{V}$ component of $G_\mu^{-1}(\mathcal{U})$ is not simply connected, hence at least, there are two critical points in $\mathcal{V}$. There are different Fatou components $\omega_0^i \mathcal{U}$ ($2m-1$) from the symmetry, where $\omega^{2m} = 1, 1 \leq i \leq 2m-1$, so at least there are two critical points in any of $\omega_0^i \mathcal{U}$. Hence at least $4m$ free critical points lies in G_μ , this is contradictory. Therefore $Q_\beta^{-1}(\mathcal{U})$ contain all connected

components also, critical value lies in $\mathcal{U}$. Therefore W_0^k contain all connected components. Hence $G_\mu^{-1}(l_0)$ contain all simply connected components because critical values do not lie in W_0^k . From Lemma 2.1, each Fatou component of G_μ are simply connected. Therefore the complement of $\bigcup_{j \geq 0} G_\mu^{-j}(l_\infty \cup l_0)$ is $J(G_\mu)$ and so is connected and compact. Because $J(G_\mu) \neq \mathbb{C}_\infty$ also from [1, Lemma 2.5], so $\overline{J(G_\mu)}^\circ = \emptyset$, thus $J(G_\mu)$ is nowhere dense. From [1, Lemma 2.5] and G_μ is a hyperbolic map. Hence $J(G_\mu)$ is locally connected.

Theorem 3.6: Each Fatou component of G_μ are Jordan disks whenever for some $c_\mu \in W_\infty^k$ (or W_0^k) for a few $k \geq 2$. Moreover, $J(G_\mu)$ is a Sierpinski carpet if $\partial l_\infty \cap \partial l_0 = \emptyset$. Otherwise $J(G_\mu)$ is a degenerate Sierpinski carpet.

Proof: From Lemma 2.1 also Proposition 3.5, we must prove ∂l_0 is simple closed curve. Since ∂l_0 is locally connected also connected. The complement of $\mathbb{C}_\infty \setminus \overline{l_0}$ contains countable Jordan disks at most. $\forall \mathcal{W}_\infty$ component of $\mathbb{C}_\infty \setminus \overline{l_0}$ contains ∞ . So that $\partial \mathcal{W}_\infty$ is a Jordan curve. To show that $G_\mu^{-1}(\mathcal{W}_\infty) \subset l_0$. Assume that $\infty \in l_\infty \subset \mathcal{W}_\infty$, to prove that $G_\mu^{-1}(\infty) \subset l_0$. By using Lemmas 2.1, the $2m$ -poles $G_\mu^{-1}(\infty) \setminus \{0\}$ lies either in $2m$ different components or in γ_0 (Fatou component) surround 0 of G_μ . Suppose that $G_\mu^{-1}(\infty) \not\subset l_0$, so that γ_0 separate $\mathcal{W}_\infty$ from l_0 , which is contradicting because $\partial \mathcal{W}_\infty \subset \partial l_0$. There exists state that $2m$ Fatou component should has $2m$ different component $\mathcal{U}_0, \dots, \mathcal{U}_{2m-1}$ of $\mathbb{C}_\infty \setminus (\overline{l_0} \cup \overline{\mathcal{W}_\infty})$. Moreover $G_\mu^{-1}(0) \setminus \{\infty\} \subset \bigcup_{j=0}^{2m-1} \mathbb{T}(\mathcal{U}_j) \subset \mathcal{W}_\infty$, where $\mathbb{T}(z) = \frac{\mu^2}{z}$. So that for any j $G_\mu(\mathcal{U}_j) = \mathcal{W}_\infty$ also $G_\mu(\bigcup_{j=0}^{2m-1} \mathbb{T}(\partial \mathcal{U}_j)) = \partial \mathcal{W}_\infty \subset \partial l_0$. So at least $2m$ -preimages on ∂l_0 lies in any point of $\partial \mathcal{W}_\infty \subset \partial l_\infty$, since $G_\mu: \partial l_\infty \rightarrow \partial l_0$ has degree m . Which is impossible. Thus $G_\mu^{-1}(\mathcal{W}_\infty) \subset l_0$ and $\mathcal{W}_\infty = \mathbb{C}_\infty \setminus \overline{l_0}$. For each not exceptional point $v \in \partial \mathcal{W}_\infty$, because $G_\mu^{-1}(\overline{\mathcal{W}_\infty}) \subset \overline{l_0}$, hence $\partial l_0 \subset J(G_\mu) = J(G_\mu^2) = \overline{\bigcup_{i \geq 0} Q_\beta^{-2i}(v)} \subset \partial \mathcal{W}_\infty$, also $\partial \mathcal{W}_\infty \subset \partial l_0$. Therefore $\partial \mathcal{W}_\infty = \partial l_0$ is simple closed curve. Since G_μ is hyperbolic and from [1, Theorem 1.1c], thus ∂l_0 is quasi circle. By the Bottcher theorem, for some $\psi: l_0 \rightarrow \mathbb{D}$, where ($\mathbb{D}$ open disk) is a conformal map contiguous at 0, also there exists a conformal map $\phi: l_\infty \rightarrow \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ contiguous at ∞ which if $\circ G_\mu(v) = (\phi(v))^{-m}$. We define the external ray in l_∞ as $\eta(\vartheta) = \{v \in l_\infty: \psi(v) = re^{2\pi i \vartheta}, r > 1\}$, for $\vartheta \in [0, 1)$. So $\eta(\vartheta)$ is landing at v if $\eta(\vartheta) \rightarrow v \in \partial l_\infty, r \rightarrow 1$. For some components U of $G_\mu^{-j}(l_\infty)$. An internal ray component E is a subset of U . Thus $G_\mu^{-j}(v) \in \partial U$, where v is a landing point of $\eta(\vartheta)$, so $G_\mu(\eta(\vartheta)) = \eta(m\vartheta)$ also ∂l_∞ is locally connected. We give three cases as:

Case 1: There is two components U and V of W_∞^k (or W_0^k) where $k \geq 1$, such that $\bar{U} \cap \bar{V}$. So that $G_\mu^k(U) = G_\mu^k(V) = l_\infty$ or $G_\mu^{k+1}(U) = G_\mu^{k+1}(V) = l_\infty$. For some E_1 is a subset of U also E_2 is a subset of V satisfying both of them land at $v \in \bar{U} \cap \bar{V}$, also a landing ray $\eta = G_\mu^k(E_1) = G_\mu^k(E_2)$ or $\eta = G_\mu^{k+1}(E_1) = G_\mu^{k+1}(E_2)$ in l_∞ at $G_\mu^k(v)$ or $G_\mu^{k+1}(v)$. Therefore G_μ^k has a critical point v . This is impossible because each critical points escape to either 0 or ∞ .

Case 2: There is two components U and V of W_∞^j and W_∞^i (or W_0^j and W_0^i) where $0 \leq j < i$ satisfying $\bar{U} \cap \bar{V} \neq \emptyset$. So that $G_\mu^i(U) = l_\infty$ or l_0 , also $G_\mu^i(E) = l_\infty$. When $G_\mu^i(U) = l_0$ also $G_\mu^i(E) = l_\infty$, thus $G_\mu^i(\bar{U} \cap \bar{V}) \neq \emptyset$. Therefore $\partial l_\infty \cap \partial l_0 \neq \emptyset$, implies a contradiction. But if $G_\mu^i(U) = l_\infty$ also $G_\mu^i(E) = l_\infty$ and then from case 1, so is contradict.

Case 3: There is two components U and V of W_0^j and W_∞^i for $0 \neq j, 0 \neq i$. Since $\partial l_\infty \cap \partial l_0 = \emptyset$, so that $\bar{U} \cap \bar{V} = \emptyset$. Hence $J(G_\mu)$ is a Sierpinski carpet from Proposition 3.5. Otherwise, $J(G_\mu)$ is a degenerate Sierpinski carpet when $\partial l_\infty \cap \partial l_0 \neq \emptyset$.

Discussion and Conclusion

The following family of maps are defined as:

$$G_\mu(z) = 2\mu^{m+1}z^{-m} + \frac{z^{2m-\mu^{3m+1}}}{z^m(z^{2m-\mu^{m-1}})}, \text{ and our get as follows:}$$

- The Julia set of G_μ is a degenerate Sierpinski carpet
- The Julia set of G_μ is a Sierpinski carpet
- The Julia set of G_μ is a cantor circle

One of the limitations of the current study is the important condition for giving parameter values to satisfying $b \in \mathbb{C} \setminus \{0\} - b^{n+1} = 1$ and $b^{2n+2} = 1$. Also The, condition one of the free critical points has an image for G_μ is diverged on ∞ or 0.

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