Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 7, pp. 1511–1516
DOI : 10.47974/JIM-1573

Pr-small R-submodules of modules and Pr-radicals

Khawlah Ahmmed *
Nuhad S. Al. Mothafar †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

The goal of this discussion is to study the twigged of pure-small (pr-small) sub-moduleof a module *W* as recirculation of a small sub-module, and we give some basic idiosyncrasy and instances of this kind of sub-module. Also, we give the acquaint of pure radical of a module *W* (pr-radical) with peculiarities.

Subject Classification: *16P70, 16S90.*

Keywords: *Pure small, Pure radicals.*

Introduction

Throughout this paper, W will be left R-mod-ule, in [1] A proper sub-module$K$ of a module$W$ is Named small $(K << W)\, if\, K + H = W$ for any sub- module$H$ of Wnamely $H = W$. also in [2], A module$W$ is named hallow module if the whole part sub- module of W is a small sub- module of W. In [3] A sub-module$P$ of a module W is named a pure sub-module of W if $P \cap IW = IP$, for whole ideal I of R .Abdul Jaleel and Yaseen in [4] evoked the twigged of large and small sub-modules as a recirculation of the class of teeny sub- mod-ule.in this paper, we evoked the twigged of pure small sub-modules, which is another recirculation of the class of small sub- mod-ule, where a sub-module$K$ of an module$W$is named pure-small $(K \ll_{pr} W), if\, K + H = W$ then, $0 \neq H$ pure sub- moduleof W. This paper consists of two parts; in part 1, we give some of the important peculiarities of this module. In part 2, we give a definition of Pr-Radical and some remarks on Pr-radical modules, that is $Rad_{pr}(W) = \sum\{N: N \ll_{pr} W\}$, which is named pure-radical of W, and we present some peculiarities of it.

* *E-mail:* **khawlahahmmed@gmail.com (Corresponding Author)**
† *E-mail:* **nuhad.salim@sc.uobaghdad.edu.iq**

© Taru Publications

1. Pr-Small R-sub modules

In this section, we introduced the twigged of a pure small R-sub module and we gave few of its peculiarities.

Definition 1.1: If W is a module proper sub- module K of W is named pure-small (pr-small) sub-module of W means $(K \ll_{pr} W)\ if\ H + K = W$, then $0 \neq H$ is pure sub- moduleof W; in [3] H is a pure sub- module of $W\ if\ H \cap IW = IH$ for each ideal I of R.

Noticing and Instances 1.2:

1. It is nakedly that the whole small sub- modules pr-small sub- mod-ule. Thus, in Z_4 as Z-module $\{0,2\} \ll_{pr} Z_4$ since $\{0,2\} + Z_4 = Z_4$ and Z_4 is non-zero pure sub- mod-ule, in Z_4.
2. The covers of the point above is false in general as the subsequent show, consider $W = Z_6$, $\{0,3\}$ is pr-small in Z_6 but not small.
3. $2Z + 3Z = Z$ as Z-module, $3Z$ is not pure in Z, so $2Z$ not pr-small in Z.
4. If W is a semisimple mod-ule, then every part sub-module of W is pr-small.

We know that in [5] an R-module W is called a F-regular if each sub-module of W is pure.

In [6],[7], A module W is named a simple if $W \neq < 0 >$, and it has no sub-modules except $< 0 >$ and W. A module is semi simple if W is a total of simple sub-modules or every sub-module of W is directly summoned of W. [8] A sub-module H of a module W is named an e-small sub-module in W $(H \ll_e W)$, if $H + K = W$, when $K \leq_e W$ implies $K = W$.

Remark 1.3: E--small sub- mod-ule, need not pr-small as the subsequent instance in Z as Z-module $2Z + 3Z = Z, 2Z$ is e-small but not pr-small since $3Z$ is not pure.

Remark 1.4: If W *is* F-regular module, then every pr-small sub- moduleof W is a small sub- mod-ule.

In the subsequent proposition, we give properties of pr-small R-sub modules:

Proposition 1.5: If W be a module and K, H are sub-modules of W so $K \leq H \leq W$ if $H \ll_{pr} W$, then $K \ll_{pr} W$.

Proof: Suppose U be R-sub module of W so $K + U = W$ since $K \leq H$, thence $H + U = W$ we defined $H \ll_{pr} W$ implies U is a pure R-sub module of W hence $K \ll_{pr} W$.

Remark 1.6: If W is a module and $K \leq H \leq W$, if $K \ll_{pr} H$, then it is not true that $K \ll_{pr} W$, as the subsequent instance. In Z as a Z-module let $K = 4Z$, $H = 2Z$ so $4Z \leq 2Z \leq Z$, we have $4Z \ll_{pr} 2Z$ since we have only $4Z + 2Z = 2Z$ and 2Zis a pure R-sub module of 2Z in Z, but 4Z is Lise Pr-small in Z because $4Z + 3Z = Z$and 3Z is not pure in Z.

Corollary 1.7: *If W is R- modulesuch that $H_1 \ll_{pr} W$, then $H_1 \cap H_2 \ll_{pr} W$.*

Proof: $H_1 \cap H_2 \leq H_1$, since $H_1 \ll_{pr} W$ implies $H_1 \cap H_2 \ll_{pr} W$ used proposition (1.5). Similarity if $H_2 \ll_{pr} W$ we have $H_1 \cap H_2 \ll_{pr} W$.

Remark 1.8: The covers of corollary (1.7) is Lise true as the subsequent instance: In Z as Z-module $K_1=2Z_{12}$, $K_2=3Z_{12}$ so $K_1 \cap K_2 = 2Z_{12} \cap 3Z_{12} = 6Z_{12} \ll_{pr} Z_{12}$ but $K_1, K_2$are Lise pr-small in Z_{12}.

Corollary 1.9: *If W is an R-module and H_1, H_2 be sub-modules of W, if $H_1 + H_2 \ll_{pr} W$, then $H_1 \ll_{pr} W$ and $H_2 \ll_{pr} W$.*

Proof: Since $H_1 \leq H_1 + H_2 \leq W$ and $H_1 + H_2 \ll_{pr} W$ implies $H_1 \ll_{pr} W$ used proposition (1.5). In the same way, we have $H_2 \leq H_1 + H_2 \leq W$ and $H_1 + H_2 \ll_{pr} W$ implies $H_2 \ll_{pr} W$.

Remark 1.10: The opposite of corollary (1.9) is Lise true as the subsequent instance, in Z_{12} as Z-module $2Z_{12} + 3Z_{12} = Z_{12}$is pr-small in Z_{12} but $2Z_{12}$ is not pure in Z_{12}.

Proposition 1.11: If W is an R-module and H, L are sub-modules of W so $H \leq L \leq W$ if $H \ll_{pr} L$ and L pure of W then $H \ll_{pr} W$.

Proof: Assume P be sub- moduleof W such that $P + H = W$, then $(P + H) \cap L = W \cap L = L$. assist modular law $H + (P \cap L) = L$ since $H \ll_{pr} L$, then $P \cap L$ is pure in L, but L is pure in Wnamely $P \cap L$ pure in Wused (1.1) so $H \ll_{pr} W$.

Remark 1.12: If $f: W \to K$ is an isomorphism where W and K are an R-module such that $H \ll_{pr} W$, then $f(H) \ll_{pr} K$.

Proof: Assume L is a sub-module of K so $f(H) + L = K$ since f is an isomorphism, then $f^{-1}(f(H) + L) = W, H + f^{-1}(L) = W$ since $H \ll_{pr} W$, then $f^{-1}(L)$ is pure in W, then $f(f^{-1}(L))$ is pure in W, but $f(f^{-1}(L)) = L$ implies $f(H) \ll_{pr} W$.

Proposition 1.13: If W is an R- module and H, L are sub-modules of W so $H \leq L \leq W$ if $\frac{\mathrm{L}}{\mathrm{H}} \ll_{pr} \frac{\mathrm{W}}{\mathrm{H}}$, then $L \ll_{pr} W$.

Proof: Assume K be a sub-module of w so $L + K = W$, since there $\frac{L}{H} \ll_{pr} \frac{W}{H}$, there exist $\frac{\mathrm{K}}{\mathrm{H}}$ is pure in $\frac{\mathrm{W}}{\mathrm{H}}$since$H$ R-sub module of W such that $l + H = w$, $\frac{L}{H} + \frac{K}{H} = \frac{W}{H}$ but $\frac{K}{H}$ is pure of $\frac{w}{H}$, then used [8] K is pure of W there fore $L \ll_{pr} W$.

Proposition 1.14: If W is an R-moduleand H, L are sub- mod-ules of W so $H \leq L \leq W$if $L \ll_{pr} W$ then $\frac{\mathrm{L}}{\mathrm{H}} \ll_{pr} \frac{\mathrm{W}}{\mathrm{H}}$.

Proof: Suppose $\frac{\mathrm{K}}{\mathrm{H}}$ be an R-sub- module of $\frac{\mathrm{W}}{\mathrm{H}}, H \leq K$ so $\frac{\mathrm{L}}{\mathrm{H}} + \frac{\mathrm{K}}{\mathrm{H}} = \frac{\mathrm{W}}{\mathrm{H}}$, then $\frac{\mathrm{L+K}}{\mathrm{H}} = \frac{\mathrm{W}}{\mathrm{H}}$ implies $L + K = W$, since $L \ll_{pr} W$ so K is pure in W Also $H \leq K$, then $\frac{\mathrm{L}}{\mathrm{H}}$ pure a R-sub- modulein $\frac{\mathrm{W}}{\mathrm{H}}$ used [9], then $\frac{L}{H} \ll_{pr} \frac{W}{H}$.

[10] An R-module W have the pure crossing idiosyncrasy (briefly PIP) if the crossing of any two pure sub-modules is pure.

Proposition 1.15: If W is an R- module with (PIP) and K_1, K_2 be sub-modules of W, then $K_1 \ll_{pr} W$ and $K_2 \ll_{pr} W$ iff $K_1 + K_2 \ll_{pr} W$.

Proof: $\Longrightarrow$) $K_1 \ll_{pr} W$ and $K_2 \ll_{pr} W$ then $K_1 + K_2 \ll_{pr} W$. Suppose H be an R-sub-module of W such that $(K_1 + K_2) + H = W$ Since $K_1 \ll_{pr} W$ and $K_2 \ll_{pr} W$ then $K_2 + H$ pure in W and $H_1 + K$ pure in W, Since W has the (PIP) idiosyncrasy $(K_1 + H) \cap (K_2 + H)$ is pure in W, But $(K_1 + H) \cap (K_2 + H) = H$, namely H is a pure R-sub module of W, then $K_1 + K_2 \ll_{pr} W$.

If $K_1 + K_2 \ll_{pr} W$, then $K_1 \ll_{pr} W$and$K_2 \ll_{pr} W$, Since $K_1 \leq K_1 \oplus K_2 \leq W$ and $K_1 \oplus K_2 \ll_{pr} W$, then used Remark (1.4); we have $k_1 \ll_{pr} w$, likeness we prove that $K_2 \ll_{pr} W$.

2. Pure Radicals (Pr-Radicals)

In [9] the Jacobson radical of a module$W$ name Rad (W) is the total of all small R-sub- modules of W.

Definition 2.1: If W is a module, the total of all pr-small sub- module of W name pure-radical (pr-radical) of W signify $Rad_{pr}(W) = \sum\{H: H \ll_{pr} W\}$.

Remark 2.2: $Rad_{pr}(W) \leq W$.

Proof: Let $H_i \ll_{pr} W$, $i \in I$ then $\oplus H\, i \in I \ll_{pr} W$ by Remark (1.4) implies Rad $_{pr}(W) \leq W$.

Instance and Remark 2.3:

1. Rad $_{pr}Z_4 = \{\bar{0}, \bar{2}\}$.
2. Rad $_{pr}Z_6 = \{\bar{0}, \bar{3}\} + \{\bar{0}, \bar{2}, \bar{4}\} = Z_6$.
3. Every radical of R-mod-ule$W$ contains in pr-radical of W i. e. Rad $(W) \leq Rad_{Pr}(W)$.

Proof: Suppose $H \in$Rad (W) implies $H \in$Rad $_{pr}W$ since every small is pr-small, then Rad $(W) \leq_{Rad_{Pr}} W$.

For instance:$Rad_{Pr}(Z_6) = Z_6$, Rad $Z_6 = \{\bar{0}\}$ clearly, Rad $(Z_6) \leq$Rad $_{pr}(Z_6)$.

Theorem 2.4: *Suppose W, H be R-modules and $\varphi: W \to H$ be an isomorphism, then φ(Rad $_{pr}(W)) \leq (Rad_{Pr}(H))$.*

Proof: Suppose $A \ll_{pr} W$, then by proposition (1.10), $\varphi(A) \ll_{pr} H$ hence $\varphi(A) \leq$ Rad $_{pr}(H)\ \forall\ A \ll_{pr} W$, then φ (Rad $_{pr}(W)) \leq (Rad_{Pr}(H))$.

Corollary 2.5: *If W is an R-module and K be a R-sub- module of W, then* $Rad_{Pr}(K) \leq Rad_{Pr}(W)$.

Proof: Suppose H be an R-sub module of Rad $_{pr}(K)$ such that $H \ll_{pr} K$since $K \ll_{pr} W$ use proposition (1.5), we get $H \ll_{pr} W$ hence K be a pr-small R-sub module of W so K R-sub- module of Rad $_{pr}(W)$, then Rad $_{pr}(K) \leq$ Rad $_{pr}(W)$.

Proposition 2.6: If $\varphi: W \longrightarrow K$ is an isomorphism where W and K be R-modules, then φ^{-1} (Rad $_{pr}(K)) \leq$ Rad $_{pr}(W)$.

Proof: Suppose $H \ll_{pr} K$, then used proposition (1.12), $\varphi^{-1}(H) \ll_{pr} W$ hence $\varphi^{-1}(H) \subseteq$ Rad $_{pr}(W)\ \forall\ H \ll_{pr} K$ thus φ^{-1} (Rad $_{pr}(K)) \subseteq Rad_{Pr}(W)$.

3. Conclusion

In this paper, we generalized small submodules as pure-small submodules and radical as pure-radical modules, and we explained:

1. If W be an R-module and K, H are R-sub- modules of W so $K \leq H \leq W$ if $H \ll_{Pr} W$, then $K \ll_{Pr} W$.
2. If W is an F-regular mod-ule, then every pr-small R-sub- module of W is a small R-sub- mod-ule.
3. If W is a semisimple module, then each part R-sub- module of W is pr-small.
4. If $W$is an R-module and K be an R-sub- module of W, then $Rad_{Pr}(K) \leq Rad_{Pr}(W) \cap K$.

References

[1] Wisam A. Ali & Nuhad S. Al. Mothafar Small prime and quasi-small prime modules, *Journal of Discrete Mathematical Sciences and Cryptography,* 24:7, 1967-1971 (2021).

[2] Abduljaleel, A. A., & Yaseen, S. M., On Large-Small submodule and Large-Hollow module. In *Journal of Physics: Conference Series*, Vol. 1818, No. 1, p. 012214, (2021, March). IOP Publishing.

[3] Artin, E., Algebraic numbers and algebraic functions, Vol. 358 (2005). *American Mathematical Soc*.

[4] Fieldhous. D. J," Pure Theories, math." Ann. 184, pp.1-18 (1969).

[5] Omar M. Al-Ragab & Nuhad S. Al-Mothafar Radical semiprime submodules, *Journal of Discrete Mathematical Siences and Cryptography*, 24:7, 1895-1899 (2021).

[6] Kasch F. Modules and Rings," *Academic Press*, London (1982).

[7] Israa Hassan Muslem, Layla Salman Mahmood, Some types of Retractable and Compressible Modules, thesis Pure Science, University of Baghdad (2016).

[8] Al-Bahraany B.H," Modules with the pure intersection property", Ph.D. Thesis, college of Science, University of Baghdad (2000).

[9] Wisbauer R, Foundation of module and ring theorem" Gordon and Breach, Philadelphia (1991).

[10] Eiber, M. S., & Kadhem, H. H., On Soft Pre-Compact Maps. *Wasit Journal of Computer and Mathematics Science*, 1(2) (2022).

[11] Omar M Al-Ragab & Nuhad S. Al-Mothafar. Radical semiprime submodules, *Journal of Discrete Mathematical Sciences and Cryptography*, 24:7, 1895-1899 (2021), DOI: 10.1080/09720529.2021.1957180.