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On S-acts and bounded linear operators

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Abstract

The relation between faithful, finitely generated, separated acts and the one-to-one operators was investigated, and the associated S-act of coshT and its attributes have been examined. In this paper, we proved for any bounded Linear operators T, V_{coshT} is faithful and separated S-act, and if a Banach space V is finite-dimensional, V_{coshT} is infinitely generated.

Subject Classification: *26A45, 30H05.*

Keywords: *Bounded linear operator, CoshT, Associated S-act.*

1. Introduction

We assume that A is a nonempty set and S is a monoid. If μ: A × S ⟶A, μ(a ,s) := (a , s) ⟼a, so that (as) t = a (s t), so a= a. 1, where s, t, ∈ S, and a ∈ A. Assume R = F[x] be a polynomial ring with coefficients in F. Define Ø: R×V⟶V so that p (T) v = p. v = Ø (p, v) where T is the linear operator. Ø form V is a left R-module which is denoted by V_T, so we called it the associated R-module. Using H as a Hilbert space over a field K (K may be real or complex), and T is a bounded linear operator on H [1- 4]. By defining the exponential operator e^T as follows:$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$, where T0 = I, the identity operator on H is well-defined [5]. However, we put T as a bounded operator, and also V as a

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Banach space, and $S = \{e^x : x \in R\}$ represent the semigroup. By determining $\mu : S \times V \to V$ by $e^T(v) = \mu(e^x, v)$. μ makes V a left act over S and is indicated by V_T[6]. In this paper, the V_{coshT} have been studied ,let $S = \{coshx : x \in R\}$ *is the semigroup. Put* μ *from* $S \times V$ *into* V *defined by* $\mu(\cosh x, v) = \cos h\, T(v)$, *V is called a left act over S, denoted by* V_{coshT}.

2. Main result

Definition: Banach space V over a field (F), and (T) bounded operator, put $S = \{\text{coshx} : x \in R\}$ is the semigroup. Put μ from S×V into V defined by $\mu(\cosh x, v) = \cos h\, T(v)$, V is called a left act over S, denoted by V_{coshT}.

We put

$$p_n(T) = \sum_{i=0}^{n} \frac{(T)^i}{i!} = I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!}) \text{ and } p_n(-T) = \sum_{i=0}^{n} \frac{(-T)^i}{i!} = I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!}).$$

Proposition: If a basis K = {vj, j∈ ∧} for V, then every element w of V_{coshT} can be composed as

$$w = lim_{n\to\infty}(\sum_{i=0}^{n} \frac{(T)^i}{i!} + \sum_{i=0}^{n} \frac{(-T)^i}{i!}) \sum_{j\in\wedge} a_j v_j = \lim_{n\to\infty}(pn\ (T) + p_n(-T)).\ v, \text{ for some } v \text{ in } V.$$

Proof: We defineμ: S×V →V, byμ (cosh x, v) = cosh T (v) = $\frac{1}{2}(e^T + e^{-T})(v)$ = $\frac{1}{2}(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!})$ *(v).For w* $\in V_{cosh\,T}$, *then w*

$$= \frac{1}{2}(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!})(v) = \frac{1}{2}[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!}](v).$$

Since *K = {v jj* ∈ ∧*}* is a basis for *V* then

$$w = \frac{1}{2}(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!})(\sum_{j\in\wedge} a_j v_j) = \frac{1}{2}\Big[I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + \cdots + I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!} + \cdots\Big](\sum_{j\in\wedge} a_j v_j) = \frac{1}{2}[2I + 2\frac{(T)^2}{2!} + 2\frac{(T)^4}{3!} + \cdots + 2\frac{(T)^{2n}}{n!} + \cdots.](\sum_{j\in\wedge} a_j v_j) = \frac{1}{2}.2[I + \frac{(T)^2}{2!} + \frac{(T)^4}{3!} + \cdots + \frac{(T)^{2n}}{n!} + \cdots](\sum_{j\in\wedge} a_j v_j = \Big[I + \frac{(T)^2}{2!} + \frac{(T)^4}{3!} + \cdots + \frac{(T)^{2n}}{n!} + \cdots\Big](\sum_{j\in\wedge} a_j v_j) = I(\sum_{j\in\wedge} a_j v_j) + \frac{(T)^2}{2!}(\sum_{j\in\wedge} a_j v_j) + \frac{(T)^4}{3!}(\sum_{j\in\wedge} a_j v_j) + \cdots + \frac{(T)^{2n}}{n!}(\sum_{j\in\wedge} a_j v_j) + \cdots = \sum_{i=0}^{\infty} \frac{(T)^{2i}}{(2i)!}.$$ While the series $\sum_{i=0}^{\infty} \frac{T^i}{i!}$ converges in

B(H), While T in *B(H).* Therefore $\sum_{i=0}^{\infty} \frac{(T)^{i2i}}{(2i)!}$ is converges. Thus $w = lim_{n\to\infty} \sum_{i=0}^{n} \frac{(T)^{2i}}{(2i)!}\left(\sum_{j\in\wedge} a_j\, v_j\right) = lim_{n\to\infty}(\, p_n(T) + p_n(-T)).V$

Examples:

1. Put $\{v_j : j \in \wedge\}$ is a basis for a Banach space *V.* For any operator *T,* we define *T0 = I,* and then we have $O^o = I,$ where *O* is the zero operator, Let $w \in V_{\,cosh\,0(v),}$ thus through proposition (2.2), $w = \frac{1}{2}[e^T(v) + e^{-T}(v)] = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!}\right) = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{(0)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-0)^i}{i!}\right)$, thus $w = \sum_{j\in\wedge} a_j\, v_j = I\left(\sum_{j\in\wedge} a_j\, v_j\right),$ because $e^0 = I$ *([3])*

2- Put *I* is the identity operator, and $\{v_j : j \in \wedge\}$ is a basis, and take $w \in V_{coshI}$, then $w = \; w = \frac{1}{2}[e^T(v)+e^{-T}(v)] = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!}\right) = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{(I)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-I)^i}{i!}\right)$ since $e^I = e.I,$ thus $\frac{1}{2}(e^1 I + e^{-1}I) = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{1}{i!} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\right) I$ proposition (2.2). Then $w = \frac{1}{2}\left(2\sum_{n=0}^{\infty} \frac{1}{2i!}\left(\sum_{j\in\wedge} a_j\, v_j\right)\right) = lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{2i!}\left(\sum_{j\in\wedge} a_j\, v_j\right), put\ a_n = \sum_{i=0}^{n} \frac{1}{2i!}$. Then $w = lim_{n\to\infty}\, a_n\left(\sum_{j\in\wedge} a_j\, v_j\right)$.

3. Put $\{v_j : j \in \wedge\}$ is a basis and also *T* is a nilpotent operator. Let $w \in V_{coshT}$, thus $w = \frac{1}{2}[e^T(v) + e^{-T}(v)] = \frac{1}{2}\left(\sum_{i=0}^{\infty} \frac{(T)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-T)^i}{i!}\right)(v) = \frac{1}{2}\Big(I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^n}{n!} + \cdots\Big) + \Big(I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^n}{n!} + \cdots\Big)(v) = \frac{1}{2}\Big[\, I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + \frac{(T)^{n-1}}{(n-1)!} + I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots + \frac{(-T)^{n-1}}{(n-1)!} + \Big]\left(\sum_{j\in\wedge} a_j\, v_j\right) = \frac{1}{2}[2\, I + 2\,\frac{(T)^2}{2!} + 2\frac{(T)^4}{4!} + 2\frac{(T)^6}{6!} \ldots + 2\frac{(T)^{n-2}}{(n-2)!})]\left(\sum_{j\in\wedge} a_j\, v_j\right) = \frac{1}{2}[2(\, I + \frac{(T)^2}{2!} + \frac{(T)^4}{4!} + \frac{(T)^6}{6!} \ldots + \frac{(T)^{n-2}}{(n-2)!})]\left(\sum_{j\in\wedge} a_j\, v_j\right) = [\, I + \frac{(T)^2}{2!} + \frac{(T)^4}{4!} + \frac{(T)^6}{6!} \ldots + \frac{(T)^{n-2}}{(n-2)!}]\left(\sum_{j\in\wedge} a_j\, v_j\right)$ proposition *(2.2), let* $p_{2n}(T) = I + \frac{(T)^2}{2!} + \frac{(T)^4}{4!} + \frac{(T)^6}{6!} \ldots + \frac{(T)^{n-2}}{(n-2)!}$. *Then* $w = p_{2n}(T)\left(\sum_{j\in\wedge} a_j\, v_j\right) = p_{2n}(T).v$

Lemma: *V_{coshT} is S-act ⇔ V_{coshT} is a module where T is a nilpotent operator.*

Proof: Put V_{coshT} is S-act, we have to show that V_{coshT} is R-module. Since T is a nilpotent operator on V, $\forall\, w \in V_{coshT}$, we get w = $p_{2n}(T).v$, then V_{coshT} is module [3], conversely, assume that V_{coshT} is a module, then V_{coshT} is S-act. [6]

Proposition: If T and S are similar bounded operators. Then V_{coshS} and V_{coshT} are isomorphic.

Proof: Put S and T are similar operators, then existent h on V is an invertible operator so that $h\,T\,h^{-1} = S$ [5], then $(h\,S\,h^{-1} = T).h$, thus $h\,S = T\,h$. Since hS = Th , therefore hSn= hSSn-1 = ThSSn-2 =TThSn-2 = $T^2hS^{n-2} = T^2hSS^{n-3}$=$T^2ThS^{n-3}$….. T^3hS^{n-3}=T^3hSS^{n-4}=$T^3ThS^{n-4} = T^4hS^{n-4}$ =Tnh, hence hSn=Tnh, thus h coshS = coshT h…..(1). $h' : V_{cosh\,s} \to V_{coshT}$ defined by $\big(cosh(v)\big)h' = coshT\big(h(v)\big)$ ……(2). To prove h' is an isomorphism we show that, $h'$is well defined, assume that $coshS(v_1) = coshT(v_2)$, therefore $h\big(coshS(v_1)\big) = h\big(coshT(v_2)\big)$(because h is well defined). Thus by (1), we have $coshT\big(h(v_1)\big) = h\big(coshS(v_1)\big) = h\big(coshS(v_2)\big) = coshT\big(h(v_2)\big)$, this give $coshS\big(T(v_1)\big) = coshT\big(h(v_2)\big)$.. (3). Therefore by (2),(3), we have $\big(coshS(v_1)\big)h' = coshS(v_2))h'$. Hence h' is well-defined, Show that h' is 1-1, suppose that $\big(coshS(v_1)\big)h' = (coshS(v_2))h'$, thus by (2), we have $coshT\big(h(v_1)\big) = coshT\big(h(v_2)\big)$. Thus by (1), we have $h\big(coshS(v_1)\big) = h\big(coshS(v_2)\big)$, but h has an inverse, thus $h^{-1}h\big(coshS(v_1)\big) = h^{-1}h\,(coshS(v_2))$, thus $\big(coshS(v_2)\big) = (coshS(v_1))$, thus h' is1-1, Show that $h'$is onto, put $coshT(v) \in v_{coshT}$ since $v \in V$ then$h^{-1}(v) \in V$and $coshS\big(h^{-1}(v)\big) \in V_{coshS}$, thus by (2), we have $(coshS\big(h^{-1}(v)\big)h' = coshT(h\big(h^{-1}(v)\big) = coshT(v)$, then h' is onto, and $\big(coshS(v)\big)h' = coshT\big(h(v)\big) = h(coshS(v))$, therefore $\big(coshS(v)\big)h' = h(coshS(v))$, while h is linear, hence h is a homomorphism, therefore h' is homomorphism, Hence $h'$is an isomorphism.

Proposition: V_{coshT} is a faithful S-act, where T is any bounded linear operator.

Proof: We want to show that$V_{coshT}$is a faithful S-act for any bounded operator. Let $\cosh x_1.\,\mathrm{coshT}(v) = \cosh x_2.\,\mathrm{coshT}(v)$. Since coshTis operator, then coshT is a linear transformation, this gives $\mathrm{coshT}(\cosh x_1.v) = \mathrm{coshT}(\cosh x_2.v)$. Since coshT is one to one [7] .Hence $\cosh x_1.v = \cosh x_2.v.$, then $\cosh x_1. = \cosh x_2, \forall x \neq 0$. Therefore V_{coshT} is faithful S-act.

Remark: $V_{coshT}$is finitely generated. If Banach space V is a finite-dimensional

Proposition: V is finite-dimensional, If V_{coshT} is finitely generated, and T is onto the operator.

Proof: Suppose that space V is not infinite. Put K(T)={ $w \in V: Tw = 0$}. Clearly, K is a subspace which is invariant (since K ⊆

$\bigvee$ and for each w in K, $Tw = 0$ but $0 \in K$ therefore $T\,w \subseteq K$), thus $TV \simeq \frac{V}{K}$ [2], therefore TV=V. Hence $\frac{V}{K} \simeq V$. K is a finite subspace, or K is an infinite subspace. K is invariant and thus considered as K_{coshT}. Thus K_{coshT} is finitely generated when K is finite, remark (2.7), the set $\{coshT(w_j) : j \in \Lambda\}$ generate the $sub-act\ K_{T+T'}$, where $\{w_j : j \in \Lambda\}$ is a basis for K. While $w_j \in K$ then $Tw_j = 0$, because the zero operator $\mathcal{O}$ is the restriction of T on K. Therefore $K_{coshT} = K_{coshO}$, thus K_{coshT}cannot be finitely generated, which is a contradiction. Thus K is infinite, while$K_{coshT}K_T$is a sub-act of$V_{coshT}$and $V_{coshT}$is finitely generated. Which is contradiction shows that V is finite-dimensional.

Proposition: V_{coshT} is separated S-act, where T is any bounded linear operator.

Proof: Put $a \neq b$ in $V_{coshT}$to prove V_{coshT} is separated, we aim to approve that s, e in S, also e≠ s when e is the identity element so that $sb \neq sa$. Put sb = sa, e ≠s ϵ S, so that s = coshx , where the identity element is e, a, b in V_{coshT}, thus $coshx.\,coshT(v_2) = coshx.\,coshT(v_1)$ v_1, v_2in V, x in R, since coshT is the operator, then coshT is transformation, thus $coshx.\,coshT(v_1) = coshx.\,coshT(v_2)$s , thus $coshT(coshx.\,v_2)$=$coshT(coshx.\,v_1)$, while coshT is one-to-one. Therefore $coshx.\,v_2 = coshx.\,v_1$, thus $(v_1 - v_2)coshx$ =0, because coshx ≠0. therefore $v_2 = v_1$, this gives either $coshT(v_1) \neq coshT(v_2)$ or$coshT(v_1) = coshT(v_2)$, but if $coshT(v_2) \neq coshT(v_1)$, thus $v_2 \neq v_1$ this contradiction with $v_1 = v_2$, then $coshT(v_2) = coshT(v_1)$, so b=a , which is a contradiction, thus V_{coshT} is separated.

The following proposition gives the converse of the last proposition.

Proposition: T is one-to-one. If V_{coshT} is separated.

Proof: Suppose that V_{coshT} is separated to show that T is 1-1 (one-to-one).

Put $v_2 \neq v_1$, we have to show that $T(v_1) \neq T(v_2)$. Because $v_2 \neq v_1$, therefore either $coshT(v_2) \neq coshT(v_1)$ or $coshT(v_2) = \cosh T(v_1)$. If $coshT(v_1) = coshT(v_2)$, thus contradiction with $v_2 \neq v_1$ (because coshT is one-to-one), hence $coshT(v_2) \neq coshT(v_1)$, when V_{coshT} is separated. Therefore there is e ≠s, s = coshxϵS so that $coshx.\,coshT(v_1) \neq coshx.\,coshT(v_2)$, since coshT is an operator, then coshT is a linear transformation, this gives $coshT(coshx.\,v_1) \neq coshT(coshx.\,v_2)$, i.e $(I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots)(coshx.\,v_1) \neq \left(I + (T) + \frac{(T)^2}{2!} + \frac{(T)^3}{3!} + \cdots + I + (-T) + \frac{(-T)^2}{2!} + \frac{(-T)^3}{3!} + \cdots\right)(coshx.\,v_2)$, we have $coshx.\,v_1 + \frac{(T)^2}{2!}(coshx.\,v_1) + \frac{(T)^4}{4!}(coshx.\,v_1) + \cdots \neq coshx.\,v_2 + \frac{(T)^2}{2!}(coshx.\,v_2) + \frac{(T)^4}{4!}(coshx.\,v_2) + \cdots$, $coshx.\,v_1 \neq coshx.\,v_2$, this gives $v_1 \neq$ v2 and $\frac{(T)^2}{2!}(coshx.\,v_1) \neq \frac{(T)^2}{2!}(coshx.\,v_2)$, therefore $\frac{T}{2!}(T(coshx.\,v_1)) \neq \frac{T}{2!}(T(coshx.\,v_2))$

thus $T(\cosh x.\, v_2)) \neq (T(\cosh x.\, v_1))$, thus we have $T(v_2) \neq T(v_1)$. Therefore T is 1-1.

Conclusion

In this paper, we have proved that if T and S are similar bounded operators, then $V_{\cosh S}$ and $V_{\cosh T}$ are isomorphic, also for any bounded Linear operators T, $V_{\cosh T}$ are faithful, separated S-act and $V_{\cosh T}$is finitely generated if Banach space V is finite-dimensional.

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