

On G_α -open sets

Tabarak A. Ali *

R. B. Esmaeel †

Department of Mathematics

College of Education for Pure Science (Ibn Al- Haitham)

University of Baghdad

Baghdad

Iraq

Abstract

The new concepts in grill topological spaces are investigated in this study by employing specified sets in which the α -open sets are defined and grill topological spaces. Properties of this set and certain relationships are studied, examining a group of functions, like open, closed, and continuous functions, determining their relation with each other and giving examples and properties associated with this set. This shall serve as the beginning of examining numerous topological properties with this set.

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1. Introduction

Choquet first proposed the idea of the grill on a topological space in 1947 [1]. A grill is a collection of non-empty subsets of $(\wp, \tau)$. If i. $\mu \in G$ and $\mu \subseteq v$ so $v \in G$, ii. $\mu; v \subseteq \wp$ and $\mu \cup v \in G$, then $\mu \in G$ or $v \in G$. a grill topological space denoted by $(\wp, \tau, G)$ [2]. investigated topological concepts and defined a unique topology via a grill. For any topological space point x , $(\wp, \tau)$, $\tau(\wp)$ represents a collection of x 's open nbhd. Let $\varphi: \mathcal{P}(\wp) \rightarrow \mathcal{P}(\wp)$ be a mapping, and it is referred to as $\varphi(\mu) = \{x \in \wp: \mu \cap U \in G \text{ for every } U \in \tau_\wp \text{ for every } \mu \in \mathcal{P}(\wp) \text{ and } \tau_\wp \text{ which is all open set containing } x \text{ .suppose that } \Theta: \mathcal{P}(\wp) \rightarrow \mathcal{P}(\wp)$

* E-mail: tabarakayad314@gmail.com (Corresponding Author)

† E-mail: Rana.b.i@ihcoedu.uobaghdad.edu.iq

§ E-mail: ranamumosa@yahoo.com

is a mapping, and it is referred to as $\Theta(\mu) = \mu \cup \varphi(\mu)$ for every $\mu \in \mathcal{P}(\wp)$. The map Θ satisfies the closure axioms of Kuratowski [3]

- i. $\Theta(\emptyset) = \emptyset$.
- ii. when $\mu \subseteq v$, then $\Theta(\mu) \subseteq \Theta(v)$.
- iii. when $\mu \subseteq \wp$, then $\Theta(\Theta(\mu)) = \Theta(\mu)$.
- iv. when $\mu, v \subseteq \wp$, then $\Theta(\mu \cup v) = \Theta(\mu) \cup \Theta(v)$.

Several kinds of grill topological spaces exist, including discrete and cofinite topologies [4]. We can locate τ_G in a grill G on a topological space

$(\wp, \tau)$ via the base given by following $B(\tau_G, \wp) = \{V - \mu; V \in \tau, \mu \notin G\}$. There exists a unique topology with the formula $\tau_G = \{u \subseteq \wp: \Psi(\wp - u) = (\wp - u)\}$, for any $\mu \subseteq \wp, \Psi(\mu) = \mu \cup \varphi(\mu) = \tau_G - \text{cl}(\mu)$ and $\tau \subseteq \tau_G$ [5]. Every set belongs to τ_G called G -open sets, and its complement called G -closed sets, and the family of all G -closed denoted by F_G [6]. For instance, suppose that $(\wp, \tau)$ is a topological space, If $G = \mathcal{P}(\wp) \setminus \{\emptyset\}$, then $\tau_G = \tau$. Let $(\wp, \tau)$ be a topological space, suppose that $\mu \subseteq \wp$, if $\mu \subseteq \text{Int}(\text{cl}(\text{Int}(\mu)))$ then μ is α -open set. Suppose $v \subseteq \wp$ is α -closed set if $(\wp - v)$ is an α -open set.

The collection of all α -open (respectively, α -closed) sets in $(\wp, \tau)$ will be symbolized by τ_α (respectively, $\alpha\mathcal{C}(\wp)$). Many academics have employed these combinations to produce novel generalizations [7]. In this study, $\text{Int}(\mu)$ is employed to represent the interior(μ), and the symbol $\text{cl}(\mu)$ represents a closure of the set μ .

2. On G_α -open sets

Definition 2.1: Suppose $(\wp, \tau, G)$ is a grill topological space and $u \in \tau$. A closure grill alpha of u which is shortly $\text{cl}_{G_\alpha}(u)$ is defined by $\text{cl}_{G_\alpha}(u) = \bigcap \{V \subseteq \wp; V \text{ is a closed set in } \tau_{G_\alpha} \text{ whenever } u \subseteq V\}$.

Definition 2.2: Let A be a subset of a grill topological space $(\wp, \tau, G)$, and A is a G_α -open set if there exist $\zeta \in \tau; \zeta \subseteq A \subseteq \text{Intcl}_{G_\alpha}(\zeta)$. the complement of

G_α -open set is G_α -closed set, the set of all G_α -open set denoted by G_α - $\mathcal{O}(\wp)$ and the set of all G_α -closed set denoted by G_α - $\mathcal{C}(\wp)$.

Example 2.3: Suppose $(\wp, \tau, G)$ is a grill topological space

$$\begin{aligned} \wp &= \{\ell_1, \ell_2, \ell_3\}, \tau = \{\emptyset, \emptyset, \{\ell_1\}\}, G = \{u \subseteq \wp; \ell_1 \in u\} \\ \tau_G &= \{\emptyset, \emptyset, \{\ell_1\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_3\}\} \quad \tau_{G_\alpha} = \{\emptyset, \emptyset, \{\ell_1\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_3\}\} \\ F_{G_\alpha} &= \{\emptyset, \emptyset, \{\ell_2, \ell_3\}, \{\ell_3\}, \{\ell_2\}\} \quad G_\alpha \text{ } \mathcal{O}(\wp) = \\ &\quad \{\emptyset, \emptyset, \{\ell_1\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_3\}\}. \end{aligned}$$

Proposition 2.4:

1. Each open set is G_α -o.
2. Each closed set is G_α -c.

Proof:

- (i) Suppose $A \in \tau$, and let $\zeta \in \tau$ such that $\zeta = A$ and let $A \subseteq \text{htcl}_{G_\alpha}(A)$ then, $A \subseteq \text{htcl}_{G_\alpha}(\zeta)$, then A is G_α -o($\emptyset$).
- (ii) Let A is a closed set, thus $A^c \in \tau$, $A^c \in G_\alpha$ -o($\emptyset$), then $A \in G_\alpha$ -c($\emptyset$).

Proposition 2.5: Whenever set A is G_α -o it is also G_α -o set.

Proof: Suppose that A is G_α -o then there is an open set denoted by ζ such that $\zeta \subseteq A \subseteq \text{htcl}_{G_\alpha}(\zeta)$ thus $\text{cl}_{G_\alpha}(\zeta) \subseteq \text{cl}_G(\zeta)$ and $\text{intcl}_{G_\alpha}(\zeta) \subseteq \text{intcl}_G(\zeta)$. and since $\zeta \subseteq A \subseteq \text{htcl}_{G_\alpha}(\zeta) \subseteq \text{htcl}_G(\zeta)$ then $\zeta \subseteq A \subseteq \text{htcl}_G(\zeta)$, then A is G_α -open set.

Theorem 2.6: Suppose $(\mathcal{P}, \tau, G)$ is a grill topological space, $A \in \tau$, A is a G_α -open set $\Leftrightarrow A \in G_\alpha$ -o set when ever $G = \mathcal{P}(\emptyset) \setminus \{\emptyset\}$.

Proof: Suppose A be a G_α -open set, so A is a G_α -open set by (Proposition (2.5)). Conversely, let A is a G_α -open set so there exists $\zeta \in \tau$ such that $\zeta \subseteq A \subseteq \text{htcl}_G(\zeta)$ since $\text{htcl}_{G_\alpha}(\zeta) \subseteq \text{intcl}_G(\zeta)$ then $\zeta \subseteq A \subseteq \text{htcl}_{G_\alpha}(\zeta)$, Therefore A is a G_α -open set.

The two sets G_α -open and α -open are independent, as in this example(2.7).

Example 2.7: Let $(\mathcal{P}, \tau, G)$ be any Grill topology and $\tau = \{\emptyset, \phi, \{1\}\}$

$G = \{\zeta \subseteq \mathcal{P} ; \zeta \text{ is an infinite set}\}, \tau_G = \{\emptyset, \phi, \{1\}\} \cup \{\zeta \subseteq \mathcal{P}; \zeta^c \text{ finite set}\}$ is an α -open set; it's not G_α -o set.

Theorem 2.8: Suppose that A is G_α -o set, let $H \subseteq \mathcal{P}$ such that $A \subseteq H \subseteq \text{htcl}_{G_\alpha}(\zeta)$ for each $\zeta \in \tau$ then H is a G_α -o set.

Proof: Whenever A is G_α -o set, then $\exists \zeta \in \tau$ such that $\zeta \subseteq A \subseteq \text{htcl}_{G_\alpha}(\zeta)$, and since $A \subseteq H$, so $\zeta \subseteq H$ and $H \subseteq \text{htcl}_{G_\alpha}(\zeta)$, then $\zeta \subseteq H \subseteq \text{htcl}_{G_\alpha}(\zeta)$, therefore H is G_α -open set.

Lemma 2.9: $\bigcup_{t \in I} (\text{htcl}_{G_\alpha}(A_t)) \subseteq (\text{htcl}_{G_\alpha}(\bigcup_{t \in I} A_t))$

Proof: $A_t \subseteq \bigcup_{t \in I} (A_t)$, so $\text{htcl}_{G_\alpha}(A_t) \subseteq \text{htcl}_{G_\alpha}(\bigcup_{t \in I} (A_t))$

$$\bigcup_{t \in I} (\text{htcl}_{G_\alpha}(A_t)) \subseteq (\text{htcl}_{G_\alpha}(\bigcup_{t \in I} A_t))$$

$$\bigcup_{t \in I} (\text{htcl}_{G_\alpha}(A_t)) \subseteq (\text{htcl}_{G_\alpha}(\bigcup_{t \in I} A_t)).$$

Theorem 2.10: The union of any family of G_α -o sets is G_α -o set.

Proof: Let A_i be G_α -o set $\forall i$ to show $\bigcup_{i \in I} A_i$ is an G_α -o set, since A_i is G_α -open set, then there exist $\varsigma_i \in \tau$ such that $\varsigma_i \subseteq A_i \subseteq \text{htcl}_{G_\alpha}(\varsigma_i)$. Now since $A_i \subseteq \bigcup_{i \in I} (A_i)$ then $(\varsigma_i) \subseteq \bigcup_{i \in I} (A_i) \forall i$, then

$$\bigcup_{i \in I} \varsigma_i \subseteq \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} \text{htcl}_{G_\alpha}(\varsigma_i), \text{ so by (lemma 2.9)}$$

$$\bigcup_{i \in I} \varsigma_i \subseteq \bigcup_{i \in I} A_i \subseteq \text{htcl}_{G_\alpha}(\bigcup_{i \in I} \varsigma_i) \text{ Then } \bigcup_{i \in I} A_i \text{ is an } G_\alpha \text{ -open set.}$$

Note 2.11: A collection of every G_α -o sets is supra topology.

3. Some Kinds of G_α -O Functions.

Definition 3.1: Suppose $f: (\wp, \tau, G) \rightarrow (Y, \tau', G')$ is a function then f is said :

1. G_α -open function denoted by G_α of if $f(s) \in G_\alpha o(Y)$ when ever, $s \in G_\alpha o(\wp)$.
2. G_α^* -open function denoted by G_α^* of if $f(s) \in G_\alpha o(Y)$ when ever, $s \in \tau$.
3. G_α^{**} -open function denoted by G_α^{**} of if $f(s) \in \tau'$ when ever, $s \in G_\alpha o(\wp)$.

Proposition 3.2: Suppose $f: (\wp, \tau, G) \rightarrow (\vartheta, \tau', G')$ is function then

1. f is an o -f whenever f is a G_α^{**} o f
2. f is a G_α of whenever f is a G_α^{**} of.
3. f is a G_α^* o f whenever f is a G_α o f.
4. f is a G_α^* o f whenever f is an o -f

Proof: (1) suppose $u \in \tau$ by (proposition 2.4 (i)), $u \in G_\alpha(\wp)$. Let f be a G_α^{**} o f, $f(u)$ is an open set (Y, τ') . Hence f is an open function. (2) suppose $\varsigma \in G_\alpha o(\wp)$ let f is G_α^{**} of so $f(\varsigma) \in \tau'$. By (proposition 2.4 (i)), $f(\varsigma) \in G_\alpha o(\vartheta)$. so f is G_α -o f. (3) Let $\varsigma \in \tau$ by (proposition 2.4 (i)), $\varsigma \in G_\alpha o(\wp)$. Whenever $f \in G_\alpha$ -o f, $f(\varsigma) \in G_\alpha o(\vartheta)$. so $f \in G_\alpha^*$ of. (4) Let $\varsigma \in \tau$ and since f is an o -f then $f(\varsigma) \in \tau'$. By (proposition 2.4 (i)), $\varsigma \in G_\alpha(\wp)$. So f is a G_α^* of.

The reverse direction of this Proposition is not true as an example.

Example 3.3: suppose $\wp = \{\ell_1, \ell_2, \ell_3\}, \tau = \{\wp, \phi, \{\ell_1\}\}, G = P(\wp) \setminus \{\emptyset\}$,

$f: (\wp, \tau, G) \rightarrow (\wp, \tau, G), f(\ell) = \ell$ for all $\ell \in \wp, G_\alpha o(\wp) = \{u \subseteq \wp; \ell_1 \in u\} \cup \{\wp\}$, there exist $\{\ell_1, \ell_2\} \in G_\alpha o(\wp), f(\{\ell_1, \ell_2\}) = \{\ell_1, \ell_2\} \notin \tau$. Then we observe that f is not G_α^{**} of.

Example 3.4: Let $\wp = \{\ell_1, \ell_2, \ell_3\}, \tau = \{\wp, \phi, \{\ell_1\}, \{\ell_1, \ell_2\}\}$

$G = \{u \subseteq \wp; \ell_3 \in u\}, \tau_G = P(\wp), G_\alpha o(\wp) = P(\wp), f: (\wp, \tau, G) \rightarrow (\wp, \tau, G), f(\ell_1) = \{\ell_2\}, f(\ell_2) = \{\ell_1\}, f(\ell_3) = \{\ell_3\}$. We observe that f is G_α of and G_α^* of, but it's not open and not G_α^{**} of since $f(\ell_1) = \{\ell_2\} \notin \tau$.

Example 3.5: Let $f: (\wp, \tau, G) \rightarrow (\wp, \tau, G)$,

$\wp = \{\ell_1, \ell_2, \ell_3\}, \tau = \{\wp, \phi, \{\ell_1, \ell_2\}\}, G = P(\wp) \setminus \{\{\emptyset\} \cup \{\ell_1\}\}, f(\ell_2) = \{\ell_3\},$
 $f(\ell_1) = \{\ell_2\}, f(\ell_3) = \{\ell_1\}, \tau_G = \{\wp, \phi, \{\ell_1, \ell_2\}, \{\ell_2, \ell_3\}, \{\ell_2\}\}, G_\alpha o(\wp) =$
 $\{\wp, \phi, \{\ell_1, \ell_2\}, \{\ell_2, \ell_3\}, \{\ell_2\}, \{\ell_1\}\}.$ f is a $G_\alpha^{**} o f$ but f is not $G_\alpha^* - o f$ since
 there exist $\{\ell_2, \ell_3\} \in G_\alpha o(\wp)$, but $f(\{\ell_2, \ell_3\}) = \{\ell_1, \ell_3\} \notin G_\alpha o(\wp)$, and it's not
 $G_\alpha^{**} o f$ since there exists $\{\ell_2\} \in G_\alpha o(\wp)$ but $f\{\ell_2\} = \{\ell_3\} \notin \tau.$

Definition 3.6: Suppose that $f: (\wp, \tau, G) \rightarrow (\wp, \tau', G')$ is a function, then f is said to be:

1. G_α -closed function, denoted by " $G_\alpha c f$ " if $f(u) \in G_\alpha c(\wp)$ whenever, $u \in G_\alpha c(\wp)$.
2. G_α^* -closed function, denoted by " $G_\alpha^* c f$ " if $f(u) \in G_\alpha c(\wp)$ whenever, u is a closed set in $(\wp, \tau)$.
3. G_α^{**} -closed function, denoted by " $G_\alpha^{**} c f$ " if $f(u)$ is closed set in $(\wp, \tau')$ whenever, $u \in G_\alpha c(\wp)$.

Proposition 3.7: Suppose that $f: (\wp, \tau, G) \rightarrow (\wp, \tau', G')$ is a function

1. f is a closed function whenever f is $G_\alpha^{**} c f$.
2. f is $G_\alpha c f$ whenever f is $G_\alpha^{**} c f$.
3. f is $G_\alpha^* c f$ whenever f is $G_\alpha c f$.
4. f is $G_\alpha^* c f$ whenever f is a closed function.

Proof: In the same way as (proposition 3.2), the proof has been completed.

The reverse direction of this Proposition is not true, as in examples (Example 3.3), (Example 3.4), and (Example 3.5).

Remark 3.8: If f is onto function, then:

1. $G_\alpha o f$ and $G_\alpha c f$ are equivalents.
2. $G_\alpha^* o f$ and $G_\alpha^* c f$ are equivalents.
3. $G_\alpha^{**} o f$ and $G_\alpha^{**} c f$ are equivalents.

4. Some kinds of G_α - Continuous Functions.

Definition 4.1: Let $f: (\wp, \tau, G) \rightarrow (\wp, \tau', G')$ be a function then:

1. f is a G_α -continuous function, denoted by " $G_\alpha -c f$ " if $f^{-1}(u) \in G_\alpha o(\wp)$ for each $u \in \tau'.$
2. f is Strongly G_α -continuous function, denoted by " $s G_\alpha -c f$ " if $f^{-1}(u) \in \tau$, for each $u \in G_\alpha o(Y).$
3. f is G_α -irresolute function, denoted by " $G_\alpha -irr f$ " if $f^{-1}(u) \in G_\alpha o(\wp)$, for each $u \in G_\alpha o(Y).$

Theorem 4.2: Suppose that $f: (\wp, \tau, G) \rightarrow (\wp, \tau', G')$ is a function denote that :

1. f is G_α -irr f when ever f is an s G_α -c f .
2. f is cont- function whenever f is an s G_α -c f .
3. f is G_α -c f whenever f is a c- f .
4. f is a G_α -c f whenever f is G_α -irr f .

Proof:

1. Let $\zeta \in G_\alpha o(\vartheta)$, Since f is an s G_α -c f , then $f^{-1}(\zeta) \in \tau$. by
(**proposition (2.4(i))**) $f^{-1}(\zeta) \in G_\alpha o(\wp)$, this implies f is a G_α -irr f .
2. Suppose that $\zeta \in \tau$. By (**proposition 2.4(i)**), $\zeta \in G_\alpha o(\vartheta)$. since f is
s- G_α -c f , then $f^{-1}(\zeta)$ is an open set in $(\wp, \tau)$, Implies that f is a c f .
3. Suppose that $\zeta \in \tau$, since f is a continuous function, then $f^{-1}(\zeta) \in \tau$ By
(**proposition (2.4(i))**) $f^{-1}(\zeta) \in G_\alpha o(\wp)$, so f is G_α -cont function.
4. Let $\zeta \in \tau$, By (**proposition 2.4(i)**), $\zeta \in G_\alpha o(\wp)$, since f is G_α -irr f .
Then $f^{-1}(\zeta) \in G_\alpha o(\wp)$, so f is G_α -cont function.
The inverse of this theorem is not true in the following example.

Example 4.3: Let $f : (\wp, \tau, G) \rightarrow (Y, \tau, G')$ be a function such that $f(\ell) = \ell$ for each $\ell \in \wp$. where $\wp = \{\ell_1, \ell_2, \ell_3\}$, $\tau = \{\wp, \emptyset, \{\ell_1\}\}$, $G = P(\wp) \setminus \{\emptyset\}$, $G' = \{u; \ell_1 \in u\}$, $G_\alpha o(\wp) = \{u; \ell_1 \in u\} \cup \{\emptyset\}$, $G'_\alpha o(Y) = P(\wp)$. then f is c f and G_α -c f . but f is not $SG_\alpha(\wp)$ -c f and f is not $G_\alpha(\wp)$ -irr f Since whenever $\{\ell_2, \ell_3\} \in G'_\alpha o(Y)$ but $f^{-1}(\{\ell_2, \ell_3\}) = \{\ell_2, \ell_3\} \notin G_\alpha o(\wp)$ and $\{\ell_2, \ell_3\} \notin \tau$.

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