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On fixed point theorems of maximal monotone operators

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Abstract

This paper aims to study the viscosity approximation approach to fixed points and the solution of a variational inequality. Also, we introduce a new iterative technique and establish the convergence theorems under certain conditions in the Hilbert space.

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1. Introduction

In 1964, Stampacchia[1] introduced variational inequality(V.I) problems and accordingly, Variable Inequality Theory (M.VI) developed into an interesting branch in applied mathematics with many applications in fields (see ([2] - [20]). Because of the rapid pace of change in theory and practice around this subject, researchers from various disciplines have picked their preferred research methods and topics. Numerous scientific disciplines use the ideals and methods of M.V.I, which are profitable and imaginative. Many researchers have lately focused on the F-point method to resolve M.V.I. F-point algorithms, including those of the Mann-type iterative form, are useful for dealing with many nonlinear problems. However, only weakly convergent iterative methods like Mann's are applicable even in Hilbert space; for details, see [21]. Economics

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[22], image recovery [18], quantum physics [23]and control theory [24]all face issues that arise in infinite dimensional space. Research into iterative algorithms with high convergence rates, such as Halpern's, has been extensive. For locating the F-points of nonlinear operators in the Hilbert space, Moudfi [25]has introduced a stickiness technique. For further details and citations, he proved that the convergence point is now more than simply a fixed point for nonlinear operations. As a result, we investigate an iterative Moudaf's viscosity approach to addressing typical M.V.I and F-point problems. For typical solutions, we establish s-convergence theorems in the context of Hilbert space. In most cases, the findings presented here improve upon those found in ([26]-[30]) A For the sake of argument, assume that $\mathbb{H}$ is Hilbert space with the imposed norm$\|z\| = \sqrt{\langle z, z\rangle}$ for $z, w \in \mathbb{H}$ and the inner product $\langle z, w\rangle$for $z, w \in \mathbb{H}$. Let $\mathcal{K}$ be a convex subset of $\mathbb{H}$ that is both closed and nonempty. In this work, the set of solutions to the V.I is denoted by the notation $\mathbb{VI}(\mathcal{K}, \mathbb{M})$. One method for resolving the V.I is the gradients algorithm $\mathcal{J}_{r_{n,i}}(I - r_n\mathbb{M})z$, $n = 0,1, \dots$, where $r_n > 0$.

Now, we recall the following lemmas.

Lemma 1.1 [27]: *Let* $\{\mathfrak{B}_i\}_{i=1}^{r}$, *where* $r > 0$, *be a sequence of nonexpansive mapping. If* $\bigcap_{i=1}^{r} z(\mathfrak{B}_i) \neq \emptyset$*and* $\acute{\mu}_i > 0$,*with* $\sum_{i=1}^{r} = 1$. *Then* $\mathbb{M}_z = \sum_{i=1}^{r} \acute{\mu}_i\mathfrak{B}_i z$ *for* $z \in \mathcal{K}$ *is well define, nonexpansive and* $z(\mathbb{M}) = \bigcup_{i=1}^{r} z(\mathfrak{B}_i)$ *holds* .

Lemma 1.2 [28]: *Let* $\{\mathfrak{v}_n\}$ *be a sequence lies in*$(0, \infty)$, *and*

$$\mathfrak{v}_{n+1} \leq (1 - \mathfrak{h}_n)\mathfrak{v}_n + \mathfrak{U}_n ,$$

where $\{\mathfrak{h}_n\}$ *is a seq. in*$(0,1)$ *and* $\{\mathfrak{U}_n\}$ *is a seq. such that*

(i) $lim_{n\to\infty} \mathfrak{h}_n = 0$ *and* $\sum_{n=1}^{\infty} \mathfrak{h}_n = \infty$ *;*

(ii) $lim\, sup_{n\to\infty} \frac{\mathfrak{U}_n}{\mathfrak{h}_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\mathfrak{U}_n| < \infty$.

Then $lim_{i\to\infty} \mathfrak{v}_n = 0$

2. Main Results

In this section, we defined new iterative technique methods via $\acute{\mu}_i$ – inverse strongly M-mappings and contraction mappings. Also, we prove the convergence theorems to a unique solution of the variational inequality.

Theorem 2.1: *Let* $\mathbb{M}_i : \mathcal{K} \to \mathbb{H}$ *be* $\acute{\mu}_i$ – *inverse strongly M-mapping for each* $1 \leq i \leq r$, *where* $r > 0$, $\mathcal{S}$ *be a nonexpansive mapping, and* $\mathfrak{f}$ *be a* $\mathfrak{v}$ – *contractive. IF* $\mathbb{F} := \bigcap_{i=1}^{r} VI(\mathcal{K}, \mathbb{M}_i) \cap z(\mathbb{M}) \neq \emptyset$,$\{\acute{\lambda}_i\}$ *be a real number in (0,* $2\acute{\mu}_i$*) and* $\{\mathfrak{v}_n\}$, $\{\mathfrak{B}_n\}$, $\{\zeta_n\}$ *be a seq in (0,1) such that* $\{x_n\}$ *be a seq define by*

$$\begin{cases} z_1 \in \mathcal{C} \\ w_{n,i} = \mathcal{J}_{r_{n,i}}\left(z_n - \acute{\lambda}_i \mathbb{M}_i z_n\right) \\ z_{n+1} = \mathfrak{v}_n \sum_{i=1}^{r} \mathfrak{p}_{c,i}(z_n) + \mathfrak{B}_n \mathfrak{f}(z_n) + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} \end{cases} \quad (1)$$

Where $\| w_{\mathfrak{n},\mathrm{i}} - \mathcal{J}_{\mathrm{r}_{\mathfrak{n},\mathrm{i}}}(z_{\mathfrak{n}} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z_{\mathfrak{n}})\| \leq e_{\mathfrak{n},\mathrm{i}}$, where $\lim_{\mathfrak{n}\to\infty}\|e_{\mathfrak{n},\mathrm{i}}\| = 0$ for each $1 \leq i \leq r$ satisfy:

(a) $\mathfrak{v}_{\mathfrak{n}} + \mathfrak{B}_{\mathfrak{n}} + \zeta_{\mathfrak{n}} = \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}} = 1 , \forall\, \mathfrak{n} \geq 1$.

(b) $\lim_{\mathfrak{n}\to\infty} \mathfrak{v}_{\mathfrak{n}} = 0$, $\sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{v}_{\mathfrak{n}} = \infty$.

Then, the sequence $\{z_{\mathfrak{n}}\}$converges to fixed point and to unique solution variational inequality.

$$\langle \mathfrak{f}(\mathfrak{p}) - \mathfrak{p}, \mathfrak{p} - \mathfrak{q}\rangle \geq 0 , \forall \mathfrak{q} \in \mathcal{T}$$

Proof: For any $z, w \in \mathcal{C}$, we have

$$\|(\mathrm{I} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}})z - (\mathrm{I} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}})w\|^2 = \|z - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z - z + \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}w\|^2$$

$$= \|z - w - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z + \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}w\|^2$$

$$= \|z - w\|^2 - 2\acute{\lambda}_{\mathrm{i}}\langle \mathbb{M}_{\mathrm{i}}z - \mathbb{M}_{\mathrm{i}}w , z - w \rangle + \acute{\lambda}_i^2\|\mathbb{M}_{\mathrm{i}}z - \mathbb{M}_{\mathrm{i}}w\|^2$$

$$\leq \|z - w\|^2 - 2\acute{\mu}_i\acute{\lambda}_i\|\mathbb{M}_{\mathrm{i}}z - \mathbb{M}_{\mathrm{i}}w\|^2 + \acute{\lambda}_{\mathrm{i}}^2\|\mathbb{M}_{\mathrm{i}}z - \mathbb{M}_{\mathrm{i}}w\|^2$$

$$\leq \|z - w\|^2 - \acute{\lambda}_i(2\acute{\mu}_i - \acute{\lambda}_i)\, \|\mathbb{M}_{\mathrm{i}}z - \mathbb{M}_{\mathrm{i}}w\|^2 \leq \|z - w\|^2 , \acute{\lambda}_i \in (0,2\acute{\mu}_{\mathrm{i}})$$

So , $(\mathrm{I} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}})$ is nonexpansive.Now , we have

$$\|z_{\mathfrak{n}+1} - z^*\| = \|\mathfrak{v}_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z_{\mathfrak{n}}) + \mathfrak{B}_{\mathfrak{n}}\mathfrak{f}(z_{\mathfrak{n}}) + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}w_{\mathfrak{n}.\mathrm{i}} - z^*\|$$

$$= \|\mathfrak{v}_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z_{\mathfrak{n}}) + \mathfrak{B}_{\mathfrak{n}}\mathfrak{f}(z_{\mathfrak{n}}) + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}w_{\mathfrak{n}.\mathrm{i}} - (\mathfrak{v}_{\mathfrak{n}} + \mathfrak{B}_{\mathfrak{n}} + \zeta_{\mathfrak{n}})z^*\|$$

$$\leq\ \mathfrak{v}_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}}\|\mathcal{P}_{\mathcal{C},\mathrm{i}}(z_{\mathfrak{n}}) - z^*\|\ + \mathfrak{B}_{\mathfrak{n}}\|\mathfrak{f}(z_{\mathfrak{n}}) - z^*\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|w_{\mathfrak{n}.\mathrm{i}} - z^*\|$$

$$\leq \mathfrak{v}_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}}\|\mathcal{P}_{\mathcal{C},\mathrm{i}}(z_{\mathfrak{n}}) - \mathcal{P}_{\mathcal{C},\mathrm{i}}(z^*)\|\ + \alpha_n\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \mathfrak{B}_{\mathfrak{n}}\|\mathfrak{f}(z_{\mathfrak{n}}) - z^*\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|w_{\mathfrak{n}.\mathrm{i}} - \mathcal{J}_{r_{\mathfrak{n},i}}(z^* - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z^*)\|$$

$$\leq\ \mathfrak{v}_{\mathfrak{n}}\, \|z_{\mathfrak{n}} - (z^*)\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \mathfrak{v}\mathfrak{B}_{\mathfrak{n}}\|z_{\mathfrak{n}} - z^*\|$$
$$+\zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|w_{\mathfrak{n}.\mathrm{i}} + \mathcal{J}_{\mathrm{r}_{\mathfrak{n},\mathrm{i}}}(z_{\mathfrak{n}} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z_{\mathfrak{n}}) - \mathcal{J}_{\mathrm{r}_{\mathfrak{n},\mathrm{i}}}(z_{\mathfrak{n}} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z_{\mathfrak{n}}) - \mathcal{J}_{r_{\mathfrak{n},i}}(z^* - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z^*)\|$$

$$\leq\ \mathfrak{v}_{\mathfrak{n}}\, \|z_{\mathfrak{n}} - (z^*)\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \mathfrak{v}\mathfrak{B}_{\mathfrak{n}}\|z_{\mathfrak{n}} - z^*\|$$
$$+\zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|\mathcal{J}_{\mathrm{r}_{\mathfrak{n},\mathrm{i}}}(z_{\mathfrak{n}} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z_{\mathfrak{n}}) - z^*\|$$

$$\leq\ \mathfrak{v}_{\mathfrak{n}}\, \|z_{\mathfrak{n}} - (z^*)\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \mathfrak{v}\mathfrak{B}_{\mathfrak{n}}\|z_{\mathfrak{n}} - z^*\|$$
$$+\zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|(z_{\mathfrak{n}} - \acute{\lambda}_{\mathrm{i}}\mathbb{M}_{\mathrm{i}}z_{\mathfrak{n}}) - z^*\|$$

$$\leq\ \mathfrak{v}_{\mathfrak{n}}\, \|z_{\mathfrak{n}} - (z^*)\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \mathfrak{v}\mathfrak{B}_{\mathfrak{n}}\|z_{\mathfrak{n}} - z^*\|$$
$$+\zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|z_{\mathfrak{n}} - z^*\|$$

$$= (\mathfrak{v}_{\mathfrak{n}} + \mathfrak{B}_{\mathfrak{n}} + \zeta_{\mathfrak{n}})\|z_{\mathfrak{n}} - z^*\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\|$$

$$\leq (1 - \mathfrak{B}_{\mathfrak{n}}(1 - \mathfrak{v})\|z_{\mathfrak{n}} - z^*\| + \mathfrak{v}_{\mathfrak{n}}\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\ (z^*) - z^*\| + \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\|$$

$$\leq (1 - \mathfrak{B}_{\mathfrak{n}}(1 - \mathfrak{v})\|z_{\mathfrak{n}-1} - z^*\| + \sum_{\mathcal{K}=0}^{\mathfrak{n}} \mathfrak{v}_{\mathfrak{n}}(1 - \mathfrak{v}_{\mathfrak{n}}) \frac{\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},\mathrm{i}}\, (z^*) - z^*\|}{(1 - \mathfrak{v}_{\mathfrak{n}})}$$
$$+ \sum_{\mathcal{K}=0}^{\mathfrak{n}} \zeta_{\mathfrak{n}} \sum_{\mathrm{i}=1}^{\mathrm{r}} \mathfrak{y}_{\mathrm{i}}\, \|e_{\mathfrak{n},\mathrm{i}}\|$$

$\vdots$

$\leq (1-\mathfrak{B}_n(1-\mathfrak{v})\|z_0 - z^*\| + \sum_{\mathcal{K}=0}^{n} \mathfrak{v}_n(1-\mathfrak{v}_n)\frac{\|\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z^*)-z^*\|}{(1-\mathfrak{v}_n)}$

$+\sum_{\mathcal{K}=0}^{n}\zeta_n\sum_{i=1}^{r}\mathfrak{y}_i\|e_{n,i}\|$

$\leq max\{\|z_0 - z^*\|, \frac{\|\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z^*)-z^*\|}{(1-\mathfrak{v}_n)}\} + \sum_{\mathcal{K}=0}^{n}\zeta_n\sum_{i=1}^{r}\mathfrak{y}_i\|e_{n,i}\|$

By mathematical induction we have

$\|z_{n+1} - z^*\| \leq$

$\{max\{\|z_0 - z^*\|, \frac{\|\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z^*)-z^*\|}{(1-\mathfrak{v}_n)}\} + \sum_{\mathcal{K}=0}^{n}\zeta_n\sum_{i=1}^{r}\mathfrak{y}_i\|e_{n,i}\|\} < \infty$

So, $\{z_n\}$ is bounded

$\|w_{n+1,i} - w_{n,i}\| = \|w_{n+1,i} - \mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1}) + \mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1}) - \mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n) + \mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n) - w_{n,i}\|$

$\leq \|w_{n+1,i} - \mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1})\| + \|\mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1}) - \mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n)\| + \|\mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n) - w_{n,i}\|$

$\leq \|e_{n+1,i}\| + \|z_{n+1} - z_n\| + \|e_{n,i}\|$

Putting $w_n = \sum_{i=1}^{r}\mathfrak{y}_i w_{n,i}$, we have

$\|w_{n+1,i} - w_{n,i}\| = \|\sum_{i=1}^{r}\mathfrak{y}_i w_{n+1,i} - \sum_{i=1}^{r}\mathfrak{y}_i w_{n,i}\|$

$\leq \sum_{i=1}^{r}\mathfrak{y}_i\|w_{n+1,i} - w_{n,i}\|$

$\leq \sum_{i=1}^{r}\mathfrak{y}_i\|w_{n+1,i} - \mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1}) + \mathcal{J}_{r_{n+1,i}}(z_{n+1} - \acute{\lambda}_i\mathbb{M}_i z_{n+1}) - \mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n) + \mathcal{J}_{r_{n,i}}(z_n - \acute{\lambda}_i\mathbb{M}_i z_n) - w_{n,i}\|$

$\leq \sum_{i=1}^{r}\mathfrak{y}_i(\|e_{n+1,i}\| + \|e_{n,i}\| + \|z_{n+1} - z_n\|)$

Put $\mathfrak{c}_n = \frac{z_{n+1} - \mathfrak{B}_n z_n}{1-\mathfrak{B}_n}$, for all $n \geq 1$. that is

$(1-\mathfrak{B}_n)\mathfrak{c}_n = z_{n+1} - \mathfrak{B}_n z_n \Rightarrow z_{n+1} = (1-\mathfrak{B}_n)\mathfrak{c}_n + \mathfrak{B}_n z_n$ for all $n \geq 1$.

Note that

$$\|\mathfrak{c}_{n+1} - \mathfrak{c}_n\| = \left\|\frac{z_{n+2} - \mathfrak{B}_{n+1}z_{n+1}}{1-\mathfrak{B}_{n+1}} - \frac{z_{n+1} - \mathfrak{B}_n z_n}{1-\mathfrak{B}_n}\right\|$$

$$= \left\|\begin{array}{c}\frac{\mathfrak{v}_{n+1}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_{n+1}) + \mathfrak{B}_{n+1}\mathfrak{f}(z_{n+1}) + \zeta_{n+1}\sum_{i=1}^{r}\mathfrak{y}_i w_{n+1,i} - \mathfrak{B}_{n+1}z_{n+1}}{1-\mathfrak{B}_{n+1}} \\ - \frac{\mathfrak{v}_n\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_n) + \mathfrak{B}_n\mathfrak{f}(z_n) + \zeta_n\sum_{i=1}^{r}\mathfrak{y}_i w_{n,i} - \mathfrak{B}_n z_n}{1-\mathfrak{B}_n}\end{array}\right\|$$

$$\leq \left\|\begin{array}{c}\frac{\mathfrak{v}_{n+1}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_{n+1}) + \mathfrak{B}_{n+1}z_{n+1} + \zeta_{n+1}\sum_{i=1}^{r}\mathfrak{y}_i w_{n+1,i} - \mathfrak{B}_{n+1}z_{n+1}}{1-\mathfrak{B}_{n+1}} \\ - \frac{\mathfrak{v}_n\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_n) + \mathfrak{B}_n z_n + \zeta_n\sum_{i=1}^{r}\mathfrak{y}_i w_{n,i} - \mathfrak{B}_n z_n}{1-\mathfrak{B}_n}\end{array}\right\|$$

$$\leq \left\|\frac{\mathfrak{v}_{n+1}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_{n+1}) + \zeta_{n+1}\sum_{i=1}^{r}\mathfrak{y}_i w_{n+1,i}}{1-\mathfrak{B}_{n+1}} - \frac{\mathfrak{v}_n\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_n) + \zeta_n\sum_{i=1}^{r}\mathfrak{y}_i w_{n,i}}{1-\mathfrak{B}_n}\right\|$$

$$\leq \left\|\frac{\mathfrak{v}_{n+1}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_{n+1}) + \zeta_{n+1}}{1-\mathfrak{B}_{n+1}} - \frac{\mathfrak{v}_n\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_n) + \zeta_n w_n}{1-\mathfrak{B}_n}\right\|$$

Since $\mathfrak{v}_n + \mathfrak{B}_n + \zeta_n = 1 \Rightarrow \zeta_n = 1 - \mathfrak{v}_n - \mathfrak{B}_n$

$= \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_{n+1}) + \frac{1-\mathfrak{B}_{n+1}-\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}}w_{n+1} - \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n}\sum_{i=1}^{r}\mathcal{P}_{\mathcal{C},i}(z_n) - \frac{1-\mathfrak{B}_n-\mathfrak{v}_n}{1-\mathfrak{B}_n}w_n$

$= \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) + \left(\frac{1-\mathfrak{B}_{n+1}}{1-\mathfrak{B}_{n+1}} - \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}}\right) w_{n+1} - \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) - \left(\frac{1-\mathfrak{B}_n}{1-\mathfrak{B}_n} - \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n}\right) w_n$

$= \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) - \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} w_{n+1} - \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) + \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} w_n + w_{n+1} - w_n$

$\leq \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) \\ -w_{n+1} \end{pmatrix} + \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) \\ -w_n \end{pmatrix} + w_{n+1} - w_n$

It follows that

$\|\mathfrak{c}_{n+1} - \mathfrak{c}_n\| \leq \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \left\| \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) \\ -w_{n+1} \end{pmatrix} \right\| + \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \left\| \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) \\ -w_n \end{pmatrix} \right\| + \|w_{n+1} - w_n\|$

Since $\|w_{n+1} - w_n\| \leq \sum_{i=1}^{r} \mathfrak{y}_i \left(\|\mathbb{e}_{n+1,i}\| + \|\mathbb{e}_{n,i}\| + \|z_{n+1} - z_n\|\right)$

So,

$\leq \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \left\| \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) \\ -w_{n+1} \end{pmatrix} \right\| + \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \left\| \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) \\ -w_n \end{pmatrix} \right\|$

$+ \sum_{i=1}^{r} \mathfrak{y}_i \left(\|\mathbb{e}_{n+1,i}\| + \|\mathbb{e}_{n,i}\| + \|z_{n+1} - z_n\|\right)$

So, $\|\mathfrak{c}_{n+1} - \mathfrak{c}_n\| - \|z_{n+1} - z_n\| \leq \frac{\mathfrak{v}_{n+1}}{1-\mathfrak{B}_{n+1}} \left\| \left(\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_{n+1}) - w_{n+1}\right) \right\| + \frac{\mathfrak{v}_n}{1-\mathfrak{B}_n} \left\| \begin{pmatrix} \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) \\ -w_n \end{pmatrix} \right\| + \sum_{i=1}^{r} \mathfrak{y}_i \left(\|\mathbb{e}_{n+1,i}\| + \|\mathbb{e}_{n,i}\|\right.$

So, we have $\lim_{n\to\infty} \mathfrak{sup} \left(\|\mathfrak{c}_{n+1} - \mathfrak{c}_n\| - \|z_{n+1} - z_n\|\right) < 0$

Since, $\lim_{n\to\infty} \mathfrak{sup} \left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|\right) \leq 0.$

Then $\lim_{n\to\infty} \|w_n - z_n\| = 0$

It follows that $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$ (2)

On the other hand,

$\left\| \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\|^2 \leq \left\langle \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z^*, \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\rangle$

$= \frac{1}{2} \left(\left\| \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z^* \right\|^2 + \left\| \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\|^2 - \left\| \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - \left(I - \acute{\lambda}_i \mathbb{M}_i\right) z^* - \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\|^2 \right)$

$\leq \frac{1}{2} \left(\|z_n - z^*\|^2 + \left\| \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\|^2 - \left\| \begin{matrix} z_n - \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n \\ -\acute{\lambda}_i (\mathbb{M}_i z_n - \mathbb{M}_i z^*) \end{matrix} \right\|^2 \right)$

$= \frac{1}{2} \left(\|z_n - z^*\|^2 + \left\| \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n - z^* \right\|^2 - \left\| z_n - \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n \right\|^2 + 2\acute{\lambda}_i \left\langle \mathbb{M}_i z_n - \mathbb{M}_i z^*, z_n - \mathcal{J}_{r_{n,i}}\left(I - \acute{\lambda}_i \mathbb{M}_i\right) z_n \right\rangle - \acute{\lambda}_i^{\,2} \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2 \right)$

So,

$\left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2 = \|z_n - z^*\|^2 - \left\|z_n - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n\right\|^2 + \mathcal{S}_i\|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$ (3)

Where $\mathcal{S}_i = max\left\{2\acute{\lambda}_i\left\|z_n - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n\right\| : \forall\, n \geq 1\right\}$.

$\|z_{n+1} - z^*\|^2$

$$= \left\|\begin{matrix}\mathfrak{v}_n \sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) + \mathfrak{B}_n\mathfrak{f}(z_n) + \zeta_n \sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} \\ -(\mathfrak{v}_n + \mathfrak{B}_n + \zeta_n)z^*\end{matrix}\right\|^2$$

$$\leq \left\|\begin{matrix}\mathfrak{v}_n \sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - \\ \mathfrak{v}_n z^*\end{matrix}\right\|^2 + \left\|\begin{matrix}\mathfrak{B}_n\mathfrak{f}(z_n) - \\ \mathfrak{B}_n z^*\end{matrix}\right\|^2 + \left\|\begin{matrix}\zeta_n \sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} \\ -\zeta_n z^*\end{matrix}\right\|^2$$

$$\leq \left\langle\begin{matrix}\sum_{i=1}^{\mathfrak{r}} \mathfrak{v}_n\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*, \\ \sum_{i=1}^{\mathfrak{r}} \mathfrak{v}_n\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\end{matrix}\right\rangle + \left\langle\begin{matrix}\mathfrak{B}_n\mathfrak{f}(z_n) - z^* \\ , \mathfrak{B}_n\mathfrak{f}(z_n) - z^*\end{matrix}\right\rangle + \left\langle\begin{matrix}\zeta_n \sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - z^* \\ , \zeta_n \sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - z^*\end{matrix}\right\rangle$$

$$\leq \mathfrak{v}_n \left\langle\begin{matrix}\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*, \\ \sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\end{matrix}\right\rangle + \mathfrak{B}_n \left\langle\begin{matrix}\mathfrak{f}(z_n) - z^*, \\ \mathfrak{f}(z_n) - z^*\end{matrix}\right\rangle + \zeta_n \left\langle\begin{matrix}\sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - z^*, \\ \sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - z^*\end{matrix}\right\rangle$$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{B}_n\|\mathfrak{f}(z_n) - z^*\|^2 + \zeta_n\left\|\sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - z^*\right\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{B}_n\|\mathfrak{f}(z_n) - z^*\|^2 + \zeta_n\left\|\sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n + \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{B}_n\|\mathfrak{f}(z_n) - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|w_{n,i} - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n\right\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{B}_n\|\mathfrak{f}(z_n) - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \|z_n - z^*\|^2 - 2\acute{\lambda}_i\langle\mathbb{M}_i z_n - \mathbb{M}_i z^*, z_n - z^*\rangle + \acute{\lambda}_i^{\,2}\|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \|\mathfrak{f}(z_n) - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2 - \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \acute{\lambda}_i\left(2\acute{\mu}_\iota - \acute{\lambda}_i\right) \|\mathbb{M}_i z - \mathbb{M}_i w\|^2$

Therefore,

$\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \acute{\lambda}_i\left(2\acute{\mu}_\iota - \acute{\lambda}_i\right) \|\mathbb{M}_i z - \mathbb{M}_i w\|^2 \leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \|z_n - z^*\|^2 - \|z_{n+1} - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \|z_n - z^* + z^* - z_{n+1}\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{\mathfrak{r}} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + (\|z_n - z^*\| + \|z^* - z_{n+1}\|)\|z_n - z_{n+1}\| + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2$

From (2), we get $\lim_{n\to\infty}\|\mathbb{M}_i z - \mathbb{M}_i z^*\|^2 = 0, \forall\, 1 \leq \mathfrak{i} \leq \mathfrak{r}$

$$\|w_n - z_n\| = \left\|\begin{matrix}\sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} - \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n \\ + \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z_n\end{matrix}\right\|$$

$$\leq \left\|\begin{matrix}\sum_{i=1}^{\mathfrak{r}} \mathfrak{y}_i w_{n,i} \\ - \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n\end{matrix}\right\| + \left\|\begin{matrix}\sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n \\ -z_n\end{matrix}\right\|$$

$\leq \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2 + \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z_n\right\|$

It follows from (3) that

Since $\left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2 \le \|z_n - z^*\|^2 - \left\|z_n - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I} - \acute{\lambda}_i\mathbb{M}_i\right)z_n\right\|^2 + \mathcal{S}_i\|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$

$$\sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$$

$$\le \|z_n - z^*\|^2 - \|w_n - z_n\| + \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$$

$$+ \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$$

Hence we have

$$\|z_{n+1} - z^*\|^2 = \left\|\begin{matrix}\mathfrak{v}_n \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) + \mathfrak{B}_n \mathfrak{f}(z_n) \\ +\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - z^*\end{matrix}\right\|^2$$

$$= \left\|\begin{matrix}\mathfrak{v}_n \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) + \mathfrak{B}_n \mathfrak{f}(z_n) + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} \\ -(\mathfrak{v}_n + \mathfrak{B}_n + \zeta_n) z^*\end{matrix}\right\|^2$$

$$\le \left\|\begin{matrix}\mathfrak{v}_n \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) \\ -\mathfrak{v}_n z^*\end{matrix}\right\|^2 + \left\|\begin{matrix}\mathfrak{B}_n \mathfrak{f}(z_n) \\ -\mathfrak{B}_n z^*\end{matrix}\right\|^2 + \left\|\begin{matrix}\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} \\ -\zeta_n z^*\end{matrix}\right\|^2$$

$$\le \mathfrak{v}_n \left\langle\begin{matrix}\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*, \\ \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\end{matrix}\right\rangle + \mathfrak{B}_n \left\langle\begin{matrix}\mathfrak{f}(z_n) - z^* \\ , \mathfrak{f}(z_n) - z^*\end{matrix}\right\rangle + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\langle\begin{matrix}w_{n,i} - z^*, \\ \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - z^*\end{matrix}\right\rangle$$

$$\le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{v}\mathfrak{B}_n \|z_n - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|w_{n,i} - z^*\right\|^2$$

$$\le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{v}\mathfrak{B}_n \|z_n - z^*\|^2 + \zeta_n \left\|\sum_{i=1}^{r} \mathfrak{y}_i\, w_{n,i} - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n + \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$$

$$\le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{v}\mathfrak{B}_n \|z_n - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|w_{n,i} - \mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n\right\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$$

$$\le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + \mathfrak{v}\mathfrak{B}_n \|z_n - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2$$

Since $\sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathcal{J}_{\mathfrak{r}_{n,i}}\left(\mathrm{I}-\acute{\lambda}_i\mathbb{M}_i\right)z_n - z^*\right\|^2 \le \|z_n - z^*\|^2 - \|w_n - z_n\| + \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\| + \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$

$$\le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + 2\|z_n - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\| - \zeta_n \|w_n - z_n\| + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\| + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2$$

$$\zeta_n \|w_n - z_n\| \le \mathfrak{v}_n \sum_{i=1}^{r} \left\|\mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + 2\|z_n - z^*\|^2 - \|z_{n+1} - z^*\|^2 + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2 + 2\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$$

$$\le \mathfrak{v}_n \left\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + 2\|z_n - z^* + z^* - z_{n+1}\|^2 +$$

$\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2 + 2\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$

$\leq \mathfrak{v}_n \left\|\sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}\ (z_n) - z^*\right\|^2 + 2(\|z_n - z^*\| + \|z^* - z_{n+1}\|)\|z_n - z_{n+1}\| + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{S}_i \|\mathbb{M}_i z_n - \mathbb{M}_i z^*\|^2 + 2\zeta_n \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$

Hence, $\lim_{n\to\infty}\|w_n - z_n\| = 0$.Let $\bar{z} = \mathcal{P}_{\mathcal{T}} \mathfrak{f}\ (\bar{z})$

Now , we show

$\limsup_{n\to\infty} \langle \mathfrak{f}(\bar{z}) - \bar{z}, z_n - \bar{z}\rangle \leq 0$

Choose a sequence $\{z_{n_i}\}$ of $\{\bar{z}\}$ such that

$\limsup_{n\to\infty} \langle \mathfrak{f}(\bar{z}) - \bar{z}, z_n - \bar{z}\rangle = \lim_{i\to\infty} \langle \mathfrak{f}(\bar{z}) - \bar{z}, z_{n_i} - \bar{z}\rangle$.

Since $\{z_{n_i}\}$ is bounded. So, there exists a subsequence $\left\{z_{n_{i_j}}\right\}$ of $\{z_{n_i}\}$ s.t converges weakly to $\bar{z}$.

Suppose that $z_{n_i} \rightharpoonup \bar{z}$. Define a mapping $\mathcal{R}: \mathcal{K} \longrightarrow \mathcal{K}$ by

$\mathcal{R}_Z = \sum_{i=1}^{r} \mathfrak{y}_i\, \mathcal{J}_{r_{n,i}}(I - \lambda_i \mathbb{M}_i) z_n$, $\forall\, z \in \mathcal{K}$ *and* $\left\|\mathcal{R}_{z_n} - w_n\right\| \to 0$

Using lemma (1.1) , we get, W is nonexpansive s.t $\mathbb{Z}(\mathcal{R}) = \bigcap_{i=1}^{r} \mathbb{Z}\left(\mathcal{J}_{r_{n,i}}(I - \lambda_i \mathbb{M}_i) z\right) = \bigcap_{i=1}^{r} \mathbb{V}I(\mathcal{K}, \mathbb{M}_i)$. But $\lim_{n\to\infty}\|z_n - \mathcal{R}z_n\| = 0$. We obtain that $\bar{z} \in \mathbb{Z}(\mathcal{R})$, we see that $\bar{z} \in \mathbb{Z}(S)$. This proves that $\bar{z} \in \mathbb{Z}(\mathcal{R}) = \bigcap_{i=1}^{r} \mathbb{V}I(\mathcal{K}, \mathbb{M}_i)$.

so, $\limsup_{n\to\infty} \langle \mathfrak{f}(\bar{z}) - \bar{z}, z_n - \bar{z}\rangle \leq 0$

$\|z_{n+1} - \bar{z}\|^2 = \langle z_{n+1} - \bar{z}, z_{n+1} - \bar{z}\rangle$

$\leq \langle \mathfrak{v}_n \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) + \mathfrak{B}_n \mathfrak{f}(z_n) + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - \bar{z}\, , z_{n+1} - \bar{z}\ \rangle$

$\leq \langle \mathfrak{v}_n \sum_{i=1}^{n} \mathcal{P}_{\mathcal{C},i}(z_n) + \mathfrak{B}_n \mathfrak{f}(z_n) + \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - (\mathfrak{v}_n + \mathfrak{B}_n + \zeta_n)\bar{z}\, , z_{n+1} - \bar{z}\ \rangle$

$\leq \langle \mathfrak{v}_n \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) - \mathfrak{v}_n \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \langle \mathfrak{B}_n \mathfrak{f}(z_n) - \mathfrak{B}_n \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \langle \zeta_n \sum_{i=1}^{r} \mathfrak{y}_i w_{n,i} - \zeta_n \bar{z}\, , z_{n+1} - \bar{z}\ \rangle$

$\leq \mathfrak{v}_n \langle \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) - \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \mathfrak{v}\mathfrak{B}_n \|z_n - \bar{z}\| \|z_{n+1} - \bar{z}\| + \zeta_n \left\|w_{n,i} - \bar{z}\right\| \|z_{n+1} - \bar{z}\|$

$\leq \mathfrak{v}_n \langle \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) - \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \mathfrak{v}\mathfrak{B}_n \|z_n - \bar{z}\| \|z_{n+1} - \bar{z}\| + \zeta_n \|w_n - \bar{z}\| \|z_{n+1} - \bar{z}\| + \zeta_n \left[\|z_n - \bar{z}\| + \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|\right] \|z_{n+1} - \bar{z}\|$

$\leq \mathfrak{B}_n \langle \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) - \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \mathfrak{v}\mathfrak{B}_n \|z_n - \bar{z}\| \|z_{n+1} - \bar{z}\| + (\mathfrak{v}_n + \zeta_n) \|z_n - \bar{z}\| + \|z_{n+1} - \bar{z}\| + \zeta_n \|z_{n+1} - \bar{z}\| \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$

$\|z_{n+1} - \bar{z}\|^2 = \left(1 - \mathfrak{B}_n(1 - \mathfrak{v})\right) \|z_n - \bar{z}\|^2 + \mathfrak{B}_n \langle \sum_{i=1}^{r} \mathcal{P}_{\mathcal{C},i}(z_n) - \bar{z}\, , z_{n+1} - \bar{z}\ \rangle + \zeta_n \|z_{n+1} - \bar{z}\| \sum_{i=1}^{r} \mathfrak{y}_i \left\|\mathbb{e}_{n,i}\right\|$

By Lemma (1.2), one has $\lim_{n\to\infty}\|z_n - \bar{z}\| = 0$.

Theorem 2.2: *Let* $\mathbb{M}_i: \mathcal{K} \to \mathbb{H}$ *be* $\acute{\mu}_i -$ *inverse stronglyM-mapping for each* $1 \leq i \leq r$, *where* $r > 0$, $\mathcal{S}$ *and* $\mathfrak{f}$ *be like contraction mappings. If* $\mathbb{F} := \bigcap_{i=1}^{r} VI(\mathcal{K}, \mathbb{M}_i) \cap z(\mathbb{M}) \neq \emptyset$, $\{\lambda_i\}$ *be a real number in (0,* $2\acute{\mu}_i$*) . and* $\{\mathfrak{x}_n\}$ *be a seq define by*

$$\begin{cases} z_1 \in \mathcal{C} \\ w_{\mathfrak{n},\mathfrak{i}} = \mathcal{P}_{\mathcal{C},\mathfrak{i}}\left(z_{\mathfrak{n}} - \acute{\lambda}_{\mathfrak{i}}\mathbb{M}_{\mathfrak{i}} z_{\mathfrak{n}}\right) \\ z_{\mathfrak{n}+1} = \mathfrak{v}_{\mathfrak{n}} \sum_{i=1}^{r} \mathcal{J}_{\mathfrak{r}_{\mathfrak{n},\mathfrak{i}}}(z_{\mathfrak{n}}) + \mathfrak{B}_{\mathfrak{n}}\mathfrak{f}(z_{\mathfrak{n}}) + \zeta_n \sum_{\mathfrak{i}=1}^{\mathfrak{r}} \mathfrak{y}_{\mathfrak{i}} w_{\mathfrak{n},\mathfrak{i}} \end{cases}$$

Where $\left\| w_{\mathfrak{n},\mathfrak{i}} - \mathcal{P}_{\mathcal{C},\mathfrak{i}}\left(z_{\mathfrak{n}} - \acute{\lambda}_{\mathfrak{i}}\mathbb{M}_{\mathfrak{i}} z_{\mathfrak{n}}\right)\right\| \leq \mathbb{e}_{\mathfrak{n},\mathfrak{i}}$, where $\lim_{\mathfrak{n}\to\infty}\left\|\mathbb{e}_{\mathfrak{n},\mathfrak{i}}\right\| = 0$ for each $1 \leq i \leq r$. If

(a) $\mathfrak{v}_{\mathfrak{n}} + \mathfrak{B}_{\mathfrak{n}} + \zeta_{\mathfrak{n}} = \sum_{\mathfrak{i}=1}^{\mathfrak{r}} \mathfrak{y}_{\mathfrak{i}} = 1 , \forall\, \mathfrak{n} \geq 1$.
(b) $\lim_{\mathfrak{n}\to\infty} \mathfrak{v}_{\mathfrak{n}} = 0$, $\sum_{\mathfrak{i}=1}^{\mathfrak{r}} \mathfrak{v}_{\mathfrak{n}} = \infty$.

Then, the sequence $\{z_{\mathfrak{n}}\}$ converges to a unique fixed point.

Proof: We get the desired result by following the same proof structure in the theorem(2.1)**.**

3. Conclusion

A new algorithm is introduced to find a solution for a family of maximal monotone operators. Also, we established some convergence theorems to the unique fixed point and zeros of finite maximal operators.

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