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On analytical convergence of multi iterative procedure for finite family of generalized contractive

Omar Mohammed Abbas Joodi *
Zena Hussein Maibed †
Department of Mathematics
College of Education for Pure Science (Ibn Al-Haithem)
University of Baghdad
Baghdad
Iraq

Abstract

This paper introduces new iterations type three steps as multi_ implicit Noor iteration, multi_explicit Noor iteration and multi_Picard S-iteration, and defines a generalized quasi-like contractive mapping in convex metric space. Also, the convergence and stability of it are studied. On the other hand, the multi_ implicit Noor iteration is faster than the multi_explicit Noor iteration and multi_Picard S-iteration

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1. Introduction

The Banach contraction principle, introduced in 1922, is a powerful tool in nonlinear analysis. It is well known that this principle plays a central part in existence, and uniqueness theorems play an important role in finding fixed points. Because of this, the result for contractive type mappings has been extensively researched. In this paper, we assume S is a nonempty closed convex subset of the convex metric space C (CMS). Represent for the set of the fixed point (f-point) of the function V by $F(V)$, for all r, s in S if $d(Vr, Vs) \leq d(r, s)$ then say V is non-expansive when $x \in F(V)$ say V quasi non-expansive if

* *E-mail:* **Omar.Mohammed1203a@ihcoedu.uobaghdad.edu.iq**
(Corresponding Author)
† *E-mail:* **zena.h.m@ihcoedu.uobaghdad.edu.iq**

$d(Vr, x) \leq d(r, x)$. In complete metric space, Banach[1] proved that V have a f-point if there exist $0 \leq q < 1$ such that $d(Vr, Vs) \leq qd(r, s)$. The iterative

$$Vr_n = r_{n+1} \; n \geq 0$$

namely Picard iteration and its convergence to x with contraction mapping, but its failure converges to x with non-expansive mapping.

Definition 1.1 [2] The Mann iteration with initial r_0

$r_{n+1} = (1 - a_n)r_n + a_n Vr_n$; $n \geq 0$, where $\langle a_n \rangle$ is a real sequence in $[0,1]$

Definition 1.2[3] The Ishikawa iteration with initial r_0

$$r_{n+1} = (1 - a_n)r_n + a_n Vr_n, s_n = (1 - b_n)r_n + b_n Vr_n$$

Where $\langle a_n \rangle$ and $\langle b_n \rangle$ are real sequences in $[0,1]$ with $n \geq 0$ and proved its convergence to f-point x with non-expansive mapping. According to a remark by Rhoades [4], the Mann iterative scheme for decreasing functions converges faster than the Ishikawa iterative scheme.

Definetion 1.3 [5] the Noor iteration with initial r_0

$$r_{n+1} = (1 - a_n)r_n + a_n Vr_n, s_n = (1 - b_n)r_n + b_n Vz_n,$$
$$z_n = (1 - c_n)r_n + c_n Vr_n$$

Where $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ are real sequences in $[0,1]$, with $n \geq 0$.

Over the years, new iterations have been defined, such as Agarwal et al. [6], Abbas and Nazir [7] and Thakur[8]. In the last years, defined a more general concept for contractive mapping.

Imoru and Olatino[9] came up with a more generalized concept of what they previously called contractive-like

Definition 1.4 [9] The operator $V: S \to S$ namely contractive-like if there exists constant $\xi \in [0,1)$ and $\emptyset: R^+ \to R^+$ with $\emptyset(0) = 0$ then

$$d(Vr, Vs) \leq \emptyset\big(d(r, Vr)\big) + \xi d(r, s) \; \forall \, r, s \in S$$

Definition 1.5 [10] The operator $V: S \to S$ namely Suzuki contractive if Vsatisfying the below condition

$\frac{1}{2}\|r - Vr\| \leq \|r - s\|$ implies $\|Vr - Vs\| \leq \|r - s\|$ for all $r, s \in S$

Suzuki proved the existence of f-points for these mappings and achieved convergence findings. Moreover, Suzuki proved that the concept of mappings satisfying generalized non-expansive mappings is more general than the concept of non-expansive mappings. In 2011, Phuengrattana [11]used the Ishikawa iterative approach in uniformly convex Banach spaces to verify convergence findings for Suzuki's generalized non-expansive mappings. In addition, a lot of research has been published over the past years about finding new types of iterative schemes with different functions and proving convergence, rate of convergence, and stability (see[12]–[18]).

In this paper, we introduce generalized quasi like contractive with three new iterative schemes from a three-step type. We will study converges, rate of convergence and stability for each of them.

Below is a set of important definitions and lemma to reach our main results

Definition1.6 [19] A map $\mathcal{X}: C \times C \times [0,1] \to C$ called convex structure on C when $d(z, \mathcal{X}(r,s,\sigma)) \leq \sigma d(z,r) + (1-\sigma)d(z,s)$, $\forall r,s,z \in C$ and $\sigma \in [0,1]$.

Called for C with convex structure $\mathcal{X}$ CMS and denoted by (C,d,σ), say for S convex subset if for all $r,s \in S$ then $\mathcal{X}(r,s,\sigma) \in S$.

Every normed space and subset of normed space is CMS, But the opposite is not true see[19],[20].

Example1.7[21] Let $S = \{(r_1,r_2) \in R^2 : r_1, r_2 > 0\}$. For each $r = (r_1,r_2)$, $s = (s_1,s_2) \in S$ and $\mu \in [0,1]$. Define $\mathcal{X}: S^2 \times [0,1]$ by

$$\mathcal{X}(r,s,\mu) = \left(\mu r_1 + (1-\mu)s_1, \frac{\mu r_1 r_2 + (1-\mu)s_1 s_2}{\mu r_1 + (1-\mu)s_1}\right)$$

and define a metric $d: S \times S \to [0,\infty)$ by

$$d(r,s) = |r_1 - s_1| + |r_1 r_2 - s_1 s_2|$$

We can show that $(S,d,\mathcal{X})$ is convex metric space, but it is not a normed space.

Definition 1.8 [22] Let V be any mapping, $x \in F(V)$, $\{r_n\}_{n=0}^{\infty}$ be a sequence defined by an iterative scheme depends on V such as $r_{n+1} = f(V, r_n)$, $n \geq 0$. where f some function and $r_0 \in C$ the initial value, if $\{r_n\}_{n=0}^{\infty}$ converge to x and $\{s_n\}_{n=0}^{\infty} \subset C$ with $\epsilon_n = d\big(s_{n+1}, f(V, s_n)\big), n \geq 0$, so the iterative is called V-stable if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} s_n = x$ provided that the opposite is true.

Lemma 1.9 [23] *For the sequence of positive numbers* $\{\epsilon_n\}_{n=0}^{\infty}$ *such that* $lim_{n\to\infty} \epsilon_n = 0$ *and the real number* ϑ *such that* $0 \leq \vartheta < 1$, *for the positive sequence* $\{z_n\}_{n=0}^{\infty}$ *satisfying*

$$z_{n+1} \leq \vartheta z_n + \epsilon_n, n \geq 0. \text{ Then we have } lim_{n\to\infty} z_n = 0$$

Rhoades[24]say $\{r_n\}_{n=0}^{\infty}$ *better than* $\{s_n\}_{n=0}^{\infty}$ *when it converges to a certain f-point* x *of the operator* V *if*

$$\|r_n - x\| \leq \|s_n - x\|, \forall n \geq 0$$

Definition 1.10 [25] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, let the limit exists

$$\ell = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}$$

say $\{a_n\}_{n=0}^{\infty}$ converge to a faster than $\{b_n\}_{n=0}^{\infty}$ converge to b if $\ell = 0$ and say $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ have the same rate if $0 < \ell < 1$.

Let the two f-point iterative schemes $\{r_n\}_{n=0}^{\infty}$, $\{s_n\}_{n=0}^{\infty}$ converge to the same f-point x of the operator V, the error estimator

$$\|r_n - x\| \le a_n, \|s_n - x\| \le b_n \ \forall n \ge 0$$

where the sequence of the real number $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ both converge to zero. If $\{a_n\}_{n=0}^{\infty}$ converge faster than $\{b_n\}_{n=0}^{\infty}$, then $\{r_n\}_{n=0}^{\infty}$ converge faster to x than $\{s_n\}_{n=0}^{\infty}$.

2. Main Results

This section defines a generalized quasi-like contractive operator and three new iterative from a three-step type. On the other hand, we state and prove the convergence, stability and rate of convergence between the last iterative with a given operator.

Definition 2.1 The operator $V: S \to S$ namely generalized quasi-like contractive when there exists a function $\emptyset: R^+ \to R^+$ with $\emptyset(0) = 0$ and the constant $\xi \in [0,1), \beta, \gamma \ge 0$, then for each $r, s \in S$

$$d(Vs, Vr) \le \xi d(s,r) + \beta\emptyset\big(d(s,Vs)\big) + \gamma \min\{d(s,Vs), d(r,Vr)\} \tag{1}$$

If we put in V, $\xi \in [0,1), \beta = 1$ and $\gamma = 0$, then we have the contractive-like mapping.

Now, we define three new iterative schemes from a three-step type

I. The multi_ implicit Noor iteration (MINI) define as follows :

$$r_n = a_{n0} g r_{n-1} + \sum_{i=1}^{k} a_{ni} V^i s_n, \ s_n = b_{n0} t_n + (1 - b_{n0}) V^0 t_n$$
$$t_n = c_{n0} \mathbb{P}_{0S} r_n + \sum_{i=1}^{k} c_{ni} \mathbb{P}_{iS} V^i r_n \ ; n \ge 0 \tag{2}$$

If $k = 1$ then we have an implicit Noor iteration

Where V^i are generalized quasi-like contractive mapping such that $i = 0,1,2,\dots,k$, and $\mathbb{P}_{0s}, \mathbb{P}_{1s}, \mathbb{P}_{2s}, \dots, \mathbb{P}_{ks}$ are projective metric mapping and g nonexpansive mapping, $\langle a_n\rangle$, $\langle b_n\rangle$ and $\langle c_n\rangle$ are real sequences in [0,1].

II. The multi_explicit Noor iteration (MENI) define as follows :

$$r_n = \sum_{i=1}^{k} a_{ni} \mathbb{P}_{iS} r_{n-1} + a_{n0} V^0 s_n, \ s_n = (1 - b_{n0}) \mathbb{P}_{0S} g r_{n-1} + b_{n0} V^0 t_n$$
$$t_n = \sum_{i=1}^{k} c_{ni} \mathbb{P}_{iS} g r_{n-1} + c_{n0} \mathbb{P}_{0S} V^0 r_{n-1} \ ; n \ge 0 \tag{3}$$

If $k = 1$ then we have explicit Noor iteration

Where V^i are generalized quasi-like contractive mapping such that $i = 0,1,2,\dots,k$, and $\mathbb{P}_{0s}, \mathbb{P}_{1s}, \mathbb{P}_{2s}, \dots, \mathbb{P}_{ks}$ are projective metric mapping and g nonexpansive mapping, $\langle a_n\rangle$, $\langle b_n\rangle$ and $\langle c_n\rangle$ are real sequences in [0,1]

III. The multi_Picard S-iteration (MPSI) define as follows:

$$r_n = \mathbb{P}_{0S} s_n, \ s_n = a_{n0} g r_{n-1} + \sum_{i=1}^{k} a_{ni} \mathbb{P}_{0S} V^i t_n$$
$$t_n = \sum_{i=1}^{k} b_{ni} \mathbb{P}_{iS} r_{n-1} + b_{n0} V^0 r_{n-1} \ ; n \ge 0 \tag{4}$$

Where V^i are generalized quasi like contractive mapping such that $i = 0,1,2,\dots,k$, and $\mathbb{P}_{0s}, \mathbb{P}_{1s}, \mathbb{P}_{2s}, \dots, \mathbb{P}_{ks}$ are projective metric and g nonexpansive mapping, $\langle a_n\rangle$ and $\langle b_n\rangle$ are real sequences in [0,1].

Now, we prove the following.

Theorem 2.2: *Let V^i be a finite generalized quasi like contractive mappings, all $i = 0,1,2,\dots,k$ and g nonexpansive mapping, If $x \in CF(V^i, \mathbb{P}_{iS}, g)$. Then, for $r_0 \in S$, the MINI $\{r_n\}_{n=0}^{\infty}$ defined by (2.2) with $\sum_{j0=1}^{\infty}(1 - a_{j0}) = \infty$, converges to the f-point x of V^i*

Proof: Since $x \in C\mathrm{F}(V^i, \mathbb{P}_{iS}, g)$, then from $\{r_\mathrm{n}\}_{\mathrm{n}=0}^{\infty}$MINI

$$d(r_n, x) = d(\mathcal{X}(gr_n, V^i s_n, a_{n0}), x)$$
$$\leq a_{n0} d(gr_{n-1}, x) + \textstyle\sum_{i=1}^{k} a_{ni}\, d(V^i s_n, x)$$
$$\leq a_{n0} d(gr_{n-1}, x) + \textstyle\sum_{i=1}^{k} a_{ni} \begin{bmatrix} \xi_i d(s_n, x) + \beta_i \emptyset_i \left(d(V^i x, x)\right) \\ +\gamma_i min\{d(s_n, V^i s_n), d(V^i x, x)\} \end{bmatrix}$$
$$\leq a_{n0} d(r_{n-1}, x) + \textstyle\sum_{i=1}^{k} a_{ni} \xi_i d(s_n, x)] \ (5)$$

$$d(s_n, x) = d(\mathcal{X}(t_n, V^0 t_n, b_{n0}), x)$$
$$\leq b_{n0} d(t_n, x) + (1 - b_{n0}) d(V^0 t_n, x)$$
$$\leq b_{n0} d(t_n, x) + (1 - b_{n0}) \begin{bmatrix} \xi_0 d(t_n, x) + \beta_0 \emptyset_0\left(d(V^0 x, x)\right) \\ +\gamma_0 min\{d(t_n, V^0 t_n), d(V^0 x, x)\} \end{bmatrix}$$
$$\leq (b_{n0} + (1 - b_{n0})\xi_0)(t_n, x) \ (6)$$

$$d(t_n, x) = d\ (\mathcal{X}(\mathbb{P}_{0S} r_n, \mathbb{P}_{iS} V^i r_n, c_{n0}), x)$$
$$\leq c_{n0} d(\mathbb{P}_{0S} r_n, x) + \textstyle\sum_{i=1}^{k} c_{ni}\, d(\mathbb{P}_{iS} V^i r_n, x)$$
$$\leq c_{n0} d(\mathbb{P}_{0S} r_n, x) + \textstyle\sum_{i=1}^{k} c_{ni} \begin{bmatrix} \xi_i d(r_n, x) + \beta_i \emptyset_i \left(d(V^i x, x)\right) \\ +\gamma_i min\{d(r_n, V^i r_n), d(V^i x, x)\} \end{bmatrix}$$
$$\leq (c_{n0} + \textstyle\sum_{i=1}^{k} c_{ni} \xi_i)\, d(r_n, x) \ (7)$$

Take $\xi = max\{\xi_i, i = 0,1,2, \dots k\}$, from (5),(6) and (7) we have

$$d(r_n, x) \leq$$
$$a_{n0} d(r_{n-1}, x) + \left(\xi \textstyle\sum_{i=1}^{k} a_{ni} \begin{pmatrix} b_{n0} \\ +(1 - b_{n0})\xi \end{pmatrix}\right) \begin{pmatrix} c_{n0} \\ +\xi \sum_{i=1}^{k} c_{ni} \end{pmatrix}\Bigg) d(r_n, x)$$
$$d(r_n, x)(1 - \xi \textstyle\sum_{i=1}^{k} a_{ni}(b_{n0} + (1 - b_{n0})\xi)(c_{n0} + \xi \sum_{i=1}^{k} c_{ni})) \leq a_{n0} d(r_{n-1}, x)$$
$$d(r_n, x) \leq \frac{a_{n0} d(r_{n-1}, x)}{1 - \xi \sum_{i=1}^{k} a_{ni}(b_{n0} + (1 - b_{n0})\xi)(c_{n0} + \xi \sum_{i=1}^{k} c_{ni})}$$

Let $\frac{R_n}{S_n} = \frac{a_{n0}}{1 - \xi \sum_{i=1}^{k} a_{ni}(b_{n0} + (1 - b_{n0})\xi)(c_{n0} + \xi \sum_{i=1}^{k} c_{ni})}$

$$\frac{R_n}{S_n} \leq a_{n0} + \xi \textstyle\sum_{i=1}^{k} a_{ni}(b_{n0} + (1 - b_{n0})\xi)(c_{n0} + \xi \sum_{i=1}^{k} c_{ni})$$
$$d(r_n, x) \leq$$
$$[a_{n0} + \xi \textstyle\sum_{i=1}^{k} a_{ni}(b_{n0} + (1 - b_{n0})\xi)(c_{n0} + \xi \sum_{i=1}^{k} c_{ni})] d(r_{n-1}, x)$$
$$\leq \left(a_{n0} + \xi \begin{pmatrix} 1 \\ -a_{n0} \end{pmatrix} \begin{bmatrix} 1 - \\ (1 - \xi)(1 - b_{n0}) \end{bmatrix} \begin{bmatrix} 1 - \\ (1 - \xi)(1 - c_{n0}) \end{bmatrix}\right) d(r_{n-1}, x)$$
$$\leq \left(1 - (1 - a_{n0})(1 - \xi)\right) d(r_{n-1}, x)$$
$$\vdots$$
$$d(r_n, x) \leq \textstyle\prod_{j=1}^{n}\left(1 - (1 - a_{j0})(1 - \xi)\right) d(r_0, x)$$

Take limit for two sides; we have $\lim_{n\to\infty} d(r_n, x) = 0$ then the MINI convergent to f-point x.

Theorem 2.3: *Let V^i be a finite generalized quasi like contractive mappings and g nonexpansive mapping, If $x \in CF(V^i, \mathbb{P}_{iS}, g)$. Then, for $r_0 \in S$, the MENI and MPSI converge to the f-point of V^i.*

Proof: We get the desired result by following the same proof structure in the theorem(2.2).

Theorem 2.4: *Let V^i be a finite generalized quasi-like contractive mappings, for all $i = 0,1,2, \dots, k$ and g nonexpansive mapping, If $x \in CF(V^i, \mathbb{P}_{iS}, g)$. Then, for $r_0 \in S$, the MINI defined by (2.2), is $V^i -$ stable.*

Proof: Let $\{w_n\}_{n=0}^{\infty} \subset S$ is an arbitrary sequence

$$\epsilon_n = d\left(w_n, \mathcal{X}(gw_{n-1}, V^i e_n, a_{n0})\right)$$

where $e_n = \mathcal{X}(f_n, V^0 f_n, b_{n0})$ and $f_n = \mathcal{X}(\mathbb{P}_{0S} w_n, \mathbb{P}_{iS} V^i w_n, c_{n0})$

$$d(w_n, x) \le d(w_n, \mathcal{X}(gw_{n-1}, V^i e_n, a_{n0}) + d(\mathcal{X}(gw_{n-1}, V^i e_n, a_{n0}), x)$$

$$\le \epsilon_n + a_{n0} d(gw_{n-1}, x) + \sum_{i=1}^{k} a_{ni} d(V^i e_n, x)$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni} \begin{bmatrix} \xi_i d(e_n, x) + \beta_i \emptyset_i \left(d(V^i x, x)\right) \\ +\gamma_i min\{d(e_n, V^i e_n), d(V^i x, x)\} \end{bmatrix}$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\, \xi_i \begin{bmatrix} b_{n0} d(f_n, x) + \\ (1 - b_{n0}) \begin{bmatrix} \xi_0 d(f_n, x) + \beta_0 \emptyset_0 \left(d(V^0 x, x)\right) \\ +\gamma_0 min\{d(f_n, V^0 f_n), d(V^0 x, x)\} \end{bmatrix} \end{bmatrix}$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni} \xi_i (b_{n0} + (1 - b_{n0}) \xi_0)(f_n, x)$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni} \xi_i (b_{n0} + (1 - b_{n0}) \xi_0)(c_{n0} d(\mathbb{P}_{0S} w_n, x) + \sum_{i=1}^{k} c_{ni} d(\mathbb{P}_{iS} V^i w_n, x))$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\, \xi_i \begin{pmatrix} b_{n0} + \\ (1 - b_{n0}) \xi_0 \end{pmatrix} \left(\begin{matrix} c_{n0} d(w_n, x) \\ + \sum_{i=1}^{k} c_{ni} \begin{bmatrix} \xi_i d(w_n, x) + \beta_i \emptyset_i \left(d(V^i x, x)\right) \\ +\gamma_i min\{d(w_n, V^i w_n), d(V^i x, x)\} \end{bmatrix} \end{matrix} \right)$$

$$\le \epsilon_n + a_{n0} d(w_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\, \xi_i \begin{pmatrix} b_{n0} \\ +(1 - b_{n0}) \xi_0 \end{pmatrix} \begin{pmatrix} c_{n0} d(w_n, x) \\ + \sum_{i=1}^{k} c_{ni} \xi_i d(w_n, x) \end{pmatrix}$$

Take $\xi = \max\{ \xi_i, i = 0,1,2, \dots k\}$,

$$d(w_n,x) \le$$
$$\epsilon_n + a_{n0} d(w_{n-1},x) + \sum_{i=1}^{k} a_{ni}\,\xi \begin{pmatrix} b_{n0} \\ +(1-b_{n0})\xi \end{pmatrix} \begin{pmatrix} c_{n0} \\ +\sum_{i=1}^{k} c_{ni}\xi \end{pmatrix} d(w_n,x)$$
$$\left[1 - \sum_{i=1}^{k} a_{ni}\,\xi \begin{pmatrix} b_{n0} \\ +(1-b_{n0})\xi \end{pmatrix} \begin{pmatrix} c_{n0} \\ +\sum_{i=1}^{k} c_{ni}\xi \end{pmatrix}\right] d(w_n,x)$$
$$\le \epsilon_n + a_{n0} d(w_{n-1},x)$$
$$d(w_n,x) \le \frac{\epsilon_n}{[1-\sum_{i=1}^{k} a_{ni}\xi(b_{n0}+(1-b_{n0})\xi)\,(c_{n0}+\sum_{i=1}^{k} c_{ni}\xi)]} + \frac{a_{n0} d(w_{n-1},x)}{[1-\sum_{i=1}^{k} a_{ni}\xi(b_{n0}+(1-b_{n0})\xi)\,(c_{n0}+\sum_{i=1}^{k} c_{ni}\xi)]}$$
$$d(w_n,x) = \frac{\epsilon_n a_{n0}}{[1-\sum_{i=1}^{k} a_{ni}\xi(b_{n0}+(1-b_{n0})\xi)\,(c_{n0}+\sum_{i=1}^{k} c_{ni}\xi)]a_{n0}} + \frac{a_{n0} d(w_{n-1},x)}{[1-\sum_{i=1}^{k} a_{ni}\xi(b_{n0}+(1-b_{n0})\xi)\,(c_{n0}+\sum_{i=1}^{k} c_{ni}\xi)]}$$

But, $\frac{a_{n0}}{[1-\sum_{i=1}^{k} a_{ni}\xi(b_{n0}+(1-b_{n0})\xi)\,(c_{n0}+\sum_{i=1}^{k} c_{ni}\xi)]} \le a_{n0} + \xi(1-a_{n0})(1-(1-\xi)(1-b_{n0}))(1-(1-\xi)(1-c_{n0}))$

$$d(w_n,x) \le \frac{\epsilon_n}{a_{n0}} \begin{bmatrix} a_{n0} \\ +\xi(1-a_{n0})\big(1-(1-\xi)(1-b_{n0})\big)\big(1-(1-\xi)(1-c_{n0})\big) \end{bmatrix}$$

$$+ \begin{bmatrix} a_{n0} \\ +\xi(1-a_{n0})(1-(1-\xi)(1-b_{n0}))(1-(1-\xi)(1-c_{n0})) \end{bmatrix} d(w_{n-1},x)$$

By lemma (1.9) $\lim_{n\to\infty} d(w_{n-1},x) = 0$ then $\lim_{n\to\infty} d(w_n,x) = 0$.

Conversely, if $\lim_{n\to\infty} w_n = x$, then by applying the contractive condition (1), it's easy to obtain that $\lim_{n\to\infty} \epsilon_n = 0$.

Hence, by definition 1.8, the MINI defined by (2.2) is $V^i -$ stable.

Theorem 2.5: *Let V^i be a finite generalized quasi-like contractive mappings, for all $i = 0,1,2,\dots,k$ and g nonexpansive mapping, If $x \in CF(V^i, \mathrm{P}_{iS}, g)$. Then, the MENI and MPSI are V^i –stable*

Proof: By following the same proof structure in the theorem(2.4), we get the desired result

Theorem 2.6: *Let V^i be a finite generalized quasi-like contractive mappings, for all $i = 0,1,\dots,k$ and g nonexpansive mapping, If. $x \in CF(V^i, \mathrm{P}_{iS}, g)$. Then, for $r_0 \in S$, the $\{r_n\}_{n=0}^{\infty}$ with MINI converging to x faster than all of MENI and MPSI*

Proof: Let $x \in C\mathrm{F}(V^i, \mathrm{P}_{iS}, g)$

from $\{r_n\}_{n=0}^{\infty}$with MINI defined by (2.2)

$$\begin{aligned} d(r_n,x) &= d(\mathcal{X}(gr_n,V^i s_n,a_{n0}),x) \\ &\le a_{n0}d(gr_{n-1},x)+\textstyle\sum_{i=1}^k a_{ni}\, d(V^i s_n,x) \\ &\le a_{n0}d(r_{n-1},x)+\textstyle\sum_{i=1}^k a_{ni}\xi_i d(s_n,x)] \end{aligned} \tag{8}$$

$$\begin{aligned} d(s_n,x) &= d(\mathcal{X}(t_n,V^0 t_n,b_{n0}),x) \\ &\le b_{n0}d(t_n,x)+(1-b_{n0})d(V^0 t_n,x) \\ &\le (b_n+(1-b_n)\xi_0)(t_n,x) \end{aligned} \tag{9}$$

$$\begin{aligned} d(t_n,x) &= d\ (\mathcal{X}(\mathbb{P}_{0S}r_n,\mathbb{P}_{iS}V^i t_n,c_{n0}),x) \\ &\le c_{n0}d(\mathbb{P}_{0S}r_n,x)+\textstyle\sum_{i=1}^k c_{ni}\, d(\mathbb{P}_{iS}V^i r_n,x) \\ &\le (c_{n0}+\textstyle\sum_{i=1}^k c_{ni}\xi_i)\, d(r_n,x) \end{aligned} \tag{10}$$

Take $\xi = max\{\xi_i, i=0,1,2,\dots k\}$, From (8),(9) and (10) we have

$$d(r_n,x) \le\ a_{n0}d(r_{n-1},x)+\xi\textstyle\sum_{i=1}^k a_{ni}\begin{pmatrix} b_{n0} \\ +(1-b_{n0})\xi\end{pmatrix}\begin{pmatrix} c_{n0} \\ +\xi\sum_{i=1}^k c_{ni}\end{pmatrix} d(r_n,x)$$

$$(1-\xi\textstyle\sum_{i=1}^k a_{ni}(b_{n0}+(1-b_{n0})\xi)(c_{n0}+\xi\sum_{i=1}^k c_{ni}))d(r_n,x) \le a_{n0}d(r_{n-1},x)$$

$$d(r_n,x) \le \frac{a_{n0}d(r_{n-1},x)}{1-\xi\sum_{i=1}^k a_{ni}(b_{n0}+(1-b_{n0})\xi)(c_{n0}+\xi\sum_{i=1}^k c_{ni})}$$

Let, $\frac{R_n}{S_n} = \frac{a_{n0}d(r_{n-1},x)}{1-\xi\sum_{i=1}^k a_{ni}(b_{n0}+(1-b_{n0})\xi)(c_{n0}+\xi\sum_{i=1}^k c_{ni})}$

$$\frac{R_n}{S_n} \le a_{n0}+\xi\textstyle\sum_{i=1}^k a_{ni}(b_{n0}+(1-b_{n0})\xi)(c_{n0}+\xi\sum_{i=1}^k c_{ni})$$

$$d(r_n,x) \le [a_{n0}+\xi\textstyle\sum_{i=1}^k a_{ni}(b_{n0}+(1-b_{n0})\xi)(c_{n0}+\xi\sum_{i=1}^k c_{ni})]d(r_{n-1},x)$$

$$\le \left(1-\begin{pmatrix}1\\-\alpha_{n0}\end{pmatrix}\left(1-\xi\begin{bmatrix}1-\\(1-\xi)(1-b_{n0})\end{bmatrix}\begin{bmatrix}1-\\(1-\xi)(1-c_{n0})\end{bmatrix}\right)\right)d(r_{n-1},x)$$

$$\vdots$$

$$d(r_n,x) \le \prod_{j=1}^n\left(1-\begin{pmatrix}1\\-\alpha_{j0}\end{pmatrix}\left(1-\xi\begin{bmatrix}1-\\(1-\xi)(1-b_{j0})\end{bmatrix}\begin{bmatrix}1-\\(1-\xi)(1-c_{j0})\end{bmatrix}\right)\right)d(r_0,x) \tag{11}$$

Now, consider the MENI define by (2.3)

$$\begin{aligned} d(r_n,x) &= d\big(\mathcal{X}\big(r_{n-1},V^0 s_n,\textstyle\sum_{i=1}^k a_{ni}\ \big),x\big) \\ &\le \textstyle\sum_{i=1}^k \mathbb{P}_{iS}\, a_{ni}\ \ d(r_{n-1},x)+\ \ a_{n0}\, d(V^0 s_n,x) \\ &\le \textstyle\sum_{i=1}^k a_{ni}\ \ d(r_{n-1},x)+a_{n0}\xi_0\ d(s_n,x) \end{aligned} \tag{12}$$

$$\begin{aligned} d(s_n,x) &=\ \ d(\mathcal{X}(\mathbb{P}_{0S}\, gr_{n-1},V^0 t_n,b_{n0}),x) \\ &\le (1-b_{n0})d(\mathbb{P}_{0S}\, gr_{n-1},x)+b_{n0}d(V^0 t_n,x) \\ &\le (1-b_{n0})d(r_{n-1},x)+\xi_0 b_{n0}d(t_n,x) \end{aligned} \tag{13}$$

$$\begin{aligned} d(t_n,x) &= d\big(\mathcal{X}\big(\mathbb{P}_{iS}\, gr_{n-1},\mathbb{P}_{0S}V^0 r_n,\textstyle\sum_{i=1}^k c_{ni}\big),x\big) \\ &\le \textstyle\sum_{i=1}^k c_{ni}d(\,\mathbb{P}_{iS}\, gr_{n-1},x)+c_{n0}\ d(\mathbb{P}_{0S}V^0 r_{n-1},x) \\ &\le \textstyle\sum_{i=1}^k c_{ni}d(\,r_{n-1},x)+c_{n0}\xi_0 d(r_{n-1},x) \end{aligned} \tag{14}$$

Put $\xi = \xi_0$, from (12),(13) and (14), we have

$$d(r_n, x) \leq \sum_{i=1}^{k} a_{ni}\ d(r_{n-1}, x) + a_{n0}\xi \begin{bmatrix} (1-b_{n0})d(r_{n-1}, x) \\ +\xi b_{n0}\left(\sum_{i=1}^{k}(c_{ni} + c_{n0}\xi)\ d(r_{n-1}, x)\right) \end{bmatrix}$$

$$\leq (1 - a_{n0}(1-\xi)\ d(r_{n-1}, x)$$

$$\vdots$$

$$d(r_n, x) \leq \prod_{j=1}^{n}(1 - a_{j0}(1-\xi))\ d(r_0, x) \tag{15}$$

Now we will compare the rate of convergence between $\{r_n\}_{n=0}^{\infty}$MINI (2.11) and $\{r_n\}_{n=0}^{\infty}$MENI (2.12)

$$\lim_{n\to\infty} \frac{\{r_n\}_{n=0}^{\infty}\ MINI}{\{r_n\}_{n=0}^{\infty}\ MENI}$$

$$= \lim_{n\to\infty} \frac{\prod_{j=1}^{n}\left(1-\left(-\overset{1}{\alpha}_{j0}\right)\left(1-\xi\left[\overset{1-}{(1-\xi)(1-b_{j0})}\right]\left[\overset{1-}{(1-\xi)(1-c_{j0})}\right]\right)\right)d(r_0,x)}{\prod_{j=1}^{n}(1-a_{j0}(1-\xi))\ d(r_0,x)} = 0 \tag{16}$$

By using definition (1.10), we have MINI faster than MENI when $1 - a_{j0} \geq a_{j0}$

Now, consider the MPSI define by (2.4)

$$d(r_n, x) = d(\mathbb{P}_{0S}s_n, x)\ \leq d(s_n, x) = d(\mathcal{X}(g\ r_{n-1}, \mathbb{P}_{0S}V^i t_n, a_{n0}\ ), x)$$

$$\leq a_{n0}d(gr_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\ d(\mathbb{P}_{0S}V^i t_n\ , x)$$

$$\leq a_{n0}d(r_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\xi_i\ d(t_n, x) \tag{17}$$

$$d(t_n, x) = d(\mathcal{X}(\mathbb{P}_{iS}r_{n-1}, V^0 r_{n-1}, b_{ni}), x)$$

$$\leq \sum_{i=1}^{k} b_{ni}\ d(\mathbb{P}_{iS}r_{n-1}, x) + b_{n0}d(V^0 r_{n-1}, x)$$

$$\leq \sum_{i=1}^{k} b_{ni}\ d(r_{n-1}, x) + b_{n0}\xi_0 d(r_{n-1}, x)\ \leq (1 - b_{n0}(1-\xi_0)d(r_{n-1}, x) \tag{18}$$

From (17) and (18), we have

$$d(r_n, x) \leq \left[a_{n0}d(r_{n-1}, x) + \sum_{i=1}^{k} a_{ni}\ \xi((1 - b_{n0}(1-\xi_0)d(r_{n-1}, x))\right]$$

$$\leq [a_{n0} + (1-a_{n0})\xi]d(r_{n-1}, x)$$

$$\leq [1 - (1-a_{n0})(1-\xi)]d(r_{n-1}, x)$$

$$\vdots$$

$$\leq \prod_{j=1}^{n}\left[1 - (1-a_{j0})(1-\xi)\right]d(r_0, x) \tag{19}$$

Now we will compare the rate of convergence between $\{r_n\}_{n=0}^{\infty}$MINI (2.11) and $\{r_n\}_{n=0}^{\infty}$ MPSI (2.16)

$$\lim_{n\to\infty} \frac{\{r_n\}_{n=0}^{\infty}\text{MINI}}{\{r_n\}_{n=0}^{\infty}\ \text{MPSI}}$$

$$\lim_{n\to\infty} \frac{\prod_{j=1}^{n}\left(1-(1-\alpha_{j0})\left(1-\xi\left[\overset{1-}{(1-\xi)(1-b_{j0})}\right]\left[\overset{1-}{(1-\xi)(1-c_{j0})}\right]\right)\right)d(r_0,x)}{\prod_{j=1}^{n}[1-(1-a_{j0})(1-\xi)]d(r_0,x)} = 0 \tag{20}$$

By using definition (1.10), we have MINI faster than MPSI. From the last proof, it is clear that MINI is faster than MENI and MPSI.

Remark 2.7: Since the generalized quasi like contractive (2.1) is more general than contractive-like mapping, then the converge rate result of MINI defined by

(2.2), MENI defined by(2.3) and MPSI defined by(2.4) with the contractive- like mapping can be easily accessible as special cases.

3. Conclusion

1- A new iterative scheme MINI, MENI and MPSI are convergent to the f-point of a generalized quasi like contractive operator. We got to this through theorems 2.2 and 2.3.
2- MINI, MENI and MPSI are $V^i - stable$ and proved this by theorem 2.4 and 2.5.
3- MINI is faster than MENI and MPSI concerning generalized quasi-like contractive operators and proved this by theorem 2.6.

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