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Julia sets are quasicircles for rational maps of degree N

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Abstract

This research aims to study complex dynamics, where the complex dynamic is considered a branch of dynamical systems, which takes maps from the Riemann sphere to the Riemann sphere. The Julia sets are also examples of fractals and the most important of them, thus being a fractal set. This work aims to study the Julia sets of a family of rational maps are quasicircles and offer the boundaries of the immediate basin of attraction of zero and the infinity are quasi-circles. Now, we introduce the family of maps are defined as:

$$G_b(z) = 2b^{n+1} z^{-n} + z^{2n} - b^{3n+1}/z^n (z^{2n} - b^{n-1}), \tag{1}$$

where $n \geq 2$ and $b \in \mathbb{C}\backslash\{0\}$ also $b^{n+1} \neq 1$ and $b^{2n+2} \neq 1$.

Subject Classification: *08C15, 20E10.*

Keywords: *Julia sets, Fatou set, Fatou components, Quasicircles, Perturbed.*

Introduction

In 1980 the first to use the concept of the singular perturbation on z^m is McMullen, he introduces the family of rational maps $F_\alpha(z) = z^2 + \frac{\alpha}{z^3}$, where $\alpha \in \mathbb{C}\backslash\{0\}$ see [8]. In [4], the authors have been studied the McMullen map F_α from the iterations of the free critical points. In [5], [6], [9], several people gave a generalization of McMullen maps. Fei and Jianxun [9] gave the maps

$$h_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}}, \text{ where } n \geq 2 \text{ and } \lambda \in \mathbb{C}\backslash\{0\}, \text{ such that } \lambda^{2n-2} \neq 1.$$

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They prove that quasicircles for Julia sets or cantor circles for Julia sets or Sierpinski carpet for Julia set by the orbits of the critical points, view [1],[7],[10-13]. Al-Salami and Al Sharaa`a studied the following maps

$Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$, where $d \geq 2$ and $\beta \in \mathbb{C}\backslash\{0\} - \beta^{1-d} = 1$ and $\beta^{2d-2} = 1$ and they get the Julia set is quasicircles see [2]. The paper is organized as follows. In section 2, we give the symmetric dynamical behaviors for G_b. Section 3 discusses a proof of Theorem A and B. Section 4 draws conclusions and considers future work.

1. Preliminary and the Main Theorem

For any rational map $G: \mathbb{C}_\infty \to \mathbb{C}_\infty$ with $n \geq 2$ and for G^m is the $m-th$ iteration of G for positive integer m. We define the set of points in the family of iterates for G^m fails to be a normal family is the Julia set of G also, the closure of the set of all repelling periodic points is Julia set, denoted by $\mathcal{J}(G)$. The Fatou set $\mathbb{F}(G)$ is $\mathbb{C}_\infty\backslash\mathcal{J}(G)$. The Fatou component is the connected component of the Fatou set. Now we study

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})} ,$$

where $n \geq 2$ and $b \in \mathbb{C}\backslash\{0\}$, but $b^{n+1} \neq 1$ and $b^{2n+2} \neq 1$. If $b^{n+1} = 1$ and $b^{2n+2} = 1$, then $G_b(z) = P_n(z) = 3z^{-n}$, or if $b = 0$, then $G_b(z) = P_n(z) = z^{-n}$, thus G_b has a singular perturbed at the polynomial P_n. The l_0 , l_∞ is the immediate basin of attraction of the origin and the infinity, respectively. The degree of G_b is $3n$. Also, there is $6n-2$ critical points for G_b (counted with multiplicity). We introduce the main Theorems:

Theorem A: *If G_b has the iterate of the free critical point c_b attracted by $the\ infinity$ (respect to 0) and $c_b \in l_\infty$ (respect to l_0), then $\mathcal{J}(G_b)$ is a quasicircle.*

Theorem B: *Assume that G_b has one of the orbits of the free critical point is attractive to either $0\ or\ \infty$, then the boundary of the immediate basin of attraction of $\mathrm{the\ origin}$ and The boundary of the immediate basin of attraction of $the\ infinity$ are quasicircles .*

2. Preliminaries

We will need the symmetric dynamical behaviors for G_b.

Lemma 2.1: *Let $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that*

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})},$$

let $\omega \in \mathbb{C}$ satisfy $\omega^{2n} = 1$. Then $G_b^k(\omega z) = \omega^{n^k}G_b^k(z)$ for some $k \geq 1$.

Proof: A direct calculation shows that by use $\omega^{2n} = 1$, we have

$$G_b(\omega z) = \omega^{-n}\left(2b^{n+1}z^{-n} + \frac{z^{2n} - b^{3n+1}}{z^n(z^{2n} - b^{n-1})}\right) = \omega^{-n}G_b(z).$$

By induction, assume that $G_b^k(\omega z) = \omega^{-n^k}G_b^k(z)$ for some $k \geq 1$ to prove $G_b^{k+1}(\omega z) = G_b\left(G_b^k(\omega z)\right) = \omega^{-n^{k+1}}G_b^{k+1}$.

Lemma 2.2: *Let* $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *be a map such that*

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})} \quad .$$

For any $\mathbb{t}(z) = \frac{b^2}{z}$*, then* G_b *holds the conjugate*

$$G_b \circ \mathbb{t}(z) = \mathbb{t} \circ G_b(z)\text{ , for any } z \in \mathbb{C}_\infty.$$

Proof: Clearly that $\mathbb{t}(z) = \frac{b^2}{z} = \mathbb{t}^{-1}(z)$.

So $G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})} = \frac{2b^{n+1}z^{2n}-2b^{2n}+z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})}$

If $b^2.\frac{1}{G_b(z)} = b^2.\frac{z^n(z^{2n}-b^{n-1})}{2b^{n+1}z^{2n}-2b^{2n}+z^{2n}-b^{3n+1}}$, then

$$\mathbb{t} \circ G_b \circ \mathbb{t}(z) = b^2.\frac{\left(\frac{b^2}{z}\right)^n(\left(\frac{b^2}{z}\right)^{2n}-b^{n-1})}{2b^{n+1}\left(\frac{b^2}{z}\right)^{2n}-2b^{2n}+\left(\frac{b^2}{z}\right)^{2n}-b^{3n+1}} = G_b(z)$$

Corollary 2.3: *Let* $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *be a map such that*

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})}.$$

Assume that $\mathbb{U}$ is a Fatou component of G_b, then $\mathbb{U} = \mathbb{t}(\mathbb{U})$. In particular, $l_\infty = \mathbb{t}(l_0)$ and $l_0 = \mathbb{t}(l_\infty)$.

Proof: Assume that $\mathbb{U}$ is a Fatou component of G_b, from Lemma 2.2, thus $\mathbb{t}(\mathbb{U}) = \mathbb{U}$, and $\mathbb{U} = \mathbb{t}(\mathbb{U}) = \mathbb{t}^{-1}(\mathbb{U})$, it follows $\mathbb{U}$ is completely invariant. There are only two components l_∞ and l_0, so from Lemma 2.2,

$$\mathbb{t}(z) = \frac{b^2}{z}\text{ , then } l_0 = \mathbb{t}(l_\infty) \text{ and } l_\infty = \mathbb{t}(l_0)\ .$$

Remark 2.4: Let $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})} \quad \text{, we have}$$

$$G_b{}'(z) = -nz^{-n-1}[\frac{-(3b^{3n+1}+4b^{2n}-b^{n-1})z^{2n}+z^{4n}(1+2b^{n+1})+b^{4n}+2b^{3n-1}}{(z^{2n}-b^{n-1})^2}]. \qquad (2)$$

If $G_b{}'(z) = 0$. Either $-(3b^{3n+1} + 4b^{2n} - b^{n-1})z^{2n} + z^{4n}(1 + 2b^{n+1}) + b^{4n} + 2b^{3n-1} = 0$

${c_b}^{2n} = \frac{(3b^{3n+1}+4b^{2n}-b^{n-1})\pm\sqrt{9b^{6n+2}+16b^{5n+1}-10b^{4n}-16b^{3n-1}+b^{2n-2}}}{2+4b^{n+1}}$, or $z = 0$ with multiplicity $n-1$, by lemma 2.2, ∞ is a critical point of G_b with multiplicity $n-1$, we have $C_b(G_b) = \{0, \infty, c_b\}$, where c_b is the free critical point. From Lemma 2.1 and 2.2, the set of critical points form

$$C_b(G_b) = \{\omega_0^k c_b, \omega_0^k \frac{b^2}{c_b} : 0 \leq k \leq 2n-1\}, \text{ where } \omega_0 = e^{\frac{\pi i}{n}}.$$

For each $U \subset \mathbb{C}_\infty$ also $\alpha \in \mathbb{C}$. We define $\alpha U = \{\alpha z : z \in U\}$.

Lemma 2.5: *If* $z \in l_\infty$ *or* l_0, *then* $\omega z \in l_\infty$ *or* l_0 *respectively, where* ω *satisfies* $\omega^{2n} = 1$. *So both* l_∞ *and* l_0 *have* $2n$ *–fold symmetry.*

Proof: Let $U = \{z \in l_0 : \omega z \in l_0\}$be a subset of l_0 , thus U is a non-empty and open set, l_0 consists of a small neighborhood of 0, and because $U = l_0 \cap \omega l_0 \cap \dots \cap \omega^{2n-1} l_0$. If $U \neq l_0$, suppose that $z_0 \in l_0 \cap \partial U$, that is $z_0 \in \partial U$, we have $z_0 \in l_0$ and $z_0 \notin l_0$. Therefore $\omega z_0 \in \partial l_0$ because $\omega z_0 \in \partial U$ and $U \subset l_0$. Then $G_b^m(z) \to 0$ but $G_b^m(\omega z_0) \nrightarrow 0$ as $m \to \infty$. While $G_b^m(\omega z_0) = \omega^{-n^m} G_b^m(z_0) \to 0$. This is a contradiction. Then $\omega l_0 = l_0$. We can prove that similarly $\omega l_\infty = l_\infty$.

Lemma 2.6: *If* G_b *contain for all Fatou components* V. *If* z_0 , $\omega^m z_0 \in V$, *where* $\omega^m \neq 1$ *with for integer* $m \neq 0 \bmod 2n$. *Then* $\omega^i z_0 \in V$ *for all* i. *In particular,* V *has* $2n-fold\ symmetry$ *around* 0.

For the proof of Lemma 2.6 is simple, one can refer to [lemma 3.6] as a similar proof. We assume z belongs to a Fatou component, while $\omega^i z$ does not lie in this component for some i.

Proposition 2.7: For any map $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ such that

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})}.$$

$\mathcal{J}(G_b)$ has the symmetry $(2n - fold)$.

The proof of proposition 2.7 from lemma 2.6, we have $2n$ Fatou components around the infinity or 0 and have $2n - fold$ symmetry.

Proposition 2.8: If the free critical orbits are attracted to ∞ and **0,** for some real no. b . Then the round circle define as $\mathrm{T}_{|b|} = \{z \in \mathbb{C}: |z| = |b|\}$. However, $\mathrm{T}_{|b|}$ a subset of $\mathcal{J}(G_b)$.

Proof: Let $z = |b|e^{2\pi i\theta}$, where $\theta \in [0,1)$. Then

$$|G_b(z)| = \left|\left(2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})}\right)\right|$$

$$\leq 2|b|^{n+1}|b|^{-n} + \frac{\left||b^{2n}e^{2n\pi i\theta} - b^{3n+1}|\right|}{|b|^n|b^{2n}e^{2n\pi i\theta} - b^{n-1}|}$$

$$\leq -(2|b| - |b|)$$
$$= -(|b|) = |-(|b|)| = |b|.$$

So $|G_b(z)| \leq |b|$. If $\mathrm{T}_{|b|} = \{z \in \mathbb{C}: |z| = |b|\}$, then $G_b(\mathrm{T}_{|b|}) \subset \mathrm{T}_{|b|}$. If b is real, we have $|b| \neq 1$ since $b^{2n+2} \neq 1$, $b^{n+1} \neq 1$, we have two cases:

First case: If $|b| > 1$, we have $|b|^{\frac{n-1}{2n}} < |b| < |b|^{\frac{3n+1}{2n}}$, that is b^{3n+1} is large such that by (1), G_b has no zeros also $3n$ poles in

$D_{|b|} = \{z \in \mathbb{C} : |z| < |b|\}$. So $G_b(\mathrm{T}_{|b|}) = \mathrm{T}_{|b|}$ because $G_b(\mathrm{T}_{|b|})$ around the origin $3n$ times clockwise from the Argument Theorem.

Second case: If $|b| < 1$, then $|b|^{\frac{3n+1}{2n}} < |b| < |b|^{\frac{n-1}{2n}}$. Note that if b^{n-1} is large and b^{3n+1} is small such that by (1), then we have

G_b has $2n$ roots and n poles in $D_{|b|}$. The Argument Theorem satisfy $G_b(\mathrm{T}_{|b|}) = \mathrm{T}_{|b|}$ because $G_b(\mathrm{T}_{|b|})$ surround 0 (n) times anticlockwise. So that $G_b: \mathrm{T}_{|b|} \to \mathrm{T}_{|b|}$ is onto. Assume that $z_0 \in \mathrm{T}_{|b|} \subset \mathbb{F}(G_b)$, then ${G_b}^m(z_0) \to \infty$ or 0 as $m \to \infty$. Moreover, do it another way,

${G_b}^m(z_0) \in G_b(\mathrm{T}_{|b|}) = \mathrm{T}_{|b|}$ $\forall\, m \geq 0$, which is impossible. So that, $\mathrm{T}_{|b|}$ subset of $\mathcal{J}(G_b)$.

For example , if $G_b(-b) = -b$ and n odd. By (2), then

$$G_b{}'(-b) = -nb^{-n-1}\left[\frac{-(3b^{3n+1}+4b^{2n}-b^{n-1})b^{2n}+b^{4n}(1+2b^{n+1})+b^{4n}+2b^{3n-1}}{(b^{2n}-b^{n-1})^2}\right]$$

If $G_b{}'(-b) = 0$, thus either $b = (-3)^{\frac{1}{n+1}}$ or $b = 1$. So G_b has super attracting fixed point at $-b$, since $-b$ does not attract to 0 or ∞. Therefore $\mathrm{T}_{|b|} \not\subset \mathcal{J}(G_b)$.

3. Proof of Theorem A and B

We introduce a necessary and sufficient condition for $\mathcal{J}(G_b)$ such is a quasicircle.

Lemma 3.1 [1]: *The Julia set is a quasicircle if the hyperbolic rational map has exactly two Fatou components.*

Proposition 3.2: $\mathcal{J}(G_b)$ is quasicircle if one of the free critical points c_b belong to l_∞ or l_0.

Proof: Suppose that $c_b \in l_0$ .Then by Lemma 2.2, $\frac{b^2}{c_b} \in l_\infty$, also by Lemma 2.1, $\omega^m c_b \in l_0$ and $\omega^m \frac{b^2}{c_b} \in l_\infty$ for $0 \leq m \leq 2n-1$, we suppose that ${G_b}^{-1}(l_0)$ has the unique component l_∞ also ${G_b}^{-1}(l_\infty)$ has the unique component l_0. If $G_b: l_0 \to l_\infty$ is proper the restriction on l_0 with degree m, then $m \geq n$ because at 0, the local degree of G_b is n. We have elements other than 0 in l_0 for preimages of 0 from lemma 2.1, so that $m \geq 3n$. So $m = 3n$ because $3n$ is the degree of G_b. Thus $m = n$, or $m = 3n$, we show that $m = n$ is impossible. Assume that $c_b \in l_0$ is the free critical points of G_b and

$\{{\omega_0}^i c_b : 0 \le i \le 2n-1\}$ lies in l_0 from Lemma 2.5. From formula in (1), thus ∃ at least 2n preimages for $G_b(c_b)$ *and* $\{{\omega_0}^i c_b : 0 \le i \le 2n-1\}$ (counted with multiplicity) in l_0. Therefore $m \ge 2n$, which is impossible with $m = n$. Hence $m = 3n$ and l_∞ is the unique component in ${G_b}^{-1}(l_0)$, and prove l_0 the unique component in ${G_b}^{-1}(l_\infty)$. Now we claim there are only. l_0 and l_∞ belong to $\mathbb{F}(G_b)$. We have either parabolic basins or super-attracting basins at least contain one of c_b, this is contradicted since any of the critical points belong to l_∞ also l_0. We have either Siegel disks or Herman rings, hence $\mathcal{J}(G_b)$ at least contains one $G_b(c_b)$, so this contradicted Proposition 3.1, so that G_b is hyperbolic. By Lemma (3.1), so $\mathcal{J}(G_b)$ is a quasicircle. See Figure (1).

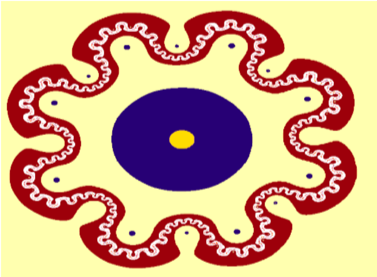

Figure 1
$b = 0.58 + 0.63i$ and $J(G_b)$ is a quasicircle.

Proposition 3.3: Let $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n} - b^{3n+1}}{z^n(z^{2n} - b^{n-1})}.$$

$\mathcal{J}(G_b)$ is a quasicircle if the modulus of b is large enough.

Proof: By Remark (2.4), we have

$${c_b}^{2n} = \frac{(3b^{3n+1} + 4b^{2n} - b^{n-1}) \pm \sqrt{9b^{6n+2} + 16b^{5n+1} - 10b^{4n} - 16b^{3n-1} + b^{2n-2}}}{2 + 4b^{n+1}}.$$

Hence $|c_b| \asymp |b|^{\frac{3n+1}{2n}}$, if $|b|$ is large enough . Since ≥ 2, so

$|b|^{\frac{3n+1}{2n}} \ge |b|^{\frac{7}{4}} > |b|^{\frac{6}{5}}$. We define $W = \{z : |z| > |b|^{\frac{6}{5}}\}$. Suppose that the absolute value of b is large enough and $\in W$, then

$$\begin{aligned} |G_b(z)| &= \left|\left(2b^{n+1}z^{-n} + \frac{z^{2n} - b^{3n+1}}{z^n(z^{2n} - b^{n-1})}\right)\right| \\ &> 2|b|^{n+1}|b|^{\frac{6n}{5}} - \frac{|z|^{\frac{12n}{5}} - |b|^{3n+1}}{|z|^{\frac{6n}{5}}\left(|z|^{\frac{12n}{5}} - |b|^{n-1}\right)} \\ &> 2|b|^{\frac{11n+5}{5}} + \frac{|b|^{\frac{6}{5}}}{5} > 2|b|^{\frac{6}{5}}. \end{aligned}$$

This means that $G_b(W) \subset W$. Hence W is contained in l_∞. Now if c_b holds $|c_b| \approx |b|^{\frac{3n+1}{2n}} > |b|^{\frac{6}{5}}$, then $c_b \in W \subset l_\infty$. From Proposition 3.2, so $\mathcal{J}(G_b)$ is a quasicircle.

Proposition 3.4: Let $G_b : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a map such that $G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n}-b^{3n+1}}{z^n(z^{2n}-b^{n-1})}$. $\mathcal{J}(G_b)$ is a quasicircle for some $b \in \mathbb{R}$ if and only if $|b| > 1$, so $\mathcal{J}(G_b) = \mathrm{T}_{|b|}$.

Proof: If $|b| > 1$, then $G_b(\mathrm{T}_{|b|}) = \mathrm{T}_{|b|}$ also $D_{|b|} = \{z \in \mathbb{C} : |z| < |b|\}$, G_b has no roots and $3n$ poles by Proposition 2.8 (counting with multiplicity). Therefore $G_b(\mathrm{D}_{|b|}) = D_{|b|}$. Hence $D_{|b|} \subset \mathbb{F}(G_b)$. By Lemma 2.2, $\mathbb{C}_\infty \backslash \overline{D}_{|b|} \subset \mathbb{F}(G_b)$. In particular, $D_{|b|} \subset l_0$ also $\mathbb{C}_\infty \backslash \overline{D}_{|b|} \subset l_\infty$. It follows that $\mathcal{J}(G_b) = \mathrm{T}_{|b|}$ because $l_\infty \cap l_0 = \emptyset$. We assume that $\mathcal{J}(G_b)$ is a quasicircle if $0 < |b| < 1$, then $\mathcal{J}(G_b) = \mathrm{T}_{|b|}$ because $G_b(\mathrm{T}_{|b|}) = \mathrm{T}_{|b|}$. Thus, $D_{|b|} = l_0$ and $G_b: D_{|b|} \to \mathbb{C}_\infty \backslash \overline{D}_{|b|}$ has a degree of $3n$ for the covering map. On the other hand, if $0 < |b| < 1$, $D_{|b|}$ contain n poles and $2n$ zeros for G_b, so that a contradiction, whenever $0 < |b| < 1$, so $\mathcal{J}(G_b)$ is not quasicircle.

Proposition 3.5: Each of the preimages of the boundaries of l_∞, l_0 and the $\partial l_\infty, \partial l_0$ are quasicircles surrounding 0.

Proof: For all closed sets $V = \mathbb{C}_\infty(l_\infty \cup l_0)$ amidst l_∞ and l_0 divided into closed sets V_0, V_1, V_2 between l_0 and a_∞, a_∞ and a_0, a_0 and l_∞ view Figure 2. $\forall$ $\mathcal{L} \subset a_0 \subset V$ smooth simple curve surrounds the origin. We claim the preimage of $\mathcal{L}$ in V_2 is smooth simple closed curve surrounding the origin. The critical values do not belong in V. Therefore we have finitely many smooth, simple closed curves in V_2 of preimage of $\mathcal{L}$. For some $\mathcal{L}_2$ is not surrounding 0. Thus in V_2, $\mathcal{L}_2$ can distort to a point in V, so $\mathcal{L} = G_b(\mathcal{L}_2)$ also can distort to a point. So that a contradicted since $\mathcal{L} \subset V$ is surrounding 0. So that the preimage of $\mathcal{L}$ are smooth, simple closed curves in V_2 around the origin and the infinity. Now in V_2, ${G_b}^{-1}(\mathcal{L})$ has two components, so the annular region includes poles or zeros between two simple closed curves, which are contracted. Hence, ${G_b}^{-1}(\mathcal{L}) \cap V_2$ is a smooth, simple closed curve ζ surrounding 0. For any $\Upsilon \subset \mathbb{C}$ be a simple closed curve and assume that the bounded component of $\mathbb{C} \backslash \Upsilon$ is Υ^{int}. So that in $\mathcal{L}^{int}$is the Jordan disk include ζ^{int} is compactly contained. From [3, Theorem p. 296], Thus

$G_b: \zeta^{int} \to \mathcal{L}^{int}$ is quasiconformally equivalent to $q(z) = z^d$. We have $\mathcal{J}(q) = S^1$. Therefore ∂l_0 is a quasicircle. We can similarly that ∂l_∞ is a quasicircle. Since each of the preimages of ∂l_0 and ∂l_∞ are belong on $V_0 \cup V_1 \cup V_2$, Therefore any the preimages of the boundaries of l_∞, l_0 are quasicircles surrounding the origin.

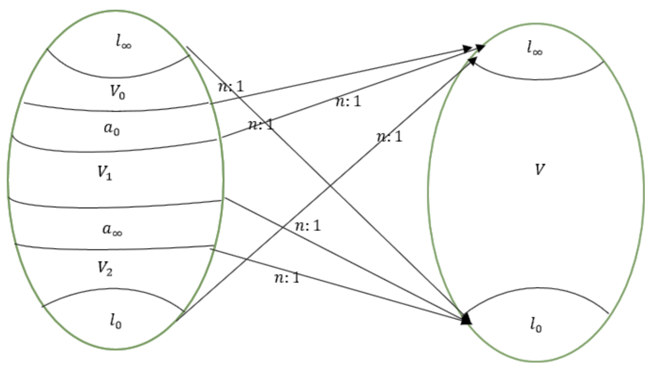

Figure 2
A description of the map G_b on Riemann sphere

4. Discussion and Conclusion

We study the family of maps are defined as:

$$G_b(z) = 2b^{n+1}z^{-n} + \frac{z^{2n} - b^{3n+1}}{z^n(z^{2n} - b^{n-1})}$$, and our get as follows:

- The Julia set of G_b is quasicircle
- The boundaries of l_∞, l_0 are quasi circle.
- The preimage of the boundaries of l_∞, l_0 are quasi circle.

One of the limitations of the current study is the important condition for giving parameter values to satisfying $b \in \mathbb{C}\backslash\{0\} - b^{n+1} = 1$ and $b^{2n+2} = 1$. The Julia set of this map is not only quasicircle, but another thing that will be considered in subsequent studies.

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