

Fuzzy homotopy theory with some result

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Abstract

This paper aims to introduce the concept of the fuzzy homotopy and give some results. It is shown that the fuzzy homotopy is an equivalence relation between two fuzzy functions. In addition, we proved the fuzzy fundamental group, some theorems and properties.

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1. Preliminaries

We recall some definitions, properties, and theorems for any fuzzy set $A \in F(X)$ and any $I = [0, 1]$, defined as the following.

Definition 1.1 [1][8]: A fuzzy set in X is a function with domain X and the codomain is valued in $I = [0, 1]$, which is an element of I^X . The membership is denoted by: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$.

Definition 1.2 [2]: Let (X, τ) be topological space, $A \in \mathcal{F}(X)$ and $\tau^* \in \mathcal{F}(X)$. Let $[\tau^* = \{G: v \rightarrow I \mid v \in \tau\}, \tau^*_\alpha = \{G_\alpha \mid G \in \tau^*, \alpha \in I\}]$. The pair (A, τ^*) is called fuzzy topological space if and only if:

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- a) $\emptyset, X \in \tau^*$.
- b) $G, H \in \tau^*$ Then $G \cap H \in \tau^*$.
- c) If $\{G_i\}_{i \in \Lambda} \subset \tau^*$ Then $\cup_{i \in \Lambda} G_i \in \tau^*$.

Definition 1.3 [2]: Let $\in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ and $C \in \mathcal{F}(Z)$. $R \subset A \times B$, $Q \subset B \times C$ are fuzzy relations from A to B and from B to C respectively and put $Q \circ R = \cup_{t \in I} (Q_t \circ R_t)$. It is called the composition of Q and R .

Definition 1.4 [4]: Let $\in \mathcal{FR}(X)$. Then R is said to be:

- 1) Reflexive if $R(x, x) = 1, \forall x \in X$.
- 2) Symmetric if $R(x, y) = R(y, x), \forall x, y \in X$. i.e. $R = R^{-1}$
- 3) Transitive if $R \circ R \subset R$.

If R satisfies (1,2 and 3), then we say R is a fuzzy equivalence relation on X . We will denote the set of all fuzzy equivalence relations on X by $\mathcal{FER}(X)$.

2. The Concept of Fuzzy Homotopy

Definition 2.1: Let (A, τ_1) and (B, τ_2) be fuzzy topological spaces, and $u, v : A \rightarrow B$ are fuzzy continuous functions. u is a fuzzy homotopy to v if there exists a fuzzy continuous function which is a family of maps $h_t : A \rightarrow B$, such that for all $t \in I$.

$$h_t(x) = u_t(x), h_0(x) = u(x) \text{ and } h_1(x) = v(x)$$

If u and v are fuzzy homotopic functions and we write $u \sim v$.

Theorem 2.2: Fuzzy homotopy relation is a fuzzy equivalence relation.

Proof: Let A, B, C be a fuzzy set, and let: $A \rightarrow B$, $v : B \rightarrow C$ and $k : A \rightarrow C$ be a continuous fuzzy function. Let (A, τ_1) , (B, τ_2) and (C, τ_3) be fuzzy topological spaces. To prove:

- 1) Reflexive: Let $u : A \rightarrow B$ be a fuzzy function, let A, B be two fuzzy sets, then there exist; $h_t : A \rightarrow B, \forall t \in I$, be a continuous fuzzy function such that satisfies the conditions:

$$h_t(x) = u_t(x), \text{ for all } t \in I, h_0(x) = u(x) \text{ and } h_1(x) = u(x)$$

Thus h_t is reflexive.

- 2) Symmetric: Let $u : A \rightarrow B$ and $v : B \rightarrow A$ be a fuzzy function such that there exist $h_t : I^A \rightarrow B$ such that:

$$h_t(x) = u_t(x), \forall t \in I, h_0(x) = u(x) \text{ and } h_1(x) = v(x)$$

And there exist $h'_t : I^B \rightarrow A$ such that:

$$h'_t(x) = u(x, 1-t), h'_0(x) = h'_1(x) = v(x) \text{ and } h'_1(x) = h_0(x) = u(x)$$

Thus h is symmetric.

- 3) Transitive: Let $u : A \rightarrow B$, $v : B \rightarrow C$ and $k : A \rightarrow C$, be fuzzy functions, then there exists a fuzzy continuous function $h_t : A \rightarrow B$, such that:

$$h_t(x) = u_t(x), \forall t \in I, h_0(x) = u(x) \text{ and } h_1(x) = v(x)$$

And $G : B \rightarrow C$, such that:

$$G_t(x) = v_t(x), \forall t \in I, G_0(x) = v(x) \text{ and } G_1(x) = k(x)$$

And let $H_t : I^A \rightarrow C$, such that

$$H_t(x) = \begin{cases} h_{2t}(x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G_{2t-1}(x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (1)$$

Such that the following are holds:

$$H_0(x) = h_0(x) = u(x) \text{ and } H_1(x) = G_1(x) = k(x), \text{ Thus } h \text{ is transitive.}$$

Properties 2.3: If (X, τ) be a fuzzy topological space, let A, B and C be a fuzzy set and let $u, v : A \rightarrow B$ be any two fuzzy continuous maps, then:

1) Let $k : A \rightarrow B$ be a fuzzy continuous map. If $u \sim v$ and $v \sim k$, then $u \sim k$.

Proof: Since $u \sim v$ and $v \sim k$ then there exist two fuzzy continuous maps $h_t, G_t : A \rightarrow B$, for all $t \in I$, such that:

$$h_0(x) = u(x), h_1(x) = v(x) \text{ and } G_0(x) = v(x), G_1(x) = k(x)$$

Define a function $H_t : A \rightarrow B$ by:

$$H_t(x) = \begin{cases} h_{2t}(x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G_{2t-1}(x) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (2)$$

This is a fuzzy continuous map and has a property such that:

$$H_0(x) = h_0(x) = u(x), \text{ While}$$

$$H_1(x) = G_1(x) = k(x), \text{ for all } t \in I, \text{ Thus } \sim k.$$

2) If $u \sim v$ and $k : B \rightarrow C$ Then $K \circ u \sim K \circ v$.

Proof: Let A, B and C be fuzzy sets and let $u, v : A \rightarrow B, k : B \rightarrow C$. To prove $k \circ u \sim k \circ v$. Since $u \sim v$, then there exists a fuzzy continuous map $h_t : I^A \rightarrow B$, such that:

$$h_0(x) = u(x) \text{ and } h_1(x) = v(x), \forall t \in I.$$

Define $G_t : A \rightarrow C$ such that $G_t(x) = k \circ h_t(x)$, be a fuzzy continuous mapping such that:

$$G_0 = k \circ u \text{ and } G_1 = k \circ v, \text{ for all } t \in I, \text{ Then } k \circ u \sim k \circ v$$

Theorem 2.4: Let $A \in h(x), B \in h(y)$ and $C \in h(z)$, be fuzzy topological spaces and $u \in A \times B, v \in B \times C$ be fuzzy continuous functions. If $v \sim k$, then $v \circ u$ and $k \circ u$ are fuzzy continuous and $v \circ u \sim k \circ u$.

Proof: If $u : A \rightarrow B$ and $v, k : B \rightarrow C$ are fuzzy functions, and $v \sim k, v \circ u$ and $k \circ u$ are fuzzy continuous. To prove $v \circ u \sim k \circ u$. Since $v \sim k$, then there exists a continuous fuzzy function $h_t : B \rightarrow C$, such that:

$$h_0(y) = v(y) \text{ and } h_1(y) = k(y), \text{ for each } t \in I$$

Define, $G_t(x) : A \rightarrow C$ be a continuous fuzzy function such that:

$$G_t(x) = h(u(x), t), \forall t \in I, G_0(x) = h(u(x), 0) = v_t \circ u_t \text{ and} \\ G_1(x) = h(u(x), 1) = k_t \circ u_t, \text{ for all } t \in I$$

Therefore:

$$v_t \circ u_t \sim k_t \circ u_t, \text{ for all } t \in I, \text{ Thus: } \circ u \sim k \circ u.$$

3. Fuzzy Fundamental Group

Definition 3.1 [3][6]: Let (X, τ) be a fuzzy topological space and $I = [0, 1]$. A continuous mapping. $f : I \rightarrow X$ is said to be a fuzzy path in X if for any two points $x, y \in X$ such that: $f(0) = x, f(1) = y$.

Definition 3.2 [1][7]: Let (X, τ) be a fuzzy topological space. Let α, β be any two fuzzy paths in X with $(1) = \beta(0)$. Then the composition of the fuzzy path is defined by:

$$(\alpha \circ \beta) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (3)$$

Definition 3.3 [1][5]: Let (X, τ) be a fuzzy topological space. For some $x_0 \in X$, then the fuzzy fundamental group of X based at x_0 is the set of all fuzzy homotopy classes of fuzzy loops in X based at x_0 if the following conditions are satisfied:

1. $[u] \cdot [v] = [uv]$, (Associative)
2. $[u] \cdot [c] = [u]$, and $[u] \cdot [u^{-1}] = [c]$ (so $[u^{-1}] = [u]^{-1}$)

And denoted by $\pi_1(X, x_0)$.

Definition 3.4: Let (X, τ) be fuzzy topological space and $I = [0, 1]$. A continuous fuzzy mapping $u : I^I \rightarrow X$ is said to be a fuzzy homotopy path if for any two points $x, y \in X$ then $(0) = x, u(1) = y$.

Theorem 3.5: A fuzzy homotopy is a group.

Proof:

- 1) Identity: Let A, B be two fuzzy sets, and let $(A, \tau_1), (B, \tau_2)$ be a fuzzy topological space. Let $\alpha_0 : A \rightarrow \{x_0\}, x_0 \in B$ be a constant mapping such that: $e = [\alpha_0]$ is the identity element of $h(x)$ with respect to $(*)$. To prove $[\alpha_0] * [\alpha] = [\alpha_0 * \alpha] = [\alpha], [\alpha] * [\alpha_0] = [\alpha * \alpha_0] = [\alpha], i.e [\alpha] \sim [\alpha * \alpha_0]$
Define a fuzzy continuous function $h_t : A \rightarrow B$, such that:

$$H_t(x) = \begin{cases} \alpha\left(\frac{2x}{1+t}\right) & \text{if } t \geq 2x - 1 \\ x_0 & \text{if } t \leq 2x - 1 \end{cases} \quad (4)$$

Then, α is a fuzzy continuous and h_t is a fuzzy continuous mapping, and we have:

$$h_0(x) = \begin{cases} \alpha(2x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ x_0 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \quad (5)$$

And, $h_1(x) = \alpha(x)$ if $0 \leq x \leq 1$

Thus $h_0(x) = (\alpha * \alpha_0)_{(x)}$ and $h_1(x) = \alpha(x)$, for all $x \in I$

We have:

$$h_t(0) = \alpha(0) = x_0 = [\alpha_0] = \alpha(1) = h_t(1), \text{ for all } x \in I$$

Hence: $[\alpha * \alpha_0] \sim [\alpha]$.

- 2) Inverse: We must prove for all fuzzy homotopy class $[\alpha] \in h(x_0)$ has an inverse $[\beta] \in h(x_0)$ such that: $[\alpha] * [\beta] = e = [x_0]$, i.e give α how to find $\beta = \alpha^{-1}$, To find $\alpha * \beta \sim e$,

Let $\beta(t) = \alpha(1-t)$, for all $t \in I$.

Since $\beta(0) = \alpha(1) = x_0 = \alpha(0) = \beta(1)$.

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (6)$$

We define fuzzy homotopy h_t between $\alpha * \beta$ and e :

$$h_t(x) = \begin{cases} x_0 & \text{if } 0 \leq t \leq \frac{x}{2} \\ \alpha(2t-x) & \text{if } \frac{x}{2} \leq t \leq \frac{1}{2} \\ \beta(2t+x-1) & \text{if } \frac{1}{2} \leq t \leq 1 - \frac{x}{2} \\ x_0 & \text{if } 1 - \frac{x}{2} \leq t \leq 1 \end{cases} \quad (7)$$

Since α and β are fuzzy continuous, h_t is fuzzy continuous, and we have:

$$h_t(0) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (8)$$

And $h_t(1) = x_0$, if $0 \leq t \leq 1$, thus

$h_t(0) = (\alpha * \beta)_t$ and $h_t(1) = \alpha_0(t) = x_0$ for all $t \in I$, Moreover, we have: $h_0(x) = x_0 = h_1(x)$ for all $x \in I$;

Hence $\alpha * \beta \sim e$.

- 3) Associative: We need to show that: $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$, By definition multiplication fuzzy path:

$$[(\alpha * \beta) * \gamma](t) = \begin{cases} \alpha(4t) & \text{if } 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (9)$$

And

$$[\alpha * (\beta * \gamma)](t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases} \quad (10)$$

We define a fuzzy homotopy between $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ as follows: $h_t : I^A \rightarrow B$, where A, B are fuzzy sets.

$$h_t(x) = \begin{cases} \alpha\left(\frac{4t}{1+x}\right) & \text{if } x \geq 4t-1 \\ \beta(4t-x-1) & \text{if } 4t-1 \geq x \geq 4t-2 \\ \gamma\left(\frac{4t-x-2}{2-x}\right) & \text{if } 4t-2 \geq x \end{cases} \quad (11)$$

Since h_t is fuzzy continuous and α, β and γ are fuzzy continuous then:

$$h_t(0) = \begin{cases} \alpha(4t) & \text{if } 0 \leq t \leq \frac{1}{4} \\ \beta(4t-1) & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad (12)$$

And

$$h_t(1) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ \gamma(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases} \quad (13)$$

Thus $h_t(0) = [(\alpha * \beta) * \gamma](t)$ and $h_t(1) = [\alpha * (\beta * \gamma)](t)$, for all $t \in I$

Also, $h_0(x) = \alpha(0) = x_0 = \gamma_1 = h_1(x)$ for each $x \in I$

Hence; $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$.

4. Conclusion:

In this paper, we have some results as shown below:

- 1- We illustrated the concept of fuzzy topology, fuzzy set, fuzzy relation and homotopy theory.
- 2- We gave a new definition of fuzzy homotopy theory.
- 3- We proved that the fuzzy homotopy relation is a fuzzy equivalence relation.
- 4- We defined a fuzzy fundamental group and proved that fuzzy homotopy is a group.

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