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Fixed point theorem for modified iteration in hyperbolic spaces

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Abstract

We focus on two problems in geodesic hyperbolic spaces in this study. We'll start by looking at how new iterative sequences of fixed point modified mappings with Fibonacci sequences behave in Banach spaces.

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Keywords: *Hyperbolic space, Modified Iteration, Fibonacci asymptotically nonexpansive mapping uniformly convex.*

1. Introduction

Let X be a hyperbolic geodesic space. Glowinski and Le Tallec [1] employed an iterative three-step technique in 1989 to develop an approximate solution to the Elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. Cho et al. [2] extended the iterative method of Xu and Noor by using a three-step iterative procedure to find approximate fixed points of asymptotically non-expansive mappings.

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Definition 1.1: [3] A metric space$(\mathscr{H}, \mathrm{d})$ is hyperbolic if and only if it can be mapped $W: \mathscr{H}^2 \times I \rightarrow \mathscr{H}$ such that:

a) $\mathrm{d}\left(e, W(a, b, \varrho)\right) \leq \varrho \mathrm{d}(e, a) + (1-\varrho)\mathrm{d}(e, b)$

b) $\mathrm{d}\left(W(a, b, \varrho), W(a, b, \rho)\right) = |\varrho - \rho| \mathrm{d}(a, b)$

c) $W(a, b, \varrho) = W(b, a, 1-\varrho)$ (1.1)

d) $\mathrm{d}\left(W(a, c, \varrho), W(b, m, \varrho)\right) \leq \varrho \mathrm{d}(a, b) + (1-\varrho)\mathrm{d}(c, m)$

For all $e, m, a, b, c \in \mathscr{H}$ and $\varrho, \rho \in \mathrm{I} = [0,1]$ (see also [4]); if only (a) is satisfied, then the space is convex[5]. If $W(a, b, \varrho)$ for every a, b, to c and ϱ *to* I, then a subset of the hyperbolic space $\mathscr{H}$ is convex, where as Hadamard mani folds are nonlinear hyperbolic spaces [6] A hyperbolic space $\mathscr{H}$ is uniformly convex$(U\mathrm{C})$[7] if for any $e, a, b \in \mathscr{H}, q > 0$ and $\varepsilon \in (0,2]$, there exist $\omega \in (0,1]$ such that $\mathrm{d}\left(W\left(a, b, \frac{1}{2}\right), e\right) \leq (1-\omega)q < q$, whenever $\mathrm{d}(a, e) \leq q, \mathrm{d}(b, e) \leq q$ and $\mathrm{d}(a, b) \geq q\varepsilon$. A modulus of uniform convexity is a mapping $\mathscr{g}: (0, \infty) \times (0,2] \rightarrow (0,1]$defined as $\mathscr{g}(q, \varepsilon) = \omega$ for give n$q > 0$ and $\varepsilon \in (0,2]$ (as definition of $(U\mathrm{C})$ If $\mathscr{g}$ declines concerning q(for a constant), we call it monotone

Definition 1.2: In [8] special sequence$\{\mathscr{F}(n)\}$ is Fibonacci sequence if $\mathscr{F}(n+1) = \mathscr{F}(n) + \mathscr{F}(n-1)$, where $\mathscr{F}(0) = \mathscr{F}(1) = 1, \forall n \geq 1$.

Definition 1.3: [8] Let C be a nonempty subset of a metric space $\mathscr{H}$. Let $\mathrm{F}: \mathrm{C} \rightarrow \mathrm{C}$ is asymptotically nonexpansive mapping if $\{f_{\mathscr{F}(n)} \geq 1\}$ with $\lim_{n\rightarrow\infty} f_{\mathscr{F}(n)} = 1$ and $\mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a, \mathrm{F}^{\mathscr{F}(n)} b\right) \leq f_{\mathscr{F}(n)} \mathrm{d}(a, b)$and $a, b \in \mathrm{C}, \mathrm{n} \geq 1$.

Lemma 1.4: [9] *Let* $\{ð_n\}, \{\omega_n\}$ *and* $\{\xi_n\}$ *be non-negative real numbers sequences where* $\sum_{n=1}^{\infty} \xi_n < \infty$ *and* $\sum_{n=1}^{\infty} \omega_n < \infty$. *If* $ð_{n+1} \leq (1+\omega_n)ð_n + \xi_n, n \geq 1$, *then* $lim_{n\rightarrow\infty} ð_n$ *exists, in this work, suppose that be a (*$U\mathrm{C}$*) hyperbolic space with uniform convexity* $\mathscr{g}$ *and monotone modulus, and* F *is asymptotically nonexpansive mapping for* C *be a nonempty, bounded, closed, and convex subset of* $\mathscr{H}$.

Lemma 1.5: [10] *Let* $\mathscr{H}$ *be a (*$U\mathrm{C}$*) space. and* $a \in \mathscr{H}$*and* $\{\varrho_n\}$ *be a sequence in*$[\ell, \delta]$ *for some* $\ell, \delta \in (0,1)$. *If* $\{a_n\}$ *and* $\{b_n\}$ *are sequences in* $\mathscr{H}$ *such that* $lim\, sup_{n\rightarrow\infty}\, d(a_n, a) \leq q, lim\, sup_{n\rightarrow\infty}\, d(b_n, a) \leq$ *and* $lim_{n\rightarrow\infty}\, d\,(W(a_n, b_n, \varrho_n), a) = q$ *for some* $q \geq 0$, *then* $lim_{n\rightarrow\infty}\, d(a_n, b_n) = 0$

2. Main Results

In this section, we have convergence fixed point to modified iteration sequence

Definition 2.1: Let $\mathscr{H}$ be a hyperbolic space $\mathscr{H}$and F: $\complement \to \complement$ be a mapping. Then, for arbitrarily chosen $a_1 \in \complement$, we construct the sequences $\{a_n\}, \{b_n\}$ and$\{c_n\}$ in $\complement$ as:

$$c_n = W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, e_n, \xi_{n1}), \eth_n\right),$$

$$b_n = W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1-\ell_n}\right), \ell_n\right) \tag{2.1}$$

$$a_{n+1} = W\left(\mathrm{F}^{\mathscr{F}(n)} b_n, W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \frac{\rho_n}{1-\varrho_n}\right), \varrho_n\right)$$

Where $\{\eth_n\}, \{\ell_n\}, \{\delta_n\}, \{\varrho_n\}, \{\rho_n\}, \{\sigma_n\}, \{\vartheta_n\}, \{\varsigma_n\}$ are sequences in$[0,1]$ and$\{e_n\}, \{t_n\}$ and $\{m_n\}$ are bounded sequences in $\complement$ and

$$\xi_{n1} = 1 - \frac{\sigma_n}{1-\eth_n}, \quad \xi_{n2} = 1 - \frac{\vartheta_n}{1-\ell_n-\delta_n} \text{ and } \xi_{n3} = 1 - \frac{\varsigma_n}{1-\varrho_n-\rho_n}$$

Theorem 2.2: *Let $\mathscr{H}$ be a $(U\complement)$ space. Let* F *satisfy definition 2.1 with* $\sum_{n=1}^{\infty}\left(f_{\mathscr{F}(n)} - 1\right) < \infty$. *For a given* $a_1 \in \complement$, *compute* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *as in (2) satisfying* $0 < \eth \leq \varrho_n, \rho_n, \eth_n, \ell_n \leq \ell < 1, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \vartheta_n < \infty$ *and* $\sum_{n=1}^{\infty} \varsigma_n < \infty$

We have:

(i) If $s = \mathrm{F}(s)$, then $\lim_{n\to\infty} \mathrm{d}(a_n, s)$ exists.

(ii) If $0 < \liminf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty}\left(\varrho_n + \rho_n + \varsigma_n\right) < 1$, then
$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} b_n, W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \frac{\rho_n}{1-\varrho_n}\right)\right) = 0$$

(iii) If $0 < \liminf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty}\left(\varrho_n + \rho_n + \varsigma_n\right) < 1$ and $0 < \liminf_{n\to\infty} \ell_n \leq \limsup_{n\to\infty}\left(\ell_n + \delta_n + \vartheta_n\right) < 1$, then
$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1-\ell_n}\right)\right) = 0$$

(iv) If $0 < \liminf_{n\to\infty} \ell_n \leq \limsup_{n\to\infty}\left(\ell_n + \delta_n + \vartheta_n\right) < 1$ and $0 < \liminf_{n\to\infty} \eth_n \leq \limsup_{n\to\infty}\left(\eth_n + \sigma_n\right) < 1$, then
$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, e_n, \xi_{n1})\right) = 0$$

Moreover, if $0 < \liminf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty}(\varrho_n + \rho_n + \varsigma_n) < 1, 0 < \liminf_{n\to\infty} \ell_n \leq \limsup_{n\to\infty}(\ell_n + \delta_n + \vartheta_n) < 1$ and $\liminf_{n\to\infty} \eth_n \leq \limsup_{n\to\infty}(\eth_n + \sigma_n) < 1$ then $\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a_n, a_n\right) = \lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} c_n, a_n\right) = \lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} b_n, a_n\right) = 0$

Proof:

(i) By (1)(a) and $e = s \in F(\mathrm{F})$, $\{c_n\}$ in (2), then

$$\mathrm{d}(c_n, s) = \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, e_n, \xi_{n1}), \eth_n\right), s\right) \leq f_{\mathscr{F}(n)}(1 - \sigma_n)$$
$$\mathrm{d}(a_n, s) + \sigma_n \mathrm{d}(e_n, s) \tag{2.2}$$

by (1.1) (a) to the sequence$\{b_n\}$ in (2.1) and (2.2), then

$$\mathrm{d}(b_n, s) = \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1 - \ell_n}\right), \ell_n\right), s\right)$$
$$\leq f^2_{\mathscr{F}(n)}(1 - \ell_n \sigma_n - \vartheta_n)\mathrm{d}(a_n, s) + \ell_n f_{\mathscr{F}(n)} \sigma_n \mathrm{d}(e_n, s) + \vartheta_n \mathrm{d}(t_n, s)$$

That is

$$\mathrm{d}(b_n, s) = f^2_{\mathscr{F}(n)}(1 - \ell_n \sigma_n - \vartheta_n)\mathrm{d}(a_n, s) + \ell_n f_{\mathscr{F}(n)} \sigma_n \mathrm{d}(e_n, s) +$$
$$\vartheta_n \mathrm{d}(t_n, s) \tag{2.3}$$

from (2.2) and (2.3)

$$\mathrm{d}(a_{n+1}, s) = \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} b_n, W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \tfrac{\rho_n}{1-\varrho_n}\right), \varrho_n\right), s\right)$$
$$\leq f^3_{\mathscr{F}(n)} \mathrm{d}(a_n, s) + \left(f^2_{\mathscr{F}(n)} + f_{\mathscr{F}(n)}\right) \sigma_n \mathrm{d}(e_n, s) + f_{\mathscr{F}(n)} \vartheta_n \mathrm{d}(t_n, s)$$
$$+ \varsigma_n \mathrm{d}(m_n, s)$$

Therefore, we have $\mathrm{d}(a_{n+1}, s) \leq f^3_{\mathscr{F}(n)} \mathrm{d}(a_n, s) + \sigma_n D + \vartheta_n G + \varsigma_n \complement$

Where $D = \sup\left\{\left(f^2_{\mathscr{F}(n)} + f_{\mathscr{F}(n)}\right) \mathrm{d}(e_n, s) : n \geq 1\right\}$,
$G = \sup\{f_{\mathscr{F}(n)} \mathrm{d}(t_n, s) : n \geq 1\}And\complement = \sup\{\mathrm{d}(m_n, s) : n \geq 1\}$. If we let $S = \max\{D, G, \complement\}$ and $\mathrm{d}(a_{n+1}, s) \leq f^3_{\mathscr{F}(n)} \mathrm{d}(a_n, s) + S(\sigma_n + \vartheta_n + \varsigma_n)$.mSince$\sum_{n=1}^{\infty}(f_{\mathscr{F}(n)} - 1) < \infty, \sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \vartheta_n < \infty$ and$\sum_{n=1}^{\infty} \varsigma_n < \infty$, by lemma 1.4 that $\lim_{n\to\infty} \mathrm{d}(a_n, s)$ exists.

(ii) Since $\complement$ is bounded, there exists $Q > 0$ such that $\max\{\mathrm{d}(a_n, e_n), \mathrm{d}(a_n, t_n), \mathrm{d}(a_n, m_n)\} \leq Q$
If $0 < \lim inf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty}(\varrho_n + \rho_n + \varsigma_n) < 1$, for some $\wp_1, \wp_2 \in (0,1)$ such that $0 < \wp_1 \leq \varrho_n \leq \varrho_n + \rho_n + \varsigma_n \leq \wp_2 < 1$ for all $n \geq 1$. In part(i), we have $\lim_{n\to\infty} \mathrm{d}(a_n, s)$ exists, therefore $\lim_{n\to\infty} \mathrm{d}(a_{n+1}, s) = \delta > 0$.

$$\lim_{n\to\infty} W\left(\mathrm{F}^{\mathscr{F}(n)} b_n, W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \tfrac{\rho_n}{1-\varrho_n}\right), \varrho_n\right) = \delta \tag{2.4}$$

From (2.3), we have that

$$\limsup_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} b_n, s\right) \leq \delta \tag{2.5}$$

Gives that

$$\limsup_{n\to\infty} \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \frac{\rho_n}{1-\varrho_n}\right), s\right) \leq \delta \tag{2.6}$$

Lemma 1.4 is satisfied in (2.4),(2.5) and (2.6),

$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} b_n, W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \frac{\rho_n}{1-\varrho_n}\right)\right) = 0 \tag{2.7}$$

(iii) For some $q_1, q_2 \in (0,1)$ such that $0 < q_1 \leq \ell_n \leq \ell_n + \delta_n + \vartheta_n \leq q_2 < 1$ for all $n \geq 1$By (2.7),

That is $\lim_{n\to\infty} \mathrm{d}(b_n, s) = \delta$ (2.8).

Obviously, $\limsup_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} c_n, s\right) \leq \delta$ (2.9)

Gives that

$$\limsup_{n\to\infty} \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1-\ell_n}\right), s\right) \leq \delta \tag{2.10}$$

by lemma 1.5 is satisfied in (2.8), (2.9) and (2.10), then

$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1-\ell_n}\right)\right) = 0. \tag{2.11}$$

(iv) Then there exist $q_1, q_2, \tau_1, \tau_2 \in (0,1)$ such that $0 < q_1 \leq \ell_n \leq \ell_n + \delta_n + \vartheta_n \leq q_2 < 1$ and $0 < \tau_1 \leq ð_n \leq ð_n + \sigma_n \leq \tau_2 < 1$ for all $n \geq 1$Since $\mathrm{d}(b_n, s) \leq \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} c_n, W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, t_n, \xi_{n2}), \frac{\delta_n}{1-\ell_n}\right), \ell_n\right), s\right)$
With the help of (2.11), That is

$$\lim_{n\to\infty} \mathrm{d}(c_n, s) = \lim_{n\to\infty} \mathrm{d}\left(W\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, e_n, \xi_{n1}), ð_n\right), s\right) = \delta \tag{2.12}$$

Obviously $\limsup_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a_n, s\right) \leq \delta$ (2.13)

Provide that $\limsup_{n\to\infty} \mathrm{d}(W(a_n, e_n, \xi_{n1}), s) \leq \delta$ (2.14)By lemma 1.5 (using (2.12), (2.13) and (2.14)), then

$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a_n, W(a_n, e_n, \xi_{n1})\right) = 0 \tag{2.15}$$

Together with (2.15) gives that

$$\lim_{n\to\infty} \mathrm{d}\left(\mathrm{F}^{\mathscr{F}(n)} a_n, a_n\right) = 0 \tag{2.16}$$

Together with (2.16) gives that $\lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} c_n, a_n\right) =$.Provides that $\lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} b_n, a_n\right) = 0$. Hence $\lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} a_n, a_n\right) =$

$$\lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} b_n, a_n\right) = \lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} c_n, a_n\right) = 0$$

Theorem 2.3: *Let $\mathscr{H}$ be a seprable (U$\complement$) space. Let*F *be a completely continuous and satisfied definition2.1 with* $\sum_{n=1}^{\infty}\left(f_{\mathscr{F}(n)} - 1\right) < \infty$. *Let* $\{ð_n\}, \{\ell_n\}, \{\delta_n\}, \{\varrho_n\}, \{\rho_n\}, \{\sigma_n\}, \{\vartheta_n\}$ *and* $\{\varsigma_n\}$ *are sequences in* $[0,1]$. *If the following conditions:*

(i) $0 < \liminf_{n\to\infty} \varrho_n \leq \limsup_{n\to\infty}\left(\varrho_n + \rho_n + \varsigma_n\right) < 1$

(ii) $0 < \liminf_{n\to\infty} \ell_n \leq \limsup_{n\to\infty}\left(\ell_n + \delta_n + \vartheta_n\right) < 1$

(iii) $0 < \liminf_{n\to\infty} ð_n \leq \limsup_{n\to\infty}\left(ð_n + \sigma_n\right) < 1$

(iv) $\sum_{n=1}^{\infty} \sigma_n < \infty, \sum_{n=1}^{\infty} \vartheta_n < \infty$ and $\sum_{n=1}^{\infty} \varsigma_n < \infty$

Then $\{a_n\}, \{b_n\}$ and $\{c_n\}$ in (2.1) converge on a fixed point ofF.

Proof: by theorem 2.2, $\lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} a_n, a_n\right) = \lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} b_n, a_n\right) = \lim_{n\to\infty} d\left(F^{\mathscr{F}(n)} c_n, a_n\right) = 0$ Since

$d(a_{n+1}, a_n) =$

$d\left(W\left(F^{\mathscr{F}(n)} b_n, W\left(F^{\mathscr{F}(n)} c_n, W(a_n, m_n, \xi_{n3}), \frac{\rho_n}{1-\varrho_n}\right), \varrho_n\right), a_n\right)$by theorem 2.2 then $\lim_{n\to\infty} d\left(a_{n+1}, F^{\mathscr{F}(n)} a_{n+1}\right) = 0$ now

$d\left(a_{n+1}, Fa_{n+1}\right) \leq d\left(a_{n+1}, F^{\mathscr{F}(n)+1} a_{n+1}\right) + d\left(Fa_{n+1}, F^{\mathscr{F}(n)+1} a_{n+1}\right) \leq d\left(a_{n+1}, F^{\mathscr{F}(n)+1} a_{n+1}\right) + f_{\mathscr{F}(1)} d\left(a_{n+1}, F^{\mathscr{F}(n)} a_{n+1}\right)$ then $\lim_{n\to\infty} d\left(a_n, Fa_n\right) = 0$ (2.17). Since F is completely continuous and$\{a_n\} \subset \complement$ is bounded, there exists a subsequence$\left\{a_{n_f}\right\}$ of$\{a_n\}$ and $\left\{Fa_{n_f}\right\}$ converges by (2.17),$\left\{a_{n_f}\right\}$ converges. Let$\lim_{f\to\infty} a_{n_f} = s$

By the continuity of F and (2.17), then $Fs = s$, so s is a fixed point of F. By lemma 2.1(i), $\lim_{n\to\infty} d(a_n, s)$ exists. But $\lim_{f\to\infty} d\left(a_{n_f}, s\right) = 0$. $\lim_{n\to\infty} d(a_n, s) = 0$. Further the inequalities$(b_n, a_n) \leq \ell_n d\left(F^{\mathscr{F}(n)} c_n, a_n\right) + \delta_n d\left(F^{\mathscr{F}(n)} a_n, a_n\right) + \vartheta_n d(t_n, a_n)$ and $d(c_n, a_n) \leq ð_n d\left(F^{\mathscr{F}(n)} a_n, a_n\right) + \sigma_n d(e_n, a_n)$ gives that $\lim_{n\to\infty} d(b_n, a_n) = 0$ and$\lim_{n\to\infty} d(c_n, a_n) = 0$, respectively, that is,$\lim_{n\to\infty} b_n = s$ and$\lim_{n\to\infty} c_n = s$.For$\sigma_n = \vartheta_n = \varsigma_n = 0$.

Corollary 2.4: *Let* $\mathscr{H}$, C, F. *satisfied the conditions theorem 2.3. Le t*$\{\eth_n\}, \{\ell_n\}, \{\delta_n\}, \{\varrho_n\}$ *and* $\{\rho_n\}$ *be in* $[0,1]$ *with*$\ell_n + \delta_n, \varrho_n + \rho_n \in [0,1]$ *for all* $n \geq 1$ *and(i), (ii), in theorem (2.3) for*$\sigma_n = \vartheta_n = \varsigma_n = 0$, *then* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *in (2.1) with* $\sigma_n = \vartheta_n = \varsigma_n = 0$ *converge on a fixed point of* F. *For* $\delta_n = \rho_n = \sigma_n = \vartheta_n = \varsigma_n = 0$ *in theorem 2.3, we obtain the following result.*

4. Open Problem:

Study the fixed point of results paper [11]- [15] under new iteration.

5. Conclusion

The idea in this research includes obtaining a modified iteration subject to a sequence and describing this iteration as obtaining the fixed point theorems in hyperbolic space.

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