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Double ZMA-transform for solving linear telegraph equation

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Abstract

Recently, a new technique to solve initial-boundary value problems (IBVP) for general linear telegraph equation (GLTE), namely Double ZMA- transform (DZMAT) introduced. This transform is an efficient and attractive technique; moreover accurate to find the exact solution by using the procedure for the transform of (LTE).

Subject Classification: *15A04, 20M20.*

Keywords: *Transform, Double ZMAT, General LTE, LTE, ZMAT, PDEs.*

1. Introduction

Most events are modeled using partial differential equations (PDEs) in mathematical physic [1], and have confirmed and calculated the connection between the double transform, as well as their applications to differential equations by [2-4].

From [5] have established the relation between the double transform and discussed their applications. Quite recently, the general linear telegraph and partial integral-differential equations have been settled by [6].

In [1],[7-8],[19-20], a lot of problems have been explained and discussed by integral transforms. Also, [9-16] applied many problems by differential transform method. In this task, discuss some majorities new transformation to find solution of PDEs and understand procedure by discuses applications of (IBVP) for (GLTE) and confirmation the Convergence Theorem of (DZMAT).

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2. Definition and derivations of Double ZMA- transform

In [17],[18] ZMAT realized as:

$$\Lambda\{\wp(t)\} = \left\{\wp(t): \exists M, \Omega_1, \Omega_2 > 0, |\wp(t)| < Me^{-\frac{t}{\Omega v}}\right\} \tag{2.1}$$

$$\mathbb{Z}_{MA}(v;s) = \Lambda\{\wp(t)\} = \frac{1}{sv}\int_0^\infty \rho(t)e^{-\frac{t}{sv}}dt \tag{2.2}$$

To find (DZMAT) for PDEs as:

Let $\wp(x,t)$where $x,t \in \Re^+$be a capacity which can be represented as convergent series:

$$\mathbb{Z}_{MA}(u,v;s) = \Lambda_2\{\wp(x,t)\} = \frac{1}{s^2uv}\int_0^\infty \int_0^\infty \wp(x,t)e^{-\left(\frac{x}{su}+\frac{t}{sv}\right)}dxdt, x,t > 0 \tag{2.3}$$

Where u,v are complex values, by (DZMAT) can be obtained the telegraph and wave equation. In the following find the 1st and 2nd- order derivatives by (DZMAT) as:

$$\Lambda_2\left\{\frac{\partial\wp(x,t)}{\partial x}\right\} = \frac{1}{su}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{su}\mathbb{Z}_{MA}(0,v;s) \tag{2.4}$$

$$\Lambda_2\left\{\frac{\partial^2\wp(x,t)}{\partial x^2}\right\} = \frac{1}{s^2u^2}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{s^2u^2}\mathbb{Z}_{MA}(0,v;s) - \frac{1}{su}\frac{\partial\mathbb{Z}_{MA}(0,v;s)}{\partial x} \tag{2.5}$$

$$\Lambda_2\left\{\frac{\partial\wp(x,t)}{\partial t}\right\} = \frac{1}{sv}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{sv}\mathbb{Z}_{MA}(u,0;s) \tag{2.6}$$

$$\Lambda_2\left\{\frac{\partial^2\wp(x,t)}{\partial t^2}\right\} = \frac{1}{s^2v^2}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{s^2v^2}\mathbb{Z}_{MA}(u,0;s) - \frac{1}{sv}\frac{\partial\mathbb{Z}_{MA}(u,0;s)}{\partial t} \tag{2.7}$$

$$\Lambda_2\left\{\frac{\partial^2\wp(x,t)}{\partial x\,\partial t}\right\} = \frac{1}{s^2uv}\wp(0,0) - \frac{1}{sv}\mathbb{Z}_{MA}(u,0;s) + \frac{1}{s^2uv}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{s^2uv}\mathbb{Z}_{MA}(0,v;s) \tag{2.8}$$

3. Methodology

3.1 *The Convergence Theorem of (DZMAT)*

This action proves the convergence of (DZMAT) informed (2.3).

Theorem 3.1: *If two variables' function $\wp(x,t)$ is continuous in the 1st- quadrant of the xt- plane. Where (DZMAT) realized in (2.3) converges at $s = s_0, u = u_0$ and $v = v_0$ .When $s > s_0, u > u_0$ and $v > v_0$ then the integral converges, assuming that $\frac{1}{su} = \frac{1}{s_0u_0} > 0$ and $\frac{1}{sv} = \frac{1}{s_0v_0} > 0$.*

Proof: Rewrite the integral (2.3) as:

$$\begin{aligned}\Lambda_2\{\wp(x,t)\} &= \frac{1}{s^2uv}\int_0^\infty \int_0^\infty \wp(x,t)e^{-\left(\frac{x}{su}+\frac{t}{sv}\right)}dxdt, x,t > 0\\ &= \frac{1}{su}\int_0^\infty e^{-\frac{x}{su}}\left[\frac{1}{sv}\int_0^\infty e^{-\frac{t}{sv}}\wp(x,t)\right]dtdx\\ &= \frac{1}{su}\int_0^\infty e^{-\frac{x}{su}}\Psi(x,t)dx\end{aligned} \tag{3.1}$$

where

$$\Psi(x,t)=\frac{1}{sv}\int_0^{\infty} e^{-\frac{t}{sv}}\wp(x,t)d \tag{3.2}$$

Firstly, prove that the integral (3.2) for $s>s_0$ and $v>v_0$, where $\frac{1}{sv}=\frac{1}{s_0v_0}>0$, suppose that integral (3.1) converges at

$$s>s_0 \text{ and } v>v_0\ \mathbb{Q}(x,t)=\frac{1}{s_0v_0}\int_0^{\infty} e^{-\frac{\kappa}{s_0v_0}}\wp(x,\kappa)d\kappa, 0<t<\infty \tag{3.3}$$

$\lim_{t\to\infty}\mathbb{Q}(x,t)$ exists and $\mathbb{Q}(x,t)=0$ since (3.2) converges at $s>s_0$ and $v>v_0$.Using the fundamental theorem of calculus, find:

$$\mathbb{Q}_t(x,t)=\frac{1}{s_0v_0}e^{-\frac{t}{s_0v_0}}\wp(x,\kappa) \tag{3.4}$$

Select α and $\beta, 0<\alpha<\beta$, then from (3.2) obtain:

$$\begin{aligned}\frac{1}{sv}\int_0^{\infty} e^{-\frac{t}{sv}}\wp(x,t)dt&=\frac{1}{sv}\int_\alpha^{\beta} e^{-\frac{t}{sv}}e^{\frac{t}{s_0v_0}}\mathbb{Q}_t(x,t)dt\\&=\frac{1}{sv}\int_\alpha^{\beta} e^{-\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)t}\mathbb{Q}_t(x,t)dt\end{aligned} \tag{3.5}$$

by the integration by parts for (3.5) get:

$$\begin{aligned}&\frac{1}{sv}\left[e^{-\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)\alpha}\mathbb{Q}(x,\alpha)-e^{-\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)\beta}\mathbb{Q}(x,\beta)\right]\\&+\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)\int_0^{\beta} e^{-\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)t}\mathbb{Q}(x,t)dt\end{aligned} \tag{3.6}$$

when $\alpha\to 0$ from ($\frac{1}{sv}=\frac{1}{s_0v_0}>0$) the 1st-term on (3.6) converges to zero, lastly, find:

$$\frac{1}{sv}\int_0^{\infty} e^{-\frac{t}{sv}}\wp(x,t)dt=\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)\int_0^{\infty} e^{-\left(\frac{1}{sv}-\frac{1}{s_0v_0}\right)}\mathbb{Q}(x,t)dt \tag{3.7}$$

It is clear that the integral on the right converges, then the integral (3.2) converges for $s>s_0$ and $v>v_0$ where $\frac{s}{s_0}=\frac{v}{v_0}>0$.

In the same way, it proves the convergence of the integral $\frac{1}{su}\int_0^{\infty} e^{-\frac{x}{su}}\Psi(x,t)dx\ s>s_0, u>u_0$ with $\frac{s}{s_0}=\frac{u}{u_0}>0$, Thus, the integral on the right-hand side of (3.1) converges for $s>s_0, u>u_0$ and $v>v_0$ with assumption $\frac{s}{s_0}=\frac{u}{u_0}>0$ and $\frac{s}{s_0}=\frac{v}{v_0}>0$, Therefore, the (DZMAT) transform defined by (2.3) converges, and the proof of the theorem is then complete.

4. Procedure

In this portion, applied (DZMAT) to solve (GLTE) as:

- Make the (DZMAT) for PDEs.
- Find the single ZMAT of conditions.
- Substitute the 2nd step in the 1st step and solve the resulting equation. I, lastly applied the double inverse of ZMA-transform to obtain the general solution.

5. Applications

This work established some applications to understand the procedure of (DZMAT):

Application 5.1: Look at the (LTE).

$$\mathrm{U_{xx}} = \mathrm{U_{tt}} + \mathrm{U_t} + \mathrm{U} \tag{5.1}$$

with the frontier condition $\mathrm{U}(0,\mathrm{t}) = \mathrm{e}^{-\mathrm{t}}, \mathrm{U_x}(0,\mathrm{t}) = \mathrm{e}^{-\mathrm{t}}$ (5.2)

And the incipient conditions $\mathrm{U}(\mathrm{x},0) = e^{\mathrm{x}}, \mathrm{U_t}(\mathrm{x},0) = -e^{\mathrm{x}}$ (5.3)

The careful solution is $\mathrm{U}(\mathrm{x},\mathrm{t}) = \mathrm{e}^{\mathrm{x}-\mathrm{t}}$ (5.4)

Solution: By (DZMAT) of (5.1), find:

$$\begin{aligned}&\tfrac{1}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s) - \tfrac{1}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(0,v;s) - \tfrac{1}{su}\tfrac{\partial \mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x}\\ &= \tfrac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s) - \tfrac{1}{sv}\tfrac{\partial \mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t} - \tfrac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,0;s)\\ &+\tfrac{1}{sv}\mathbb{Z}_{\mathrm{MA}}(u,v;s) - \tfrac{1}{sv}\mathbb{Z}_{\mathrm{MA}}(u,0;s) + \mathbb{Z}_{\mathrm{MA}}(u,v;s)\end{aligned} \tag{5.5}$$

By (ZMAT) of (5.2) and (5.3) obtain:

$$\mathbb{Z}_{\mathrm{MA}}(0,v;s) = \tfrac{1}{1+sv} \tag{5.6}$$

$$\tfrac{\partial \mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x} = \tfrac{1}{1+sv} \tag{5.7}$$

$$\mathbb{Z}_{\mathrm{MA}}(u,0;s) = \tfrac{1}{1-su} \tag{5.8}$$

$$\tfrac{\partial \mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t} = -\tfrac{1}{1-su} \tag{5.9}$$

Now , substituting (5.6)-(5.9) in (5.5)

$$\left[\tfrac{1}{s^2u^2} - \tfrac{1}{s^2v^2} - \tfrac{1}{sv} - 1\right]\mathbb{Z}_{\mathrm{MA}}(u,v;s) = \tfrac{1}{1+sv}\left[\tfrac{1}{s^2u^2} + \tfrac{1}{su}\right] - \tfrac{1}{s^2v^2}\left[\tfrac{1}{1-su}\right] \tag{5.10}$$

$$\Rightarrow \mathbb{Z}_{\mathrm{MA}}(u,v;s) = \tfrac{1}{(1-su)(1+sv)} \tag{5.11}$$

Now, take the double inverse of ZMA- transform for (5.11) and get:

$$\mathrm{U}(\mathrm{x},\mathrm{t}) = \mathrm{e}^{x}\ \mathrm{e}^{-\mathrm{t}} = \mathrm{e}^{\mathrm{x}-\mathrm{t}} \tag{5.12}$$

Application 5.2: Compose the (LTE).

$$\mathrm{U_{xx}} = \mathrm{U_{tt}} + \mathrm{U_t} - \mathrm{U} \tag{5.13}$$

with the frontier conditions $\mathrm{U}(0,\mathrm{t}) = \mathrm{e}^{-2\mathrm{t}}, \mathrm{U_x}(0,\mathrm{t}) = \mathrm{e}^{-2\mathrm{t}}$ (5.14)

And the incipient conditions $\mathrm{U}(\mathrm{x},0) = e^{\mathrm{x}}, \mathrm{U_t}(\mathrm{x},0) = -2\mathrm{e}^{\mathrm{x}}$ (5.15)

The careful solution is $\mathrm{U}(\mathrm{x},\mathrm{t}) = \mathrm{e}^{\mathrm{x}-2\,\mathrm{t}}$ (5.16)

Solution: By (DZMAT) of (5.13) obtain:

$$\begin{aligned}&\frac{1}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{1}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(0,v;s)-\frac{1}{su}\frac{\partial\mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x}\\&=\frac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{1}{sv}\frac{\partial\mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t}-\frac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,0;s)\\&+\frac{1}{sv}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{1}{sv}\mathbb{Z}_{\mathrm{MA}}(u,0;s)-\mathbb{Z}_{\mathrm{MA}}(u,v;s)\end{aligned}\tag{5.17}$$

By (ZMAT) of (5.14) and (5.15), get:

$$\mathbb{Z}_{\mathrm{MA}}(0,v;s)=\frac{1}{1+2sv}\tag{5.18}$$

$$\frac{\partial\mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x}=\frac{1}{1+2sv}\tag{5.19}$$

$$\mathbb{Z}_{\mathrm{MA}}(u,0;s)=\frac{1}{1-su}\tag{5.20}$$

$$\frac{\partial\mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t}=-\frac{2}{1-su}\tag{5.21}$$

Now , substituting (5.18)-(5.20) in (5.17)

$$\left[\frac{s^2v^2-s^2u^2-(s^2u^2)sv+(s^2u^2)(s^2v^2)}{(s^2u^2)(s^2v^2)}\right]\mathbb{Z}_{\mathrm{MA}}(u,v;s)=\frac{1+su}{s^2u^2(1+2sv)}+\frac{-1+sv}{s^2v^2(1-su)}\tag{5.22}$$

$$\Rightarrow\mathbb{Z}_{\mathrm{MA}}(u,v;s)=\frac{1}{(1-su)(1+2sv)}\tag{5.23}$$

Know, by the double inverse of ZMA- transform for (5.23), find:

$$\mathrm{U(x,t)}=\mathrm{e^{x}\,e^{-2\,t}}\;=\;\mathrm{e^{x-2\,t}}\tag{5.24}$$

Application 5.3: Take the (LTE).

$$\mathrm{U_{tt}+aU_{t}+bU=c^{2}U_{xx}}\tag{5.25}$$

with the frontier conditions $\mathrm{U(0,t)}=\ell_1(\mathrm{t}),\mathrm{U_z(0,t)}=\mathrm{k_1(t)}$ (5.26)

And the incipient conditions $\mathrm{U(x,0)}=\ell_2(\mathrm{t}),\mathrm{U_t(x,0)}=\mathrm{k_2(t)}$ (5.27)

Solution: Occupy (DZMAT) of (5.25) find:

$$\begin{aligned}&\frac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{1}{s^2v^2}\mathbb{Z}_{\mathrm{MA}}(u,0;s)-\frac{1}{sv}\frac{\partial\mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t}\\&+\frac{a}{sv}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{a}{sv}\mathbb{Z}_{\mathrm{MA}}(u,0;s)+\frac{b}{sv}\mathbb{Z}_{\mathrm{MA}}(u,v;s)\\&=\frac{c^2}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(u,v;s)-\frac{c^2}{s^2u^2}\mathbb{Z}_{\mathrm{MA}}(0,v;s)-\frac{c^2}{sv}\frac{\partial\mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x}\end{aligned}\tag{5.28}$$

By (ZMAT) of (5.26) and (5.27) resulted:

$$\mathbb{Z}_{\mathrm{MA}}(0,v;s)=L_1(v)\tag{5.29}$$

$$\frac{\partial\mathbb{Z}_{\mathrm{MA}}(0,v;s)}{\partial x}=K_1(v)\tag{5.30}$$

$$\mathbb{Z}_{\mathrm{MA}}(u,0;s)=L_2(u)\tag{5.31}$$

$$\frac{\partial\mathbb{Z}_{\mathrm{MA}}(u,0;s)}{\partial t}=K_2(u)\tag{5.32}$$

Know, exchange (5.29)-(5.32) in (5.28) as:

$$\left[\frac{1}{s^2v^2}+\frac{a}{sv}+\frac{b}{sv}-\frac{c^2}{s^2u^2}\right]\mathbb{Z}_{\mathrm{MA}}(uv;s)=\frac{1}{s^2v^2}L_2(u)+\frac{1}{sv}K_2(u)+\frac{a}{sv}L_2(v)-\frac{c^2}{s^2u^2}L_1(v)-\frac{c^2}{sv}K_l(v) \tag{5.33}$$

$$\mathbb{Z}_{MA}(u,v;s)=\frac{\frac{1}{s^2v^2}L_2(u)+\frac{1}{sv}K_2(u)+\frac{a}{sv}L_2(v)-\frac{c^2}{s^2u^2}L_1(v)-\frac{c^2}{sv}K_1(v)}{\left[\frac{1}{s^2v^2}+\frac{a}{sv}+\frac{b}{sv}-\frac{c^2}{s^2u^2}\right]}=\lambda(u,v) \tag{5.34}$$

by the double inverse of ZMA- transform for (5.23) obtain (GLTE) for (5.25):

$$\mathrm{U(x,t)}=\Lambda_2^{-1}[\lambda(\mathrm{u,v})]=\mathbb{Z}_{\mathrm{MA}}(\mathrm{x,t;s}) \tag{5.35}$$

where

$$\Lambda\{e^{\mathrm{ax+bt}}\}=\frac{1}{(1-\mathrm{asu})(1-\mathrm{asv})} \tag{5.36}$$

6. Conclusion

The primary goal of this research was to define a novel method called (DZMAT). After establishing that this transformation note is useful and accurate for locating the solution to PDEs, we then reviewed and proved the primary and theorem for this technique, and finally demonstrated the application of (IBVP) for (GLTE) using (DZMAT).

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