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ϰϢ-connected Spaces

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Abstract

In this paper, we introduced a new type of open set in generalized topological spaces: ϰϢ-open set. A new generalization of connectedness in a generalized topological space using ϰϢ-open namely ϰϢ-connectedness is defined. Also, new generalizations of continuous functions in generalized topological space will be studied here as weakly ϰϢ- continuous, strongly ϰϢ- continuous and ϰϢ- irresolute functions.

Subject Classification: *54D05, 54G05.*

Keywords: *ϰϢ-open set, ϰϢ-closed set, ϰϢ-connected space, ϰϢ- continuous function.*

1. Introduction

The pair $(Z, \mathcal{X})$ symbolizes a topological space, and $\mathcal{X}'$ symbolizes the collection of all closed sets in $(Z, \mathcal{X})$. If Ꮋ is a subset of Z, then

1. the interior of Ꮋ is the set $\text{int}(Ꮋ) = \cup\{Ꮗ: Ꮗ \in \mathcal{X} \; and \; Ꮗ \subseteq Ꮋ\}$.
2. the closure of Ꮋ is the set $\text{cl}(Ꮋ) = \cap\{Ꮀ: Ꮀ \in \mathcal{X}' \; and \; Ꮋ \subseteq Ꮀ\}$ [1].

A topological space $(Z, \mathcal{X})$ is disconnected space if $Z = A \cup B$ where A and B are open or closed non-empty disjoint subsets of Z , Z is said to

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be connected if and only if disconnected [2]. A function $f : (Z_1, \chi_1) \to (Z_2, \chi_2)$ is a continuous function if $f^{-1}(V) \in \chi_1 \forall V \in \chi_2$ [2] and [3]. The continuous function preserves connectedness [4]. By a component of a topological space (Z, χ), we mean a connected subset of Z, which is not properly contained in any larger connected set in Z [5]. A subset M of a topological space (Z, χ) is connected if and only if the subspace (M, χ_M) is connected [2].

A generalized topological space (GTS) is a pair (Z, ω), where Z is a non-empty set, and ω is a collection of subsets of Z satisfying the conditions: $\emptyset \in \omega$ and $\bigcup_{\alpha \in \Lambda} N_\alpha \in \omega$ where $N_\alpha \in \omega \forall \alpha \in \Lambda$. Members of ω are $\omega - open$ and members of ω' are $\omega - closed$, which are the complement of $\omega - open$ [5]and [6].

1. The $\omega - interior$ of H is the set $\text{int}_\omega(H) = \bigcup\{W : W \in \omega \; and \; W \subseteq H\}$.
2. The $\omega - closure$ of H is the set $\text{cl}_\omega(H) = \bigcap\{F : F \in \omega' and \; H \subseteq F\}$ [7].

Definition 1: [8] A GTS (Z, ω) is said to be $\omega - connected$ if there are no non-empty disjoint $\omega - open$ subsets U, V of Z such that $U \cup V = Z$.

Definition 2: [9] A function $f : (Z_1, \omega_1) \to (Z_2, \omega_2)$ is called $(\omega_1, \omega_2) - continuous$ if $f^{-1}(H) \in \omega_1 \forall H \in \omega_2$.

Theorem 1: *The* $(\omega_1, \omega_2) - continuous$ *image of* $\omega_1 - connected$ *space is* $\omega_2 - connected$.

Definition 3: Take a topological space (Z, χ) with generalized topology ω. A subset H of Z is said to be $\chi(\omega) - \text{open}$ set if there exists an ω –open set N such that $N \subseteq H \subseteq \text{cl}(N)$.The family of all $\chi(\omega) - open$ sets of Z will be denoted by $\chi(\omega)O(Z)$. The complement of $\chi(\omega) - open$ is $\chi(\omega) - closed$ set, the class of all $\chi(\omega) - closed$ sets in Z will be denoted by $\chi(\omega)C(Z)$.

Definition 4: Take a topological space (Z, χ) with generalized topology ω.

1. $\text{int}_{\chi(\omega)}(H) = \bigcup\{G : G \; is \; \chi(\omega) - \text{open}, \; G \subseteq H\}$ is the $\chi(\omega) -$ interior of H.
2. $\text{cl}_{\chi(\omega)}(H) = \bigcap\{F : F \; is \; \chi(\omega) - \text{closed}, H \subseteq F\}$ is the $\chi(\omega) -$ closure of H.

2. $\chi(\omega) -$ connected space

Definition 5: A topological space (Z, χ) with generalized topology ω is said to be $\chi(\omega) - disconnected$ space if it is the union of two non-empty disjoint $\chi(\omega) - open$ sets, Z is said to be $\chi(\omega) - connected$ if and only if it is not $\chi(\omega) - disconnected$.

Theorem 2: *A topological space* (Z, χ) *with generalized topology* ω *is* $\chi(\omega) - disconnected$ *if and only if it is the union of two non-empty disjoint* $\chi(\omega) - closed$ *sets.*

Proof: **The " only if" part**

Assume that Z is $\chi(\omega) -$ disconnected space, then there exist non-empty $\chi(\omega) -$ open subsets $U, \dot{U}$ of Z such that $U \cap \dot{U} = \emptyset$ and $U \cup \dot{U} = Z$. This means that U and $\dot{U}$ are complementary to each other, therefor U and $\dot{U}$ are $\chi(\omega) -$ closed sets.

The "if" part

Suppose that there exist non-empty $\chi(\omega) - closed$ subsets $U, \dot{U}$ of Z $\ni U \cap \dot{U} = \emptyset$ and $U \cup \dot{U} = Z$. This means that U and $\dot{U}$ are complementary to each other, therefor U and $\dot{U}$ are $\chi(\omega) - open$ sets. Hence Z is $\chi(\omega) -$ disconnected.

Theorem 3: *A topological space* (Z, χ) *with generalized topology* ω *is* $\chi(\omega) - disconnected$ *if and only if there exists a non-empty proper subset of* Z *which is both* $\chi(\omega) - open$ *and* $\chi(\omega) - closed$ *set.*

Proof: **The " only if" part**

Assume that Z is $\chi(\omega) -$ disconnected space, then there exist non-empty $\chi(\omega) -$ open subsets $H_1, \dot{U}$ of Z such that $U \cap \dot{U} = \emptyset$ and $U \cup \dot{U} = Z$. It follows that U is a complement of $\dot{U}$, therefor U is $\chi(\omega) -$ closed set; also, we have that $U \neq \emptyset$ and since $\dot{U} \neq \emptyset$, so U is a proper subset of Z. So U is the required set.

The "if" part

Suppose that there exists a non-empty proper subset U of Z which is both $\chi(\omega) - open$ and $\chi(\omega) - closed$ set, and let $\dot{U}$ be the complement of U. Now we have two non-empty disjoint subsets $U, \dot{U}$ of Z such that $U \cup \dot{U} = Z$. Thus Z is $\chi(\omega) -$ disconnected.

Corollary 1: *A topological space* (Z, χ) *with generalized topology* ω *is* $\chi(\omega) -$ *connected if and only if the only non-empty subset of* Z *which is both* $\chi(\omega) -$ $open$ *and* $\chi(\omega) - closed$ *set, is* Z *itself.*

Proposition 1: Every $\chi(\omega) - connected$ space is $\omega) - connected$.

Definition 6: Take a topological space (Z, χ) with generalized topology ω, and let M be a subset of Z the $\chi(\omega)$ – relative generalized topology for M is the collection $\chi(\omega)_M$ given by $\chi(\omega)_M = \{H \cap M : H \in \chi(\omega) O(Z)\}$. The generalized topological space $(M, \chi(\omega)_M)$ is called an $\chi(\omega)$ – $subspace$ of (Z, ω).

Theorem 4: *Take a topological space* (Z, χ) *with generalized topology* ω *such that* $Z = U \cup \dot{U}$ *where* $U, \dot{U}$ *are two non- empty disjoint* $\chi(\omega) - open$ *sets and let* M *be* $\chi(\omega) -$ *connected subset of* Z. *Then* $M \subseteq U$ *or* $M \subseteq \dot{U}$.

Proof: Suppose that $M \nsubseteq U$ and $M \nsubseteq \dot{U}$, but $M \subseteq U \cup \dot{U}$, so $M \cap \dot{U} \neq \emptyset$ and $M \cap U \neq \emptyset$. On the other hand, we have $(M \cap \dot{U}) \cap (M \cap U) = M \cap$

$(\mathcal{V} \cap \mathcal{U}) = \mathcal{M} \cap \emptyset = \emptyset$, and $(\mathcal{M} \cap \mathcal{V}) \cup (\mathcal{M} \cap \mathcal{U}) = \mathcal{M} \cap (\mathcal{V} \cup \mathcal{U}) = \mathcal{M} \cap Z = \mathcal{M}$. Also $\mathcal{M} \cap \mathcal{V}$ and $\mathcal{M} \cap \mathcal{U}$are $\chi(\omega) - \text{open}$ in $\mathcal{M}$ since $\mathcal{U}, \mathcal{V}$ are $\chi(\omega) - \text{open}$ in Z. Thus $\mathcal{M}$ is $\chi(\omega) -$ disconnected space, which is a contradiction.

Theorem 5: *Take a topological space* (Z, χ) *with generalized topology* ω*, and let* $\mathcal{M}$ *be* $\chi(\omega) - connected\ subset\ of\ Z.\ If$ $\mathcal{M} \subseteq \mathcal{F} \subseteq cl_{\chi(\omega)}(\mathcal{M})$*, then* $\mathcal{F}$ *is also* $\chi(\omega) - connected.$

Proof: Suppose that $\mathcal{F}$ is $\chi(\omega) -$ disconnected, then there exist non-empty $\chi(\omega) -$ open subsets $\mathcal{U}, \mathcal{V}$ of $\mathcal{F}$ $\ni \mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{F} = \mathcal{U} \cup \mathcal{V}$, but $\mathcal{M} \subseteq \mathcal{F}$, therefore $\mathcal{M} \subseteq \mathcal{U} \cup \mathcal{V}$, it follows from Theorem 2.12 that $\mathcal{M} \subseteq \mathcal{U}$ or $\mathcal{M} \subseteq \mathcal{V}$ since $\mathcal{M}$ is $\chi(\omega) -$ connected. Let $\mathcal{M} \subseteq \mathcal{U}$, so $\text{cl}_{\chi(\omega)}(\mathcal{M}) \subseteq \text{cl}_{\chi(\omega)}(\mathcal{U})$, since $\text{cl}_{\chi(\omega)}(\mathcal{U}) \cap \mathcal{V} = \emptyset$, so $\mathcal{F} \cap \mathcal{V} = \emptyset$, but $\mathcal{V} \subseteq \mathcal{F}$. Hence $\mathcal{V} = \emptyset$, which is a contradiction.

Corollary 2: *Take a topological space* (Z, χ) *with generalized topology* ω*, and let* $\mathcal{M}$ *be* $\chi(\omega) -$ *a connected subset of* Z*. Then* $cl_{\chi(\omega)}(\mathcal{M})$ *is also* $\chi(\omega) -$ *connected.*

1. Theorems of $\chi(\omega) -$ connected space

Definition 7: Take a topological space (Z, χ_1) with generalized topology ω_1 and let (Y, χ_2) be a topological space with generalized topology ω_2. A function $f: Z \to Y$ is said to be

1. $\boldsymbol{weakly}\ \chi(\omega) - \boldsymbol{continuous\ function}$ if $f^{-1}(\mathcal{H})$ is $\chi_1(\omega_1) - open$ in Z for each $\omega_2 - open$ subset $\mathcal{H}$ of Y.
2. $\boldsymbol{strongly}\ \chi(\omega) - \boldsymbol{continuous\ function}$ if $f^{-1}(\mathcal{H})$ is $\omega_1 - open$ in Z for each $\chi_2(\omega_2) - open$ subset $\mathcal{H}$ of Y.
3. $\chi(\omega) - \boldsymbol{irresolute\ function}$ if $f^{-1}(\mathcal{H})$ is $\chi_1(\omega_1) - open$ in Z for each $\chi_2(\omega_2) - open$ subset $\mathcal{H}$ of Y.

Proposition 2: Every $\chi(\omega) - irresolute$ function is a $weakly\ \chi(\omega) - continuous.$

Proposition 3: Every $(\omega_1, \omega_2) - continuous$ function is a $weakly\ \chi(\omega) - continuous.$

Proposition 4: Every $strongly\ \chi(\omega) - continuous$ function is an $\chi(\omega) - irresolute.$

Proposition 5: Every $strongly\ \chi(\omega) - continuous$ function is an $(\omega_1, \omega_2) - continuous.$

Corollary 2: *Every* $strongly\ \chi(\omega) - continuous$ *function is a* $weakly\ \chi(\omega) - continuous.$

Theorem 8: *The strongly* $\chi(\omega) - continuous$ *image of* $\chi_1(\omega)_1 - connected$ *space is* $\chi_2(\omega)_2 - connected$.

Proof: Take a topological space (Z, χ_1) with generalized topology $(\omega)_1$ and let (Y, χ_2) be a topological space with generalized topology $(\omega)_2$ and $f: Z \to Y$ be strongly $\chi(\omega)$ −continuous and surjective function and assume that Z is $\chi_1(\omega)_1$ −connected space, we have to show that Y is $\chi_2\ (\omega)_2$ −connected space suppose that Y is not $\chi_2\ (\omega)_2$ −connected, then there are non-empty disjoint $\chi_2(\omega)_2$ –open subsets H_1, H_2 of $Y \ni H_1 \cup H_2 = Y$. It follows from Definition 3.2 and the properties of functions that $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are non-empty disjoint $(\omega)_1$ –open subsets of Z such that $f^{-1}(H_1 \cup H_2) = f^{-1}(Y)$, which implies that $f^{-1}(H_1) \cup f^{-1}(H_2) = Z$. Now from Remark 1.6 we get $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are $\chi_1(\omega)_1 - \boldsymbol{an}$ open subset of Z. Thus Z is not $\chi_1(\omega)_1$ −connected space, this contradicts our hypothesis. Hence Y must be $\chi_2(\omega)_2$ −connected space.

4. The conclusion

From this work, we obtained many results, the most important of which are:

- Every $\chi(\omega) - connected$ space is $(\omega) - connected$.
- The $strongly\ \chi(\omega) - continuous$ image of $\chi_1(\omega)_1 - connected$ space is $\chi_2(\omega)_2 - connected$.
- Every $strongly\ \chi(\omega) - continuous$ function is an $\chi(\omega) - irresolute$.

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