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Certain subclass of meromorphic univalent positive coefficients defined by q-difference operator

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Abstract

This paper introduces a Certain Subclass of Meromorphic Univalent Positives Coefficients Defined by the q-Difference Operator. Coefficient estimates are investigated and obtained, and the upped bound is calculated.

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1. Introduction

For $0 \leq \omega < 1$, let $£_{\omega}$ indicate the class of the form's univalent meromorphic functions:

$$f(\mathrm{z}) = \frac{1}{z-\omega} + \sum_{n=1}^{\infty} a_n z^n \,, f(\omega) = \infty \,,$$

defined in the unit disk $\mathrm{U} = \{\mathrm{z} \in C \,;\, \omega < |z| < 1\}$ .Also let $£_{\omega,\beta}$,$0< \beta \leq 1$ be the subclass of function f in $£_{\omega}$which has the expansion:

$$f(\mathrm{z}) = \frac{\beta}{z-\omega} + \sum_{n=1}^{\infty} a_n z^n \,, \tag{1}$$

where $= Res(f,\omega)$, with $0 \leq \omega < 1$, $\mathrm{z} \in C$.

The function f given in (1) is studied by Jinxi Ma [5] .The function $f \in £_{\omega}$ is said to be a meromorphically starlike (convex) function of order σ if and only if

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$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma, 0 \le \sigma < 1, z \in U \tag{2}$$

$$-\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \sigma, 0 \le \sigma < 1, z \in U \tag{3}$$

The subclass of such functions has been studied by [3](see also [1, 2],[4],[6-7],[9-14]). Let $£^{+}_{\omega,\beta} < £_{\omega,\beta}$ consisting of functions of the form:

$$f(z) = \frac{\beta}{z-\omega} + \sum_{n=1}^{\infty} a_n z^n, \tag{4}$$

It is known that calculus without the idea of limits is called q-calculus. Because it has so many important uses, it has impacted many scientific areas. Tang et al. [8] defined the q-derivative for meromorphic functions $f \in £_0$ by:

$$\partial_q f(z) = \frac{f(z)-f(qz)}{(1-q)z} = -\frac{1}{qz^2} + \sum_{n=1}^{\infty}[n]_q a_n z^{n-1}, \tag{5}$$

where

$$[j]_q = \frac{1-q^j}{1-q}. \tag{6}$$

As q→ $1-$,$[j]_q = j$ and $\partial_q f(z) = f^{'}(z)$. For $f \in £_{\omega,\beta}$, let

$£^0_q f(z) = f(z)$,

$$£^1_q f(z) = z\,\partial_q f(z) + \frac{\beta((q+1)z-\omega)}{(z-\omega)(qz-\omega)}$$

$$£^2_q f(z) = z\partial_q(£^1_q f(z)) + \frac{\beta((q+1)z-\omega)}{(z-\omega)(qz-\omega)},$$

and for $n \in N = \{1,2,3,\dots\}$, we can write

$$£^n_q f(z) = z\partial_q(£^{n-1}_q f(z)) + \frac{\beta((q+1)z-\omega)}{(z-\omega)(qz-\omega)}$$

$$= \frac{\beta}{z-\omega} + \sum_{k=1}^{\infty}[k]^n_q\, a_k z^k. \tag{7}$$

Now, we define the following

Definition 1: The function $f \in £^n_q(\omega,\beta)$ if it satisfies

$$\left|\frac{q\left(\frac{z\left(£^n_q f(z)\right)'}{£^n_q f(z)}\right)+1}{q\left(\frac{z\left(£^n_q f(z)\right)'}{£^n_q f(z)}\right)-1}\right| < 1, (\mathrm{n} \in N_\circ = N \cup \{0\}. \tag{8}$$

2. Coefficients Estimates

Unless indicated, for some σ $(0 \le \sigma < 1)$,let $0< q < 1, (\mathrm{n} \in N_\circ)$,$0< \beta \le 1$, $0< \sigma \le 1$, $z \in U$ and $£^n_q f(z) = \frac{\beta}{z-\omega} + \sum_{k=1}^{\infty}[k]^n_q\, a_k z^k$.

Theorem 1: *Let f (z) be defined by (4). Then f∈* $£^n_q(\omega,\beta)$ *if and only if*

$$\sum_{k=1}^{\infty}[k]^n_q\, a_k \le \frac{\beta}{1-\omega}. \tag{9}$$

Proof: Assume $f \in £^n_q(\omega,\beta)$, then (8) holds true.

Therefore

$$\left|\frac{q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)+1}{q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)-1}\right| < 1 .$$

Also,

$$\left|\frac{q\left(z\left(£_q^n f(z)\right)'\right)+£_q^n f(z)}{q\left(z\left(£_q^n f(z)\right)'\right)-(£_q^n f(z))}\right| < 1.$$

Then

$$\left|\frac{q\left(z\frac{-\beta}{(z-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)+\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k}{q\left(z\frac{-\beta}{(z-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)-(\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k)}\right| < 1.$$

So ,

$$\operatorname{Re}\left(\frac{q\left(z\frac{-\beta}{(z-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)+\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k}{q\left(z\frac{-\beta}{(z-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)-(\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k)}\right) < 1.$$

Now , when $z\rightarrow 1-$ we get:

$$\frac{q\frac{-\beta}{(1-\omega)^2}+q\sum_{k=1}^{\infty}[k]_q^n k a_k+\frac{\beta}{1-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k}{q\left(\frac{-\beta}{(1-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k\right)-\left(\frac{\beta}{1-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k\right)} < 1.$$

So,

$$q\frac{-\beta}{(1-\omega)^2}+ q\sum_{k=1}^{\infty}[k]_q^n k a_k - \frac{\beta}{1-\omega}+ \sum_{k=1}^{\infty}[k]_q^n a_k <$$
$$q\frac{-\beta}{(1-\omega)^2}+q \sum_{k=1}^{\infty}[k]_q^n k a_k+\frac{\beta}{1-\omega}-\sum_{k=1}^{\infty}[k]_q^n a_k.$$

Hence

$$\sum_{k=1}^{\infty}[k]_q^n a_k < \frac{\beta}{1-\omega}$$

Conversely, let condition (9) be true to prove that $f\in £_q^n(\omega,\beta)$ we have to show that

$$\left|q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)+1\right| - \left|q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)-1\right| \leq 0.$$

Now,

$$\left|q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)+1\right| - \left|q\left(\frac{z\left(£_q^n f(z)\right)'}{£_q^n f(z)}\right)-1\right| =$$
$$\left|q\left(z\frac{-\beta}{(z-\omega)^2}+\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)+\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k\right| - \left|q\left(z\frac{-\beta}{(z-\omega)^2}+\right.\right.$$
$$\left.\left.\sum_{k=1}^{\infty}[k]_q^n k a_k z^k\right)-\left(\frac{\beta}{z-\omega}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k\right)\right|.$$

$= \left|\sum_{k=1}^{\infty}[k]_q^n\ a_k z^k[qk+1] - \frac{\beta zq}{(z-\omega)^2} + \frac{\beta}{z-\omega}\right| - \left|\sum_{k=1}^{\infty}[k]_q^n\ a_k z^k[qk-1] - \frac{\beta zq}{(z-\omega)^2} - \frac{\beta}{z-\omega}\right|$

$= \left|\sum_{k=1}^{\infty}[k]_q^n\ a_k z^k[qk+1] - \left(\frac{\beta zq}{(z-\omega)^2} - \frac{\beta}{z-\omega}\right)\right| - \left|\sum_{k=1}^{\infty}[k]_q^n\ a_k z^k[qk-1] - \left(\frac{\beta}{z-\omega} + \frac{\beta zq}{(z-\omega)^2}\right)\right|.$

When $|z|$=1, therefore

$\leq \sum_{k=1}^{\infty}[k]_q^n\ a_k[qk+1] + \frac{\beta q}{(1-\omega)^2} - \frac{\beta}{1-\omega} - \sum_{k=1}^{\infty}[k]_q^n\ a_k[qk-1] - \frac{\beta}{1-\omega} - \frac{\beta q}{(1-\omega)^2}$

$$\leq \sum_{k=1}^{\infty}[k]_q^n\ a_k[qk+1-qk+1] - \frac{2\beta}{1-\omega}$$

$$\leq 2\sum_{k=1}^{\infty}[k]_q^n\ a_k - \frac{2\beta}{1-\omega} \leq 0$$

Therefore by the maximum modulus principle, we have $f \in £_q^n(\omega,\beta)$.

Corollary 1: *If* $f \in £_q^n(\omega,\beta)$ *, then we have*

$$a_k \leq \frac{\beta}{[k]_q^n\,(1-\omega)} \tag{10}$$

Equality is attained for the function f:

$$f(z) = \frac{\beta}{z-\omega} + \frac{\beta}{[k]_q^n\,(1-\omega)} z^k$$

Let $£_q^n(\omega,\beta,m) \subset £_q^n(\omega,\beta)$ consisting of functions:

$$f(z) = \frac{\beta}{z-\omega} + \frac{\beta m}{(1-\omega)} z + \sum_{k=2}^{\infty} a_k z^k, \tag{11}$$

with $0 \leq m < 1$.

Theorem 2: *Let f(z) be defined by (4), then f (z)* $\in £_q^n(\omega,\beta,m)$ *if and only if*

$$\sum_{k=2}^{\infty}[k]_q^n\, a_k \leq \frac{\beta}{1-\omega}(1-m), \tag{12}$$

Proof: Assume that

$$a_1 = \frac{\beta m}{(1-\omega)}, (0 \leq m < 1)\,, \tag{13}$$

In (9), we have

$$\frac{\beta m}{(1-\omega)} + \sum_{k=2}^{\infty}[k]_q^n\, a_k \leq \frac{\beta}{(1-\omega)}, \tag{14}$$

Which implies (13), the equality accurse for

$$f(z) = \frac{\beta}{z-\omega} + \frac{\beta m}{(1-\omega)} z + \sum_{k=2}^{\infty} \frac{\beta(1-m)}{[k]_q^n(1-\omega)} z^k \tag{15}$$

Corollary 2: *If f (z)* $\in £_q^n(\omega,\beta,\sigma,m)$

$$a_k \leq \frac{\beta(1-m)}{[k]_q^n\,(1-\omega)}\,, \qquad (k \geq 2) \tag{16}$$

Theorem 3: *If $f(z) \in £_q^n(\omega, \beta, m)$, then*

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta}{[2]_q^n\,(1-\omega)}(1-m), \tag{17}$$

Proof: Let $f(z) \in £_q^n(\omega, \beta, m)$.Then in view of (13), we have

$$\sum_{k=2}^{\infty} [k]_q^n\, a_k \leq \frac{\beta}{1-\omega}(1-m). \tag{18}$$

Then $[2]_q^n(1-\omega)\sum_{k=2}^{\infty} a_k \leq \beta(1-m)$

which is immediately yields the result.

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