

Approximately maximal submodules

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Abstract

As a generalization of the maximal sub-module concept, we introduce and investigate the idea of approximately maximal sub-modules in this paper, in which an actual sub-module S of D is named approximately maximal, if whenever a sub-module $S \subseteq K \subseteq D$ implies that $K + Soc(D) = D$. Numerous features of maximal sub-modules are taken into consideration. Additionally, the term AP-module refers to a module in which each suitable sub-module is roughly maximal. Also, we examine certain characteristics, characterizations, and examples about this class of these modules.

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1. Introduction

Let D be a unitary left R -module, and R have a commute ring with the unit. It is widely recognized that an actual sub-module S of D is referred to as maximal if and only if W is a sub-module of D with the implication that $S \subsetneq K \subseteq D$ implies that $K = D$, or if there is no actual sub-module of D that contains S correctly [1]. In [2], as a diversification of the maximal class sub-modules, Inaam and Riyadh invented the concept of almost maximal sub-modules, where a sub-module N of D is referred to as maximal if every time W is a good sub-module of D with $S \subsetneq K \subseteq D$ implies that $K = D$, where a sub-module K of D is considered as essential for each sub-module L of D with $K \cap L = (0)$ then $L = (0)$ [1]. In [3], Hatem developed a new generalization

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known as semi-maximal sub-modules, where a sub-module S of an R -module D is named semi maximal, if $\frac{D}{S}$ is a semi-simple R -module. The idea of approximately maximal sub-modules, a generalization of the class of maximal sub-modules that refers to an actual sub-module S of D as approximately maximal, is introduced in this study if a sub-module K of D containing S actually implies that $K + \text{Soc}(D) = D$, where $\text{Soc}(D)$ is the Socle of D and define as the sum of all simple sub-module of D [4]. We examine the primary characteristics of the class of roughly maximal sub-modules in section 2. Additionally, we investigate the connection between this idea and maximal sub-modules. The AP module where every non-zero sub-module is approximately maximal, is defined in Section 3. We look at the addition of two AP modules directly.

Approximately Maximal Sub-modules

The class of approximately maximal sub-modules, which is a generalization of maximal sub-modules, is introduced in this section. We investigate the important features of these sub-modules. First, let's define what we mean by it.

Definition 1: Approximately maximal (approximately max) describes an actual sub-module S of an R -module D , if $S \subsetneq K$ implies $K + \text{Soc}(D) = D$, where $\text{Soc}(D)$ is the Socle of D . An ideal J of a ring R is named approximately max if J is an approximately max sub-module of the R -module R .

Remark 1: Every maximal sub-module is approximately max, while the opposite is frequently untrue.

Example 1: The sub-module $(\bar{0})$ in the Z_6 as a Z module is approximately max since $(\bar{0}) \subsetneq \{\bar{0}, \bar{2}, \bar{4}\}$ and $\{\bar{0}, \bar{2}, \bar{4}\} + \text{Soc}(Z_6) = Z_6$, where $\text{Soc}(Z_6) = Z_6$ but not maximal.

Example 2: Let D be a simple module; hence $\{0\}$ is its maximal sub-module.

Example 3: If $\text{Soc}(D) = D$, then D is semi-simple and hence every approximately max sub-module is maximal when D is named semi-simple R -module if all sub-module of D is a summand of D , such as Z_6 as Z module [3].

Example 4: S is a maximal sub-module if and only when D has no simple sub-module $\text{Soc}(D) = (0)$, and S is an almost maximal sub-module.

Example 5: If S and W are actual sub-modules of an R -module D where $S \subseteq W$. If S is an approximately max sub-module of D , hence W is an approximately max sub-module of D .

Proof: Suppose K is an actual sub-module of D and $W \subsetneq K$. Beside $S \subseteq W$, thus $S \subsetneq K$. However, S is an approximately max sub-module; thus $K + \text{Soc}(D) = D$.

Example 6: Let A and B be two actual sub-modules of an R -module D where $A \cap B$ is an approximately max sub-module so that both of A and B are approximately max sub-modules. In general, the inverse is not true, as in the $\mathbb{Z}$ module $\mathbb{Z}_{12}$, each of $A = 2\mathbb{Z}_{12}$ and $B = 3\mathbb{Z}_{12}$ are approximately max sub-module but $A \cap B = 6\mathbb{Z}_{12}$ is not approximately the max sub-module. The actual sub-module containing $A \cap B$ is A and $A + \text{Soc}(\mathbb{Z}_{12}) \neq \mathbb{Z}_{12}$, where $\text{Soc}(\mathbb{Z}_{12}) = 2\mathbb{Z}_{12}$.

Example 7: For instance, an approximately max sub-module need not be an almost maximal sub-module, the sub-module $4\mathbb{Z}_{12}$ in the $\mathbb{Z}$ module $\mathbb{Z}_{12}$ is almost Maximal since, $4\mathbb{Z}_{12} \subsetneq 2\mathbb{Z}_{12}$ and $2\mathbb{Z}_{12}$ is an essential sub-module of $\mathbb{Z}_{12}$. but it is not approximately max since $2\mathbb{Z}_{12} + \text{Soc}(\mathbb{Z}_{12}) \neq \mathbb{Z}_{12}$.

Proposition 1: Let $f: D_1 \rightarrow D_2$ be an epimorphism. If S is an approximately max sub-module of D_1 such that $\text{Ker} \subseteq S$, then $f(S)$ is an approximately max sub-module of D_2 .

Proof: Assume K be a sub-module of D_2 and $f(S) \subsetneq K$, to show that $K + \text{Soc}(D_2) = D_2$. Since S is an approximately max sub-module of D_1 and S is a suitable sub-module of D_1 , also since $\text{ker} f \subseteq S$, then we get $f(S)$ is an actual sub-module of D_2 . In contrast, $f(S) \subsetneq K$, then $N \subsetneq f^{-1}(K)$, hence $f^{-1}(K) + \text{Soc}(D_1) = D_1$. This implies that $f(f^{-1}(K) + \text{Soc}(D_1)) = f(D_1)$. Thus $f(f^{-1}(K)) + f(\text{Soc}(D_1)) = D_2$, hence $D_2 = K + f(\text{Soc}(D_1)) \subseteq K + \text{Soc}(D_2)$, so $K + \text{Soc}(D_2) = D_2$. Thus $f(S)$ is an approximately max sub-module of D_2 .

Corollary 1: If S is an approximately max sub-module of an R -module D , hence $\frac{S}{K}$ is an approximately max sub-module of $\frac{D}{K}$ for each module K of D such that module K contained S .

Remark 2: It is just not required that $[S: D]$ is an approximately max ideal of R if S is an approximately max sub-module of D . Example the sub-module $(\bar{0})$ in the $\mathbb{Z}$ module $\mathbb{Z}_6$ is an approximate max, but the $[(\bar{0}) :_{\mathbb{Z}} \mathbb{Z}_6] = 6\mathbb{Z}$ is not approximately the max ideal in the ring $\mathbb{Z}$. Observe that an R -module D is considered to be multiplied where if an ideal of R exists and $S = D$ [5].

Lemma 1: Let D be a faithful multiplication R -module, then $\text{Soc}(R)D = \text{Soc}(D)$, [5, Coro. 2.14].

We now include the following elements.

Proposition 2: Let D a reliable R -module for multiplication. If and just if there is an approximately max ideal J of R and $S = JD$, then a sub-module S of D is approximately max.

Proof: $\Rightarrow$) Consider the case where S is an approximately max sub-module. Since D is a module for multiplication, then $S = JD$ for some ideal J of R . There is an ideal K of R that, if I is not nearly Maximal ideal, such that $J \subsetneq K + \text{Soc}(R) \subseteq R$, then $JD \subsetneq KD + \text{Soc}(R)D \subseteq RD$. But $\text{Soc}(R)D \subseteq \text{Soc}(D)$, thus $S \subsetneq KD + J(D) \subseteq D$, that is S is not approximately maxed sub-module which is in opposition to what we assumed.

$\Leftarrow$) If S isn't exactly the approximately max sub-module, then there is an actual sub-module L of D with $S \subsetneq L$ and $L + \text{Soc}(D) \neq D$. Since D is multiplication, so $L = KD$ for a few ideal K of R , and through supposition, it is an approximately ideal I of R such that $S = ID$. Now $ID \subsetneq KD + \text{Soc}(R)D \subseteq RD$. D being faithful and multiplication, so by [6], (Corollary (2.4)), we obtain $I \subsetneq K + \text{Soc}(R) \subseteq R$, such is I is not approximately max as which inappropriate.

Corollary 1: Let D be a multiplication and faithful module. A sub-module S of D is approximately max only in the case of $[S:D]$ is approximately the max ideal of R .

Proof: As of D is a multiplication, hence for a suitable ideal I of R . Notice that $S = [S:D]D$ [6], and using Prop (2), the outcome follows.

Recall that an actual sub-module S of an R module D is named prime or every time $rx \in S$ for $r \in R$ and $x \in D$, then neither $x \in S$ nor $r \in [S:D]$, and an ideal J of a ring R is named prime if every time $a, b \in R$ where $ab \in J$, demonstrates that neither $a \in J$ nor $b \in J$, [7].

Remark 3: A approximately max sub-module is not necessarily a prime sub-module. Examples include: $(\bar{0})$ is an approximately max sub-module of the Z module Z_6 , but it is not a prime sub-module of Z_6 .

This section ends with the example below that shows the direct sum of two approximately max sub-modules.

Example 8: Consider the Z module Z_6 , each of these sub-modules $(\bar{2})$ and $(\bar{3})$ are approximately max sub-modules of Z_6 , but $(\bar{2}) \oplus (\bar{3}) = Z_6$ is not approximately the max sub-module of Z_6 .

2. AP-Modules

This section introduces the concept of the AP module, beginning with the following definition.

Definition 2: An R -module D is named AP-module, If each actual sub-module of D is approximately max. A ring R is named AP-ring if all actual ideal of R is an approximate max.

Examples 9: $\mathbb{Z}_6$ is AP-module as a $\mathbb{Z}$ module.

Example 10: since the sub-module (6) of $\mathbb{Z}$ is not approximately max, $\mathbb{Z}$ as $\mathbb{Z}$ is not an AP module.

Proposition 5: An AP module is an epimorphic form of an AP module.

Proof: Results from Prop (1).

The heredity trait varies between an AP module over a ring R , and R itself is provided by the following theorems.

Theorem 2: Let D be a finite generate faithfully multiplicative module. Hence D is an AP-model if, and just if R is AP-model.

Proof: $\Rightarrow$) Suppose this D is an AP module, and J be an ideal of R . Put $S = JD$. By hypothesis, S is an approximately max sub-module of D ; thus, by (Prop 2), J is an approximately max ideal.

$\Leftarrow$) Consider that R is an AP module and let S be an actual sub-module of D . For D is a multiplication. Hence, there is an ideal J of R where $S = JD$. By observation, J is an approximately max ideal of R , so by (Prop. 2) S is an approximately max sub-module; thus D is an N AP-module.

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