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Approximate solutions of a certain initial value problem

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Abstract

This work aims to discuss the approximation of an initial value problem for integro-differential equation (IDEqs). of Volterra-Fredholm type (V-FIDEq), with given initial conditions (IC). These solutions are a class of conditions functions $C(R_+ R^n)$, where R_+, = [0, ∞), and R^n represents the n-dimensional Euclidean space and stands in for the norm $|.|$.

We are presenting several types of particular inequality with candid appreciation. These types are applied to find out the results of this paper.

Subject Classification: *81Q05, 37C50.*

Keywords: *Volterra type, Fredholm, Integral inequality, Closeness of solution.*

1. Introduction

We are presenting several types of particular inequality with candid appreciation. These types are applied to find out the results of this paper.

Most (IVP) and boundary value problems (BVP) about ordinary differential equations (ODEqs) and partial differential equations (PDEqs) can be

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converted into Volterra and Fredholm integral equations (VFDEq) [1, 2]. Integral equations (IEqs) have appeared widely in applied Mathematics, Statistics, Economics, and Control systems [3, 4, 5].

This research aims to expand the results of Pachpatte, G. [6]. Several other approaches have been presented recently to deal with IVPs and integro-differential equations approximately, see [7, 8, 9, 10]. This paper studied basic properties, such as the convergence of solutions to the IVP and obtaining the conditions that achieve convergence, using integral inequalities with an explicit estimate to prove the results.

2. Main Results

Take into consideration the initial value problem (IVP), second-order Volterra-Fredholm of integro-differential equation (VFIDEq),

$$\psi''(\mathfrak{z}) = g\big(\mathfrak{z}, \psi(\mathfrak{z}), \psi'(\mathfrak{z}), \psi''(\mathfrak{z}), \mathbb{E}\psi(\mathfrak{z}), \mathbb{F}\psi(\mathfrak{z})\big) \tag{1}$$

for $t \epsilon R_+ = [0, \infty)$, with the given (IC):

$$\psi(0) = \psi_0, \psi'(0) = \overline{\psi_{0,}} \tag{2}$$

where

$$\mathbb{E}\psi(\mathfrak{z}) = \int_0^t [\![\mathbb{E}(\mathfrak{z}, \sigma, \psi(\sigma),]\!]\psi'(\sigma), \psi''(\sigma)\big) d\sigma \; (3)$$

And

$$\mathbb{F}\psi(\mathfrak{z}) = \int_0^{\propto} \mathbb{F}(\mathfrak{z}, \sigma, \psi(\sigma), \psi'(\sigma), \psi''(\sigma)) d\sigma, \tag{4}$$

and $g, \mathbb{E}, \mathbb{F}$ provided with functions, ψ finds the unidentified function and indicates a derivative with respect to σ.

Let $g \epsilon C(R_+ \times R^n \times R^n \times R^n \times R^n \times R^n, R^n$, and for $\sigma \leq \mathfrak{z}$, $\mathbb{E}, \mathbb{F} \epsilon C(R_+^2 \times R^n \times R^n \times R^n, R^n)$. Let $\psi_i(\mathfrak{z}) \epsilon C(R_+, R^n), i = 1,2$ be structured so that $\psi''_i(\mathfrak{z})$ exists $\mathfrak{z} \in R_+$.

$$\left|\psi''_i(\mathfrak{z}) - g\big(\mathfrak{z}, \psi_i(\mathfrak{z}), \psi_i{}'(\mathfrak{z}), \psi_i{}''(\mathfrak{z}), \mathbb{E}\psi_i(\mathfrak{z}), \mathbb{F}\psi_i(\mathfrak{z})\big)\right| \leq \varepsilon_{i,} \tag{5}$$

$$\psi_i(0) = \psi_i, \psi_i{}'(0) = \overline{\psi_{\iota,}}. \tag{6}$$

Then we call $\psi_i(\mathfrak{z})$ the $\varepsilon_{i,} -$ approximation answers in relation to (1, 2).

In this paper, we need the following Lemma.

Lemma 1: *Let* $\mathbb{U}, a \epsilon C(R_+, R_+)$ *, and for all* $\sigma \leq s \leq \mathfrak{z}$, $r(\mathfrak{z}, s), Q_i(\mathfrak{z}, s) \epsilon C(R_{+,}^2 R_+)$, *and* $a(\mathfrak{z}), r(\mathfrak{z}, s)$ *is nondecreasing for* $\mathfrak{z} \epsilon R_+$, *and* $\mathbb{L} \geq 0$, *is a constant. If*

$$\mathbb{U}(\mathfrak{z}) \leq a(\mathfrak{z}) + \frac{1}{1-\mathbb{L}}\left(\int_0^{\alpha} Q_2(\mathfrak{z}, \sigma)\, \mathbb{U}(\sigma) d\sigma\right) + \int_0^{\mathfrak{z}} \{Q_1(\mathfrak{z}, s)\mathbb{U}(s) ds + r(\mathfrak{z}, s)\big[\mathbb{L}\mathbb{U}(s) + \int_0^s Q_1(s, \sigma)\mathbb{U}(\sigma) d\sigma + \int_0^{\alpha} Q_2(s, \sigma)\mathbb{U}(\sigma) d\sigma\big]\} ds \tag{7}$$

Then

$$\mathbb{U}(\mathfrak{z}) \leq a(\mathfrak{z})\mathcal{N}(\mathfrak{z})\exp(\int_0^{\mathfrak{z}} \mathbb{k}(t, s) ds, \tag{8}$$

where

$$\mathcal{N}(\mathfrak{z}) = \mathbb{L} + \frac{\mathbb{L}}{\gamma}\left\{\frac{1}{1-\mathbb{L}} \int_0^{\propto} Q_2(\mathcal{C},\sigma)e^{\int_0^{\sigma}\mathbb{k}(\mathfrak{z},s)ds} d\,\sigma\right\} (9)$$

$$\gamma = 1 - \frac{1}{1-\mathbb{L}} \int_0^{\propto} Q_2(\psi,\sigma)e^{\int_0^{\sigma}\mathbb{k}(\mathfrak{z},s)ds} d\,\sigma \tag{10}$$

Proof: Let $a(\mathfrak{z}) > 0$, we fixed $T, where\ 0 < \psi < \propto$, from (7), we obtain:

$$\frac{\mathbb{U}(\mathfrak{z})}{a(\mathfrak{z})} \leq 1 + \frac{1}{1-\mathbb{L}}{}'\left(\int_0^{\alpha} Q_2(\psi,\sigma)\frac{\mathbb{U}(\sigma)}{a(\mathfrak{z})}d\sigma + \int_0^{\mathfrak{z}}\left\{Q_1(\psi,s)\frac{\mathbb{U}(s)}{a(\mathfrak{z})}ds + r(\psi,s)\left[\mathbb{L}\frac{\mathbb{U}(s)}{a(\mathfrak{z})} + \int_0^{s} Q_1(s,\sigma)\frac{\mathbb{U}(\sigma)}{a(\mathfrak{z})}d\sigma + \int_0^{\alpha} Q_2(s,\sigma)\frac{\mathbb{U}(\sigma)}{a(\mathfrak{z})}d\sigma\right]\right\}ds\right) \tag{11}$$

Since $a(\mathfrak{z})$ is a non-decreasing function on R_+, we get $a(s) \leq a(\mathfrak{z})$, for $0 \leq s \leq \mathfrak{z}$, therefore $\frac{1}{a(\mathfrak{z})} \leq \frac{1}{a(s)}$, and in the same way, we have $\frac{1}{a(s)} \leq \frac{1}{a(\sigma)}, for\ 0 \leq \sigma \leq s$, then for $0 \leq \mathfrak{z} \leq \psi$, we get

$$\frac{\mathbb{U}(\mathfrak{z})}{a(\mathfrak{z})} \leq 1 + \frac{1}{1-\mathbb{L}}\left(\int_0^{\alpha} Q_2(\psi,\sigma)\frac{\mathbb{U}(\sigma)}{a(\sigma)}d\sigma + \int_0^{\mathfrak{z}}\left\{Q_1(\psi,s)\frac{\mathbb{U}(s)}{a(s)}ds + r(\psi,s)\left[\mathbb{L}\frac{\mathbb{U}(s)}{a(s)} + \int_0^{s} Q_1(\psi,\sigma)\frac{\mathbb{U}(\sigma)}{a(\sigma)}d\sigma + \int_0^{\alpha} Q_2(\psi,\sigma)\frac{\mathbb{U}(\sigma)}{a(\sigma)}d\sigma\right]\right\}ds\right) \tag{12}$$

We define $\mathbb{Z}(\mathfrak{z})$ on the right-hand side of (12). Therefore, we have:

$$\frac{\mathbb{U}(\mathfrak{z})}{a(\mathfrak{z})} \leq \mathbb{Z}(\mathfrak{z}), \text{ and } \mathbb{Z}(0) = \frac{1}{1-\mathbb{L}}\int_0^{\alpha} Q_2(\psi,\sigma)\frac{\mathbb{U}(\sigma)}{a(\sigma)}d\sigma,$$

$$\mathbb{Z}'(\mathfrak{z}) = \frac{1}{1-\mathbb{L}}\left(Q_1(T,\mathfrak{z})\mathbb{Z}(\mathfrak{z}) + r(\psi,\mathfrak{z})\{\,\mathbb{L}\mathbb{Z}(\mathfrak{z}) + \int_0^{\mathfrak{z}} Q_1(\psi,\sigma)\mathbb{Z}(\mathfrak{z})d\sigma + \int_0^{\alpha} Q_2(\psi,\sigma)\mathbb{Z}(\mathfrak{z})d\sigma\,\}\right), \mathfrak{z}\epsilon[0,\psi], \tag{13}$$

Define the function

$$\mathfrak{E}(\mathfrak{z},\psi) = \mathbb{L}\mathbb{Z}(\mathfrak{z}) + \int_0^{\mathfrak{z}} Q_1(\psi,\sigma)\mathbb{Z}(\sigma)d\sigma + \int_0^{\alpha} Q_2(\psi,\sigma)\mathbb{Z}(\sigma)d\sigma, \tag{14}$$

$$\mathbb{Z}'(\mathfrak{z},\psi) = \frac{1}{1-\mathbb{L}}\left(Q_1(\psi,\mathfrak{z})\mathbb{Z}(\mathfrak{z}) + r(\psi,\mathfrak{z})\mathfrak{E}(\mathfrak{z},\psi)\right), \mathfrak{E}(0,\psi) = \mathbb{L} + \frac{1}{1-\mathbb{L}} \tag{15}$$

$$\mathfrak{E}'(\mathfrak{z},\psi) = \left(\frac{\mathbb{L}}{1-\mathbb{L}}\{Q_1(\psi,\mathfrak{z}) + r(\psi,\mathfrak{z})\} + Q_1(\psi,\mathfrak{z})\right)\mathfrak{E}(\mathfrak{z},\psi),$$

Let

$$\mathbb{k}(\psi,\mathfrak{z}) = \left(\frac{\mathbb{L}}{1-\mathbb{L}}\{Q_1(\psi,\mathfrak{z}) + r(\psi,\mathfrak{z})\} + Q_1(\psi,\mathfrak{z})\right)\frac{\mathfrak{E}'(\mathfrak{z},\psi)}{\mathfrak{E}(\mathfrak{z},\psi)} \leq \mathbb{k}(\psi,\mathfrak{z}),$$

$\mathfrak{E}(\mathfrak{z},\psi) \leq \mathfrak{E}(0,\psi)\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\psi,s)ds)$, Since ψ is fixed, therefore, for all $\psi = \mathfrak{z}$, we obtain:

$$\mathbb{Z}(\mathfrak{z}) \leq \mathfrak{E}(0,\mathfrak{z})\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds), \tag{16}$$

$$\mathfrak{E}(0,\mathfrak{z}) \leq \mathbb{L} + \mathfrak{E}(0,\mathfrak{z})\left[\frac{1}{1-\mathbb{L}}\int_0^{\alpha} Q_2(\mathfrak{z},\sigma)\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds)\,d\sigma\right], \mathfrak{E}(0,\mathfrak{z}) \leq \frac{\mathbb{L}}{\gamma}.$$

$$\mathfrak{E}(0,\mathfrak{z}) \leq \mathbb{L} + \frac{\mathbb{L}}{\gamma}\left[\frac{1}{1-\mathbb{L}}\int_0^{\alpha} Q_2(\mathfrak{z},\sigma)\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds)\,d\sigma\right] = \mathcal{N}(\mathfrak{z}) \tag{17}$$

By using (17) into (16)

$$\mathbb{Z}(\mathfrak{z}) \leq \mathbb{L} + \frac{\mathbb{L}}{\gamma}\left[\frac{1}{1-\mathbb{L}}\int_0^{\alpha} Q_2(\mathfrak{z},\sigma)\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds)\,d\sigma\right]\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds),$$

$$\mathbb{U}(\mathfrak{z}) \leq a(\mathfrak{z})\mathcal{N}(\mathfrak{z})\exp(\int_0^{\mathfrak{z}}\mathbb{k}(\mathfrak{z},s)ds)$$

Theorem 1: *Let* $\mathscr{g}, \mathbb{E}$ *and* $\mathbb{F}$ *in equation (1) satisfy the hypotheses*

$$|\mathscr{g}(\mathfrak{z}, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5) - \mathscr{g}(\mathfrak{z}, \overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3}, \overline{\omega_4}, \overline{\omega_5})| \leq \mathbb{L}\{|\omega_1 - \overline{\omega_1}| + |\omega_2 - \overline{\omega_2}| + |\omega_3 - \overline{\omega_3}| + |\omega_4 - \overline{\omega_4}| + |\omega_5 - \overline{\omega_5}|\} \quad (18)$$

$$|\mathbb{E}(\mathfrak{z}, \sigma, \omega_1, \omega_2, \omega_3) - \mathbb{E}(\mathfrak{z}, \sigma, \overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3})| \leq \mathcal{P}_1(\mathfrak{z}, \sigma)\{|\omega_1 - \overline{\omega_1}| + |\omega_2 - \overline{\omega_2}| + |\omega_3 - \overline{\omega_3}|\}, \quad (19)$$

$$|\mathbb{E}(\mathfrak{z}, \sigma, \omega_1, \omega_2, \omega_3) - \mathbb{E}(\mathfrak{z}, \sigma, \overline{\omega_1}, \overline{\omega_2}, \overline{\omega_3})| \leq \mathcal{P}_1(\mathfrak{z}, \sigma)\{|\omega_1 - \overline{\omega_1}| + |\omega_2 - \overline{\omega_2}| + |\omega_3 - \overline{\omega_3}|\}, \quad (20)$$

Where $\mathbb{L} \geq 0$ is a constant, $\mathbb{L} \leq 1$, and for $\mathcal{P}_1(\mathfrak{z}, \sigma), \frac{\partial}{\partial t}\mathcal{P}_1(\mathfrak{z}, \sigma), \mathcal{P}_2(\mathfrak{z}, \sigma)$, $\frac{\partial}{\partial t}\mathcal{P}_2(\mathfrak{z}, \sigma) \,\epsilon C(R_+^2, R_+)$. Assume $\psi_i(\mathfrak{z})$ $(i = 1,2), \varepsilon_{i,}$ −approximate solution of (1, 2) and (6) on $R_{+,}$ such that

$$|\psi_1 - \psi_2| \leq \delta_{,} |\overline{\psi_1} - \overline{\psi_2}| \leq \bar{\delta}_{,} \quad (21)$$

$$|\psi_1(\mathfrak{z}) - \psi_2(\mathfrak{z})| + |\psi_1{}'(\mathfrak{z}) - \psi_2{}'(\mathfrak{z})| + |\psi_1{}''(\mathfrak{z}) - \psi_2{}''(\mathfrak{z})| \leq \beta(\mathfrak{z}) \exp\left(\frac{1}{1-\mathbb{L}}\right)\left\{\int_0^\alpha \mathcal{P}_2(\mathfrak{z}, \sigma)\mathbb{A}(\sigma)d\sigma + \int_0^{\mathfrak{z}}(\mathfrak{z} - s + 1)\left[\mathbb{L}\mathbb{A}(s) + \mathcal{P}_1(\mathfrak{z}, s)\mathbb{A}(s) + \int_0^s \mathcal{P}_1(s, \sigma)\mathbb{A}(\sigma)d\sigma + \int_0^\alpha \mathcal{P}_2(s, \sigma)\mathbb{A}(\sigma)d\sigma\right]ds\right\} \quad (22)$$

where

$$\beta(\mathfrak{z}) = \frac{(\varepsilon_1 + \varepsilon_2)\left(\frac{\mathfrak{z}^2}{2} + \mathfrak{z} + 1\right) + \delta + \bar{\delta}(1 + \mathfrak{z})}{1 - \mathbb{L}} \quad (23)$$

Proof: Since $\psi_i(\mathfrak{z}), (i = 1,2)$, $t \in R_+$, the ϵ_i (1, 2), and (6), we have (5). By taking $\mathfrak{z} = \tau$, in (5) and integrating both sides with respect to τ for 0 to, we get both with respect to τ, we get

$$\epsilon_i \geq \left|\int_0^{\mathfrak{z}} \psi_i{}''(\tau) - \mathscr{g}\big(\tau, \psi_i(\tau), \psi_i{}'(\tau), \psi_i{}''(\tau), \mathbb{E}\psi_i(\tau), \mathbb{F}\psi_i(\tau)\big)d\tau\right|$$

$$= \left|\psi'_i(\mathfrak{z}) - \overline{\psi_i} - \int_0^{\mathfrak{z}} \mathscr{g}\big(\tau, \psi_i(\tau), \psi_i{}'(\tau), \psi_i{}''(\tau), \mathbb{E}\psi_i(\tau), \mathbb{F}\psi_i(\tau)\big)d\tau\right| \quad (24)$$

Integrating (24), we have $\epsilon_i \frac{\mathfrak{z}^2}{2} \geq |\psi_i(\mathfrak{z}) - \{\psi_i + \overline{\psi_i}\mathfrak{z}\} -|$

$$\left|- \int_0^{\mathfrak{z}}(\mathfrak{z} - s)\mathscr{g}\big(s, \psi_i(s), \psi_i{}'(s), \psi_i{}''(s), \mathbb{E}\psi_i(s), \mathbb{F}\psi_i(s)\big)ds\right| \quad (25)$$

using equation (24) and (25), and using

$$|\mathfrak{M} - \mathfrak{N}| \leq |\mathfrak{M}| + |\mathfrak{N}|, |\mathfrak{M}| - |\mathfrak{N}| \leq |\mathfrak{M} - \mathfrak{N}| \quad (26)$$

We have observed that

$$(\epsilon_1 + \epsilon_1)\mathfrak{z} \geq \left|\psi'_1(\mathfrak{z}) - \psi'_2(\mathfrak{z})\right| - |\overline{\psi_1} - \overline{\psi_2}| - \int_0^{\mathfrak{z}}\Big|\mathscr{g}\big(s, \psi_1(s), \psi_1{}'(s), \psi_1{}''(s), \mathbb{E}\psi_1(s), \mathbb{F}\psi_1(s)\big) - \mathscr{g}\big(s, \psi_2(s), \psi_2{}'(s), \psi_2{}''(s), \mathbb{E}\psi_2(s), \mathbb{F}\psi_2(s)\big)ds\Big|, \quad (27)$$

$$(\epsilon_1 + \epsilon_1)\frac{\mathfrak{z}^2}{2} \geq |\psi_1(\mathfrak{z}) - \psi_2(\mathfrak{z})| - |\psi_1 - \psi_2| - |(\overline{\psi_1} - \overline{\psi_2})\mathfrak{z}| - \int_0^{\mathfrak{z}}\Big|(\mathfrak{z} - s)\mathscr{g}\big(s, \psi_1(s), |\psi_1{}'(s)|, \psi_1{}''(s), \mathbb{E}\psi_1(s), \mathbb{F}\psi_1(s)\big) - (\mathfrak{z} - s)\mathscr{g}\big(s, \psi_2(s), \psi_2{}'(s), \psi_2{}''(s), \mathbb{E}\psi_2(s), \mathbb{F}\psi_2(s)\big)\Big|\, ds \quad (28)$$

$$(\epsilon_1+\epsilon_1)\geq |\psi_1''(\mathfrak{z})-\psi_2''(\mathfrak{z})|-\big|\mathcal{g}\big(\mathfrak{z},\psi_1(\mathfrak{z}),|\psi_1'(\mathfrak{z})|,\psi_1''(\mathfrak{z}),\mathbb{E}\psi_1(\mathfrak{z}),\mathbb{F}\psi_1(\mathfrak{z})\big)-\mathcal{g}\big(\mathfrak{z},\psi_2(\mathfrak{z}),|\psi_2'(\mathfrak{z})|,\psi_2''(\mathfrak{z}),\mathbb{E}\psi_2(\mathfrak{z}),\mathbb{F}\psi_2(\mathfrak{z})\big)\big|, \tag{29}$$

Let $\mathbb{A}(\mathfrak{z})=|\psi_1(\mathfrak{z})-\psi_2(\mathfrak{z})|+|\psi_1'(\mathfrak{z})-\psi_2'(\mathfrak{z})|+|\psi_1''(\mathfrak{z})-\psi_2''(\mathfrak{z})|$. from (27-29), and using hypotheses, we have:

$$\mathbb{A}(\mathfrak{z})\leq\beta(\mathfrak{z})+\frac{1}{1-\mathbb{L}}\Big(\int_0^{\alpha}\mathcal{P}_2(\mathfrak{z},\sigma)\mathbb{A}(\sigma)d\sigma+\int_0^{\mathfrak{z}}\big\{\mathcal{P}_1(\mathfrak{z},s)\mathbb{A}(s)+(\mathfrak{z}-s+1)\big[\mathbb{L}\mathbb{A}(s)+\int_0^{s}\mathcal{P}_1(s,\sigma)\mathbb{A}(\sigma)d\sigma+\int_0^{\alpha}\mathcal{P}_2(s,\sigma)\mathbb{A}(\sigma)d\sigma\big]\big\}\,ds\Big) \tag{30}$$

where $\beta(\mathfrak{z})$ given in (23)., $\alpha(\mathfrak{z})$ is not declining for $\mathfrak{z}\in R_+$. Now a suitable use of the Lemma to (30), (22).

3. Conclusion

In this paper, we have studied the IVP for integro-differential equation (IODE) of Volterra-Fredholm type. The integral inequality has been derived and used in the context of this work. We have also obtained the estimation of the outcomes between the two solutions of the IODE that has one variable.

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