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A fractional models of some partial differential equations and analytical approximate solution arising in mathematical sciences

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Abstract

The present paper, a combination of Sumudu Adomian Decomposition approach (STADM)is apply to investigate the analytical solutions for some Fractional Differential Equations(FDEs). The STADM is a new technique using the starting causes. To evaluate the effectiveness and powerful of the suggested technique, several applications for examples are provided to get analytical or numerical results of Gas Dynamics equation, Fornberg - Whitham equation and Klein- Gordon equation. The Caputo sense definition applies to fractional derivatives. In addition, the approaches that were proposed gave convergence series solutions that had components that were simple to determine, and they did so without making use of perturbation or linearization.

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Keywords: *Gas dynamics, Fractional calculus, Klein-gordon equation, Caputo operator, Analytical technique.*

1. Introduction

Integral transforms and Fractional Differential Equations have gained attracted recently in number of scientific fields. In order to solve some applications in applied science, biology, mathematical biology, and dynamics, the fractional calculus has become more and more popular [1]. Nonlinear Partial differential equations (NPDES) describe various physical and chemical events and significantly impact several science and engineering sectors because many fractional techniques cannot be solved easily. Numerous methods, including

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Adomian Decomposition technique(ADM) [5], homotopy perturbation method [4], homotopy analysis method [2], the Fractional variational iteration method [6], the fractional reduced differential transform method (RDTM) [3], and others [7–16] are described in the literature as ways to solve NFPDEs.The purpose of this work is to apply STADM for some applications to obtain analytical or numerical solutions of Gas- Dynamics, Fornberg - Whitham equation and Klein Equation. Thus, it is clear that the analytical solutions obtain both efficiently and facility via a suggested approach.

2. The Methodology of Sumudu Adomian Decomposition Technique

Consider

$$D_\tau^w y(\xi,\tau) + Ly(\xi,\tau) + Ny(\xi,\tau) = Q(\xi,\tau) \tag{1}$$

$$y(\xi,0) = P(x) \tag{2}$$

Applying *Sumudu* transform of the Eq. (1)

$$S_\tau[D_\tau^w y(\xi,\tau)] + S_\tau\big[L[y(\xi,\tau)]\big] + S_\tau\big[N[y(\xi,\tau)]\big] = S_\tau[Q(\xi,\tau)] \tag{3}$$

Then

$$\frac{S_\tau[y(\xi,\tau)]}{u^w} = \sum_{l=0}^{n-1}\frac{y(\xi,0)^l}{u^{w-l}} + S_\tau[Q(\xi,\tau)] - S_\tau[L(y(\xi,\tau)) + N(y(\xi,\tau))]$$

$$S_\tau[y(\xi,\tau)] = y(\xi,0) + u^w\, S_\tau[Q(\xi,\tau)] - u^w S_\tau\big[L\big(y(\xi,\tau)\big) + N\big(y(\xi,\tau)\big)\big] \tag{4}$$

Taking *Sumudu* inverse transform on Eq. (4)

$$y(\xi,\tau) = f(x) + S^{-1}(u^w S\,[q(\xi,\tau)]) - S^{-1}(u^w S[L(y(\xi,\tau) + N(y(\xi,\tau)]) \tag{5}$$

So

$$y(\xi,\tau) = \sum_{i=0}^{\infty} y_i(\xi,\tau) \tag{6}$$

That implies

$$\sum_{i=0}^{\infty} y_i(\xi,\tau) = \sum_{l=0}^{\infty}\frac{y(\xi,0)^{(l)}}{u^{w-l}} + S^{-1}(u^w S[q(\xi,\tau)]) \\ -S^{-1}\big[u^w S[L(\textstyle\sum_{i=0}^{\infty} y_i(\xi,\tau)) + \sum_{i=0}^{\infty} A_i]\big] \tag{7}$$

Finally, we get:

$$y_0(\xi,\tau) = \sum_{k=0}^{\infty}\frac{y(\xi,\tau)^{(k)}}{u^{w-k}} + S^{-1}(s^w S[Q(\xi,\tau)])$$

$$y_1(\xi,\tau) = -S^{-1}\big[u^w S[L(y_0(\xi,\tau)) + A_0]\big]$$

$$y_2(\xi,\tau) = -S^{-1}\Big[u^w S\big[L\big(y_1(\xi,\tau)\big) + A_1\big]\Big]$$

$$\vdots$$

$$y_i(\xi,\tau) = -S^{-1}\big[u^w S[L(y_{i-1}(\xi,\tau)) + A_i]\big]$$

3. Numerical Results

Example 3.1: Let the linear nonhomogeneous Klein -Gordon equation with time -fractional as follows:

$$D_t^\alpha y(x,t) - y_{xx}(x,t) + y(x,t) = 2\sin x \quad 1 < \alpha \le 2 \tag{9}$$

With $y(x\,,0) = \sin x$ and $y_x(x, o) = 1$

Solution: Apply the Sumudu transform to above equation (9), we have:

$$S[D_t^{\alpha} y(\,x, t)] - S[\,y_{xx}(\,x\,, t) - y(\,x, t) + 2\sin x] = 0$$

By using the fractional derivative properties of the Sumudu transform, then

$$S[y(\,x, t)] = \frac{1}{u^{-\alpha}} \sum_{k=0}^{1} u^{k-\alpha}\ [\,y^{\alpha}(\,x\,,0)] + \frac{1}{u^{-\alpha}} S\left[\frac{\partial^2 y(\,x\,,t)}{\partial x^2}\ y(\,x\,, t) + 2\sin x\right]$$

Apply inverse Sumudu transform to above equation we have

$$y_0(\,x, t) = \sum_{k=0}^{1} \left(\frac{\partial^k y(\,x,t)}{\partial t^k}\right)\ u^k + S^{-1}(u^{\alpha} S(u^{\alpha} S(2sinx))$$

$$= sinx + t + 2sinx\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$y_1(x, t) = -2sinx\ \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 4sinx\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$y_2(x, t) = 4sinx\,\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} + 8\ sinx\ \frac{t^{3\alpha}}{\Gamma(2+3\alpha)}$$

And so on, thus, the series solution is given by

$$y(x, t) = \sin x + \left[\,t - \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} +..\right]$$

when $\alpha = 2$ then $y(x, t\,) = \sin x + \sin t$ This is the same exact solution

Example 3.2: Consider homogeneous fractional Gas - Dynamics equation

$$\frac{\partial^{\alpha} X(r,t)}{\partial t^{\alpha}} + X\ \frac{\partial X(r,t)}{\partial t} - X + X^2 = 0\,, 0 < \alpha\ \leq 1 \tag{10}$$

with initial corditon $X(r, 0) = e^{-r}$ (11)

Solution: By using STADM for Eq.(10) and Eq.(11),we get :

$$X_0(r, t\,) = e^{-r}$$

$$X_1(r, t) = \ e^{-r} +\ \frac{e^{-r} t^{\alpha}}{\Gamma(\,\alpha+1)}$$

$$X_2(\,r, t) = e^{-r} + \frac{e^{-r} t^{\alpha}}{\Gamma(\alpha+1)} + \ \frac{e^{-r}\,t^{2\alpha}}{\Gamma(2\alpha+1)}$$

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$$X\ (r\ +) = e^{-r}\ \sum_{m=0}^{\infty}\ \frac{(t^{\alpha})^m}{\Gamma(m\alpha+1)}$$

We can get when α=1, the same exact solution $X(r + t) = e^{-r+t}$

5. Conclusion

We have been successfully applied STADM to provide an approximate analytical solution to different numerical applications with initial conditions such as time Fractional Gas dynamics equation and Klein- Gordon equation. The suggested technique is easy to implement, suitable and effective for achieving the results of these problems and STADM provide the converge series results with easy. The STADM gives us a solution of an infinite series that converges at rapidly and has a low rate of error. In the future, we propose to use STADM in combination with other operators, such as Caputo-Fabrizio and Conformable fractional derivative, to solve other problems that come up in different fields of science.

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