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The complex integral transform complex Sadik transform of error function

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Abstract

Recently, the focus of mathematics scientists on the use of integral transforms in solving problems in various fields such as engineering, science, and other scientific fields has led to the need to know the integral transforms of the error function in order to evaluate improper integrals. In this paper, we will use transform (complex Sadik transform) and prove its ability to evaluate improper integrals that have error functions.

Subject Classification: *44-XX.*

Keywords: *Complex Sadik transform, Error function, Complementary error function, Improper integral.*

1. Introduction

Integrated transformations have proven their superior ability in finding solutions to various problems in science, engineering, radioactive decay, and many other fields. When using any integral transformation to solve any problem, we need to know the integral transformation of error functions in order to evaluate the incorrect integrals containing the error function. Recently, many integrated transformations have been developed by many researchers, such as

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the transformation of Kamal, Blas, Muhannad, Sadiq, Aboud, Rishi, Tariq, and Zaki, which had a great role in solving many problems such as the problems of heat conduction, electrical circuits and the movement of particles under Gravity and other problems. Newly, we notice that there is competition among researchers, as some are working to find new integrative transformations in order to use them in solving the problems facing us in various scientific and engineering journals. Where E.A.Mansoura et al introduced the complex integral transform[1] that applies to decrease the primary issue of an unpretentious algebraic equation. In 2020, the Zughair transform was used to solve the differential equation of radioactive decay[2]. In 2021 suggested properties of the complex SEE integral change that is utilized to resolve the systems of(ODEs)[3]. In 2022 the Rangaig integral transform was suggested and used to solve the first- and second-order ordinary differential equations[4]. In[5] E.A. Mansour et al presented the complex SEE integral transform of Bessel's functions and its applications. E.A. Mansour et al utilized the SEE to puzzle out the accurate solution to Abel's integral equation [6]. In[7,8], E.A. Mansour et al utilized the SEE to resolve the Faltung type Volterra integrodifferential equation of the first kind. In[9], the capability of the complex SEE (Sadiq-Emad-Eman) transform was discussed to solve Abel's integral equation. E.Kuffi et al employed the Al-Tememe transform to resolve the developed problem of automatic missile control [10]. E.Kuffi et al suggested the Emad-Falih Transform and used it to solve linear ordinary differential equations with constant coefficients[11]. In[12] Elaf Sabah Abbas et al to solve the linear partial differential equations with variable coefficients utilized the Al-Tememe transformation. In[13], the authors suggest the Kuffi-Abbas-Jawad KAJ integral transform that is used to solve ordinary differential equations. In[14], the new general complex integral transform was used to solve ordinary differential equations. In[16] Sara Falih Maktoof et al suggested the Emad-Sara Transform solve the linear ordinary differential equations with constant coefficients.

The main objective of this work is to define the integral transform (complex Sadik transform) of the error function and to prove the ability of this transform and its importance in the error function through the applications that were addressed in this work.

2. Some Beneficial Properties of The complex Sadik transform,[15]:

The complex Sadik transform is defined by,[16]:

$$S_\varphi^c[o(t)]= F^c(z^\varphi,\gamma) = \frac{1}{z^\gamma}\int_0^\infty o(t)\, e^{-iz^\varphi t}\, dt.$$

Where $z \in C,\ \varphi \in R^*, \gamma \in R$.

2.1 *Linearity property* [15]:

The complex Sadik transform of functions $G_1(t)$ and $G_2(t)$:

$$S_\varphi^c[bG_1(t) + dG_2(t)]= bS_\varphi^c[G_1(t)] + dS_\varphi^c[G_2(t)],$$

Where *b* and *d* are arbitrary constants.

Proof: By using the Complex Sadik transform, we get

$$S_\varphi^c[G(t)]=\frac{1}{z^\gamma}\int_0^\infty G(t)\, e^{-iz^\varphi t}\, dt,$$

$$S_\varphi^c[bG_1(t)+dG_2(t)]=\frac{1}{z^\gamma}\int_0^\infty [bG_1(t)+dG_2(t)]\, e^{-iz^\varphi t}\, dt,$$

$$S_\varphi^c[bG_1(t)+dG_2(t)]=b\frac{1}{z^\gamma}\int_0^\infty [G_1(t)\, dt+d\frac{1}{z^\gamma}\int_0^\infty G_2(t)]\, e^{-iz^\varphi t}\, dt,$$

$$S_\varphi^c[bG_1(t)+dG_2(t)]=bS_\varphi^c[G_1(t)]+dS_\varphi^c[G_2(t)].$$

2.2 *Change of scale property* [15]:

If complex Sadik transform of functions $G(t)$ is $S_\varphi^c[G(t)]=F^c(z)=\frac{1}{z^\gamma}\int_0^\infty G(t)\, e^{-iz^\varphi t}\, dt$ then the complex Sadik transform of functions $G(at)$ is given by

$$S_\varphi^c[G(at)]=\frac{1}{z^\gamma}\int_0^\infty G(at)\, e^{-iz^\varphi t}dt.$$

Put $q=at$ then t=$\frac{a}{q}$, $adt=dq$ and $dt=\frac{1}{q}\, dq$. Now

$$S_\varphi^c[G(at)]=\frac{1}{a^\gamma z^\gamma}\int_{q=0}^\infty G(q)\, e^{-iz^\varphi\frac{q}{a}}\,\frac{1}{a}dq,$$

$S_\varphi^c[G(at)]=\frac{1}{a^{\gamma+1}}\, F^c(\frac{z}{a})$. Thus, if $S_\varphi^c[G(t)]=F^c(z)$ then $S_\varphi^c[G(at)]=\frac{1}{a^{\gamma+1}}$ $F^c(\frac{z}{a})$.

2.3 *Shifting Property* [15]:

$$S_\varphi^c[e^{bt}G(t)]=\frac{1}{z^\gamma}\int_0^\infty e^{bt}G(t)\, e^{-iz^\varphi t}\, dt=\frac{1}{z^\gamma}\int_0^\infty G(t)\, e^{i(ib+z^\varphi)t}\, dt,$$

$$=\frac{(ib+z^\varphi)^\gamma}{z^\gamma}.\frac{1}{(ib+z^\varphi)^\gamma}\int_0^\infty G(t)\, e^{i(ib+z^\varphi)t}\, dt=\frac{(ib+z^\varphi)^\gamma}{z^\gamma}.F^c(ib+z^\varphi).$$

2.4 *The complex Sadik transform of The Derivatives of The Function of* $G(t)$:

- $S_\varphi^c[G'(t)]=\frac{1}{z^\gamma}\int_0^\infty G'(t)\, e^{-iz^\varphi t}\, dt=iz^\gamma F^c(z)-\frac{G(0)}{z^\gamma}.$
- $S_\varphi^c[G''(t)]=(iz^\varphi)^2F^c(z)-\frac{G'(0)}{z^\gamma}-\frac{iz^\varphi G(0)}{z^{\alpha\gamma}}.$
- $S_\varphi^c[G'''(t)]=(iz^\varphi)^3F^c(z)-\frac{1}{z^\gamma}[G''(0)+iz^\varphi G'(0)+(iz^\varphi)^2G(0)].$
- $S_\varphi^c\left[G^{(n)}(t)\right]=(iz^\varphi)^nF^c(z)-\frac{1}{z^\gamma}\left[G^{(n-1)}(0)+iz^\varphi G^{(n-2)}(0)+(iz^\varphi)^2G^{(n-3)}(0)+\cdots+(iz^\varphi)^{(n-2)}G'(0)+(iz^\varphi)^{(n-1)}G(0)\right].$

2.5 *The complex Sadik transform of Integral of a Function* $G(t)$:

If the complex Sadik transform of function $G(t)$ *is* $S_\varphi^c[G(t)]=F^c(z)$ then $S_\varphi^c[\int_0^t G(t)\, dt]=\frac{1}{iz^\gamma}F^c(z).$

Proof: let $F(t) = \int_0^t G(t)\ dt$. Then $F'(t) = G(t)$ and $F(0) = 0$

By using the complex Sadik transform property of the derivative of a function, we get $S_\varphi^c[F'(t)] = \frac{1}{z^\gamma}\int_0^\infty F'(t)\, e^{-iz^\varphi t}\ dt = iz^\gamma F^c(z) - \frac{F(0)}{z^\gamma} = iz^\gamma S_\varphi^c[F(t)]$,

$$S_\varphi^c[F(t)] = \frac{1}{iz^\gamma}\, S_\varphi^c[F'(t)] = \frac{1}{iz^\gamma}\, S_\varphi^c[G(t)] = \frac{1}{iz^\gamma} F^c(z).$$

Then $S_\varphi^c[\int_0^t G(t)\ dt] = \frac{1}{iz^\gamma} F^c(z)$.

2.6 *The complex Sadik transform* $tG(t)$:

If the Complex Sadik transform of functions $G(t)\ is\ S_\varphi^c[G(t)] = F^c(z) = \frac{1}{z^\gamma}\int_0^\infty G(t)\, e^{-iz^\varphi t}\, dt$ then $S_\varphi^c[tG(t)] = -\left[\frac{1}{i\,\varphi z^{\varphi-1}}\frac{d}{dz} + \frac{\gamma}{i\,\varphi z^\varphi}\right] F^c(z)$.

Proof: By using the complex Sadik transform, we get

$$S_\varphi^c[G(t)] = \frac{1}{z^\gamma}\int_0^\infty G(t)\, e^{-iz^\varphi t}\, dt = F^c(z),$$

$$\frac{d}{dz} F^c(z) = \frac{d}{dz}\left[z^{-\gamma}\int_0^\infty G(t)\, e^{-iz^\varphi t}\ dt\right],$$

$$\frac{d}{dz} F^c(z) = -\gamma z^{-\gamma-1}\left[\int_0^\infty G(t)\, e^{-iz^\varphi t}\ dt\right] + z^{-\gamma}\int_0^\infty G(t)\frac{d}{dz}\, e^{-iz^\varphi t}\ dt,$$

$$\frac{d}{dz} F^c(z) = \frac{-\gamma}{z} F^c(z) + z^{-\gamma}\int_0^\infty e^{-iz^\varphi t}(-it * \varphi z^{\varphi-1})G(t)\ dt,$$

$$\frac{d}{dz} F^c(z) = \frac{-\gamma}{z} F^c(z) \text{ -}i\ \varphi z^{\varphi-1}\ z^{-\gamma}\int_0^\infty e^{-iz^\varphi t} tG(t)\ dt,$$

$$\frac{d}{dz} F^c(z) = \frac{-\gamma}{z} F^c(z) \text{ -}i\ \varphi z^{\varphi-1}\ S_\varphi^c[tG(t)].$$

$$S_\varphi^c[tG(t)] = \frac{1}{i\,\varphi z^{\varphi-1}}\left[-\frac{d}{dz}\ F^c(z) - \frac{\gamma}{z} F^c(z)\right] = -1\left[\frac{1}{i\,\varphi z^{\varphi-1}}\frac{d}{dz} + \frac{\gamma}{i\,\varphi z^\varphi}\right] F^c(z).$$

2.7 *Convolution Theorem for complex Sadik transform*

If $S_\varphi^c[G_1(t)] = F^c{}_1(z)$ and $S_\varphi^c[G_2(t)] = F^c{}_2(z)$ then:

$$S_\varphi^c[G_1(t) * G_2(t)] = z^\gamma S_\varphi^c[G_1(t)] * S_\varphi^c[G_2(t)] = z^\gamma F^c{}_1(z).\, F^c{}_2(z).$$

3. The complex Sadik transform of Frequently Encountered Functions[15]:

- ❖ $S_\varphi^c\{B\} = -\frac{iB}{z^{\varphi+\gamma}}$. B constant
- ❖ $S_\varphi^c\{t^h\} = (-i)^{h+1}\frac{h!}{z^{h\varphi+(\varphi+\gamma)}}$, where $h \in N$
- ❖ $S_\varphi^c\{e^{ct}\} = \frac{-1}{z^\varphi}\left[\frac{c}{z^{2\varphi}+c^2} + i\frac{z^\varphi}{z^{2\varphi}+c^2}\right]$, c is constant
- ❖ $S_\varphi^c\{sin(ct)\} = \frac{-c}{z^{\delta\varphi}(z^{2\varphi} - c^2)}$.
- ❖ $S_\varphi^c\{cos(ct)\} = \frac{-iz^\varphi}{z^\gamma(z^{2\varphi} - c^2)}$.

- ❖ $S_\varphi^c\{sinh(ct)\}=\frac{-c}{z^{\delta\varphi}(z^{2\varphi}+c^2)}$.
- ❖ $S_\varphi^c\{cosh(ct)\}=\frac{-iz^{\varphi}}{z^{\gamma}(z^{2\varphi}+c^2)}$.

4. Several Essential Particulars of Error and Complementary Error Functions:

We define error and complementary error function as follows:

$$erf(y)=\frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}\ dx. (1)\ \text{ and }\ erfc(y)=\frac{2}{\sqrt{\pi}}\int_y^\infty e^{-x^2}\ dx. (2)$$

4.1 *The Sum of Error and Complementary Error Functions is Unity:*

$$erf(y)+erfc(y)=1.$$

Proof: We have

$$\int_0^\infty e^{-x^2}\ dx=\frac{\sqrt{\pi}}{2}.$$

Then $\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\ dx=1$,

$$So\ \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}\ dx+\frac{2}{\sqrt{\pi}}\int_y^\infty e^{-x^2}\ dx=1.$$

Then $erf(y)+erfc(y)=1$.

4.2 *Error function is an odd function:*

$$erf(-y)=-erf(y).$$

4.3 *The value of the error function at* $y=0$ *is* 0

$$erf(0)=0.$$

4.4 *The value of the complementary error function at* $y=0$ *is* 1

$$erfc(1)=0.$$

4.5 *The domain of error and complementary error functions is* $(-\infty,\infty)$

4.6 $\boldsymbol{erf(y)\to 1\ as\ y\ \to\infty.}$

4.7 $\boldsymbol{erfc(y)\to 0\ as\ y\ \to\infty.}$

5. The complex Sadik transform of Error Function:

By equation (1), we have

$$erf(\sqrt{x})=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-y^2}\ dy,$$

$$\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}}[1-\frac{y^2}{1!}+\frac{y^4}{2!}-\frac{y^6}{3!}+\cdots]\ dy,$$

$$\frac{2}{\sqrt{\pi}}\left[y-\frac{y^3}{3.1!}+\frac{y^5}{5.2!}-\frac{y^7}{7.3!}+\cdots\right]_0^{\sqrt{x}},$$

$$\frac{2}{\sqrt{\pi}}\left[x^{1/2}-\frac{x^{3/2}}{3.1!}+\frac{x^{5/2}}{5.2!}-\frac{x^{7/2}}{7.3!}+\cdots\right].(3)$$

By employing the complex Sadik transform on both sides of equation (3), we have:

$$S_\varphi^c[erf(\sqrt{x})]=\frac{2}{\sqrt{\pi}}S_\varphi^c\left(\left[x^{1/2}-\frac{x^{3/2}}{3.1!}+\frac{x^{5/2}}{5.2!}-\frac{x^{7/2}}{7.3!}+\cdots\right]\right).\ (4)$$

By using the linearity property of complex Sadik transform on equation (4), we have:

$$S_\varphi^c[erf(\sqrt{x})]=\frac{2}{\sqrt{\pi}}\left[(-i)^{3/2}\frac{\Gamma(\frac{1}{2}+1)}{z^{1/2\varphi+(\varphi+\gamma)}}-(-i)^{\frac{5}{2}}\frac{\Gamma(\frac{3}{2}+1)}{z^{3/2\varphi+(\varphi+\gamma)}.\,3.1!}+(-i)^{\frac{7}{2}}\frac{\Gamma(\frac{5}{2}+1)}{z^{5/2\varphi+(\varphi+\gamma)}.\,5.2!}-(-i)^{9/2}\frac{\Gamma(\frac{7}{2}+1)}{z^{7/2\varphi+(\varphi+\gamma)}.\,7.3!}+\cdots\right],$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{2}{\sqrt{\pi}}\left[(-i)^{3/2}\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{z^{3/2\varphi+\gamma}}-(-i)^{\frac{5}{2}}\frac{\frac{3}{2}\Gamma(\frac{3}{2})}{z^{5/2\varphi+\gamma}.\,3.1!}+(-i)^{\frac{7}{2}}\frac{\frac{5}{2}\Gamma(\frac{5}{2})}{z^{7/2\varphi7\gamma}.\,5.2!}-(-i)^{9/2}\frac{\frac{7}{2}\Gamma(\frac{7}{2})}{z^{9/2\varphi+\gamma}.\,7.3!}+\cdots\right],$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{2}{\sqrt{\pi}}\left[(-i)^{\frac{3}{2}}\frac{\frac{1}{2}\sqrt{\pi}}{z^{\frac{3}{2}\varphi+\gamma}}-(-i)^{\frac{5}{2}}\frac{\frac{3}{4}\sqrt{\pi}}{z^{\frac{5}{2}\varphi+\gamma}.\,3.1!}+(-i)^{\frac{7}{2}}\frac{\frac{15}{8}\sqrt{\pi}}{z^{\frac{7}{2}\varphi+\gamma}.\,5.2!}-(-i)^{\frac{9}{2}}\frac{\frac{105}{16}\sqrt{\pi}}{z^{\frac{9}{2}\varphi+\gamma}.\,7.3!}+\cdots\right],$$

$$S_\varphi^c[erf(\sqrt{x})]=\left[\frac{-1}{z^{\frac{3}{2}\varphi+\gamma}}+\frac{\frac{3}{2}}{z^{\frac{5}{2}\varphi+\gamma}.3.1!}-\frac{\frac{15}{4}}{z^{\frac{7}{2}\varphi+\gamma}.5.2!}+\frac{\frac{105}{8}}{z^{\frac{9}{2}\varphi+\gamma}.7.3!}+\cdots\right],$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{-z^{-\gamma}}{z^{\frac{3}{2}\varphi}}\left[1-\frac{1}{2z^\varphi}+\frac{3}{8z^{2\varphi}}-\frac{5}{16z^{3\varphi}}+\cdots\right],$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{-z^{-\gamma}}{z^{\frac{3}{2}\varphi}}\left[1+\frac{1}{z^\varphi}\right]^{-\frac{1}{2}},$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{-z^{-\gamma}}{z^{\frac{3}{2}\varphi}}\left[\frac{z^\varphi+1}{z^\varphi}\right]^{-\frac{1}{2}},$$

$$S_\varphi^c[erf(\sqrt{x})]=\frac{-z^{-\gamma}}{z^{\frac{3}{2}\varphi}}\left[\frac{\sqrt{z^\varphi}}{\sqrt{z^\varphi+1}}\right].$$

$$Then\ S_\varphi^c[erf(\sqrt{x})]=\frac{-z^{-\gamma}}{z^\varphi\sqrt{z^\varphi+1}}=\frac{i^2z^{-\gamma}}{z^\varphi\sqrt{z^\varphi+1}}\ (5)$$

6. Complex Sadik transform of Complementary Error Function:

We have,

$$erf\left(\sqrt{x}\right) + erfc(\sqrt{x}) = 1.$$

$$erfc\left(y\sqrt{x}\right) = 1 - \ erf\left(\sqrt{x}\right). \quad (6)$$

Applying complex Sadik transform on both sides of equation(6), we have

$$S_{\varphi}^{c}\left[erfc\left(\sqrt{x}\right)\right] = S_{\varphi}^{c}\left[1 - \ erf\left(\sqrt{x}\right)\right], \quad (7)$$

Using the linearity property of the complex Sadik transform on equation (7), we have

$$S_{\varphi}^{c}\left[erfc\left(\sqrt{x}\right)\right] = S_{\varphi}^{c}[1] - \ S_{\varphi}^{c}\left[\ erf\left(\sqrt{x}\right)\right],$$

$$S_{\varphi}^{c}\left[erfc\left(\sqrt{x}\right)\right] = -\frac{iz^{-\gamma}}{z^{\varphi}} - \frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}} = \frac{-iz^{-\gamma}\sqrt{z^{\varphi}+1}+z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}} = \frac{-z^{-\gamma}(i\sqrt{z^{\varphi}+1}-1)}{z^{\varphi}\sqrt{z^{\varphi}+1}}.$$

Then $S_{\varphi}^{c}\left[erfc\left(\sqrt{x}\right)\right] = \frac{-z^{-\gamma}(i\sqrt{z^{\varphi}+1}-1)}{z^{\varphi}\sqrt{z^{\varphi}+1}}$. (8)

7. Applications

In this part, several implementations are given in order to interpret the characteristics of the complex Sadik transform of the error function for evaluating the incorrect integral, which contains the error function.

7.1 *Evaluate The Incorrect Integral*

$$I = \int_{0}^{\infty} e^{-x} er\left(\sqrt{x}\right)\ dx.$$

We have $S_{\varphi}^{c}\left[erf\left(\sqrt{x}\right)\right] = \frac{1}{z^{\gamma}}\int_{0}^{\infty} erf\left(\sqrt{x}\right) e^{-iz^{\varphi}x}\, dx.$

$$S_{\varphi}^{c}\left[erf\left(\sqrt{x}\right)\right] = \frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}. \quad (9)$$

Taking z^{φ}=1 in the above equation, applying a new general integral transform, we get:

$$\frac{1}{z^{\gamma}}\int_{0}^{\infty} efr\left(\sqrt{x}\right) e^{-ix}\ dx = \frac{-z^{-\gamma}}{1\sqrt{1+1}} = \frac{-z^{-\gamma}}{1\sqrt{1+1}},$$

$$\int_{0}^{\infty} efr\left(\sqrt{x}\right) e^{-ix}\ dx = \frac{-z^{-\gamma} * z^{\gamma}}{1\sqrt{1+1}} = \frac{-1}{\sqrt{2}}.$$

Then $\int_{0}^{\infty} efr\left(\sqrt{x}\right) e^{-ix}\ dx = \frac{-1}{\sqrt{2}}$.

7.2 *Evaluate the Incorrect integral*

$$I = \int_{0}^{\infty} xe^{-3x} er\left(\sqrt{x}\right)\ dx.$$

We have

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right] = \frac{1}{z^{\gamma}}\int_{0}^{\infty} xefr\left(\sqrt{x}\right) e^{-iz^{\varphi}x}\, dx. \quad (10)$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=-\left[\frac{1}{i\,\varphi z^{\varphi-1}}\frac{d}{dz}+\frac{\gamma}{i\,\varphi z^{\varphi}}\right]F^{c}(z).$$

$$S_{\varphi}^{c}\left[erf\left(\sqrt{x}\right)\right]=\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}.$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=-\left[\frac{1}{i\,\varphi z^{\varphi-1}}\frac{d}{dz}+\frac{\gamma}{i\,\varphi z^{\varphi}}\right].\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}},$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=-\left[\frac{1}{i\,\varphi z^{\varphi-1}}.\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}\frac{d}{dz}+\frac{\gamma}{i\,\varphi z^{\varphi}}.\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{d}{dz}\frac{-1}{z^{2\varphi+\gamma-1}\sqrt{z^{\varphi}+1}}+\frac{\gamma}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{z^{2\varphi+\gamma-1}\left(\frac{1}{2}\varphi z^{\varphi-1}(z^{\varphi}+1)^{\frac{-1}{2}}\right)+\sqrt{z^{\varphi}+1}(2\varphi+\gamma-1)z^{2\varphi+\gamma-2}}{(z^{2\varphi+\gamma-1})^{2}(z^{\varphi}+1)}+\frac{\gamma}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{z^{2\varphi+\gamma-1}\left(\frac{1}{2}\varphi z^{\varphi-1}(z^{\varphi}+1)^{\frac{-1}{2}}\right)}{(z^{2\varphi+\gamma-1})^{2}(z^{\varphi}+1)}+\frac{\sqrt{z^{\varphi}+1}(2\varphi+\gamma-1)z^{2\varphi+\gamma-2}}{(z^{2\varphi+\gamma-1})^{2}(z^{\varphi}+1)}+\frac{\gamma}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{\varphi z^{\varphi-1}}{2(z^{2\varphi+\gamma-1})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)*z^{-1}}{z^{2\varphi+\gamma-1}\sqrt{z^{\varphi}+1}+}+\frac{\gamma}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{\varphi z^{\varphi-1}*z}{2(z^{2\varphi+\gamma})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1)}}+\frac{\gamma}{z^{2\varphi+\gamma}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-1}{i\,\varphi}\left[\frac{\varphi z^{\varphi}z^{-\gamma}}{2(z^{2\varphi})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)z^{-\gamma}}{z^{2\varphi}\sqrt{z^{\varphi}+1)}}+\frac{\gamma z^{-\gamma}}{z^{2\varphi}\sqrt{z^{\varphi}+1}}\right],$$

$$S_{\varphi}^{c}\left[xerf\left(\sqrt{x}\right)\right]=\frac{-z^{-\gamma}}{i\,\varphi}\left[\frac{\varphi z^{\varphi}}{2(z^{2\varphi})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)}{z^{2\varphi}\sqrt{z^{\varphi}+1)}}+\frac{\gamma}{z^{2\varphi}\sqrt{z^{\varphi}+1}}\right], \quad (11)$$

From equations (10) & (11) We get,

$$\frac{1}{z^{\gamma}}\int_{0}^{\infty}xefr\left(\sqrt{x}\right)e^{-iz^{\varphi}x}\,dx=\frac{-z^{-\gamma}}{i\,\varphi}\left[\frac{\varphi z^{\varphi}}{2(z^{2\varphi})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)}{z^{2\varphi}\sqrt{z^{\varphi}+1)}}+\frac{\gamma}{z^{2\varphi}\sqrt{z^{\varphi}+1}}\right],$$

$$\int_{0}^{\infty}xefr\left(\sqrt{x}\right)e^{-iz^{\varphi}x}\,dx=\frac{-1}{i\,\varphi}\left[\frac{\varphi z^{\varphi}}{2(z^{2\varphi})(z^{\varphi}+1)^{\frac{3}{2}}}+\frac{(2\varphi+\gamma-1)}{z^{2\varphi}\sqrt{z^{\varphi}+1)}}+\frac{\gamma}{z^{2\varphi}\sqrt{z^{\varphi}+1}}\right],$$

$$\int_{0}^{\infty}xefr\left(\sqrt{x}\right)e^{-iz^{\varphi}x}\,dx=\left[\frac{-z^{\varphi}}{2i(z^{2\varphi})(z^{\varphi}+1)^{\frac{3}{2}}}-\frac{(2\varphi+\gamma-1)}{i\,\varphi z^{2\varphi}\sqrt{z^{\varphi}+1)}}-\frac{\gamma}{i\,\varphi z^{2\varphi}\sqrt{z^{\varphi}+1}}\right],$$

Taking z^{φ}=3 in the above equation

$$\int_0^{\infty} xefr\left(\sqrt{x}\right) e^{-iz^{\varphi}x}\, dx=\left[\frac{-3}{2i(9)(3+1)^{\frac{3}{2}}}-\frac{(2\varphi+\gamma-1)}{3i\,\varphi\sqrt{3+1})}-\frac{\gamma}{3i\,\varphi\sqrt{3+1})}\right].$$

$$\int_0^{\infty} xefr\left(\sqrt{x}\right) e^{-iz^{\varphi}x}\, dx=\left[\frac{-1}{144i}-\frac{(2\varphi+\gamma-1)}{6i\,\varphi}-\frac{\gamma}{6i\,\varphi}\right].$$

7.3 *Evaluate the improper integral*

$$I=\int_0^{\infty} e^{-(z-2)}er\left(\sqrt{x}\right)\, dx.$$

We have $S_{\varphi}^{c}\left[erf\left(\sqrt{x}\right)\right]=\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}$.

$$S_{\varphi}^{c}\left[e^{2x}erf\left(\sqrt{x}\right)\right]=\frac{1}{z^{\gamma}}\int_0^{\infty} erf\left(\sqrt{x}\right) e^{i(ib+z^{\varphi})x}\, dx. \quad (12)$$

By shifting the theorem of the complex Sadik transform, we get:

$$S_{\varphi}^{c}\left[e^{2x}erf\left(\sqrt{x}\right)\right]=\frac{(ib+z^{\varphi})^{\gamma}}{z^{\gamma}}.F^{c}(ib+z^{\varphi}),$$

$$S_{\varphi}^{c}\left[e^{2x}erf\left(\sqrt{x}\right)\right]=\frac{(i2+z^{\varphi})^{\gamma}}{z^{\gamma}}\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}} \quad (13)$$

Now by equations (12) and (13), we get:

$$\frac{1}{z^{\gamma}}\int_0^{\infty} erf\left(\sqrt{x}\right) e^{i(ib+z^{\varphi})x}\, dx=\frac{-(i2+z^{\varphi})^{\gamma}}{z^{2\gamma}z^{\varphi}\sqrt{z^{\varphi}+1}},$$

$$\int_0^{\infty} erf\left(\sqrt{x}\right) e^{i(ib+z^{\varphi})x}\, dx=\frac{-(i2+z^{\varphi})^{\gamma}}{z^{\gamma}z^{\varphi}\sqrt{z^{\varphi}+1}}=\frac{-(i2+z^{\varphi})^{\gamma}}{z^{\varphi+\gamma}\sqrt{z^{\varphi}+1}}.$$

Then

$$I=\int_0^{\infty} e^{-(z-2)}er\left(\sqrt{x}\right)\, dx=\frac{-(i2+z^{\varphi})^{\gamma}}{z^{\varphi+\gamma}\sqrt{z^{\varphi}+1}}.$$

7.4 *Evaluate the improper integral*

$$I=\int_0^{\infty} e^{-x}\left[\int_0^{x} er\left(\sqrt{v}\right) dv\right] dx.$$

We have

$$S_{\varphi}^{c}\left[\left[\int_0^{x} er\left(\sqrt{v}\right)\, dv\right]\right]=\frac{1}{z^{\gamma}}\int_0^{\infty} e^{-iz^{\varphi}x}\,\left[\int_0^{x} er\left(\sqrt{v}\right) dv\right]dx. \quad (14)$$

$$S_{\varphi}^{c}\left[erf\left(\sqrt{x}\right)\right]=\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}.$$

By using the property of complex Sadik transform of integral of a function, we get:

$$S_{\varphi}^{c}\left[\left[\int_0^{x} er\left(\sqrt{v}\right)\, dv\right]\right]=\frac{1}{iz^{\gamma}}F^{c}(z)=\frac{1}{iz^{\gamma}}*\frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}=\frac{-1}{iz^{2\gamma}z^{\varphi}\sqrt{z^{\varphi}+1}} \quad (15)$$

Now by equations (14) and (15), we get

$$\frac{1}{z^{\gamma}}\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\int_0^{x} er(\sqrt{v})\ dv\ \right]dx = \frac{-1}{iz^{2\gamma}z^{\varphi}\sqrt{z^{\varphi}+1}},$$

$$\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\int_0^{x} er(\sqrt{v})\ dv\ \right]dx = \frac{-1}{iz^{\gamma}z^{\varphi}\sqrt{z^{\varphi}+1}} = \frac{-1}{iz^{\varphi+\gamma}\sqrt{z^{\varphi}+1}}.$$

Taking $z^{\varphi} = 1$ in the above equation, we have:

Then $I = \int_0^{\infty} e^{-x}\left[\int_0^{x} er(\sqrt{v})\ dv\ \right]dx = \frac{-1}{iz^{\gamma}\sqrt{2}}$.

7.5 *Evaluate the improper integral*

$$I = \int_0^{\infty} e^{-2x}\left[\frac{d}{dx}erf(2\sqrt{x})\right]dx.$$

We have $S_{\varphi}^{c}[erf(\sqrt{x})] = \frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}}$.

Now by change of scale property of complex Sadik transform, we have:

$$S_{\varphi}^{c}[erf(2\sqrt{x})] = \frac{1}{4^{\gamma+1}}\left[\frac{-z^{-\gamma}/4}{z^{\varphi}/4\sqrt{z^{\varphi}+4}}\right] = \frac{1}{4^{\gamma+1}}\left[\frac{-2z^{-\varphi^{-\gamma}}}{\sqrt{z^{\varphi}+4}}\right] = \left[\frac{-2z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+4}}\right],$$

Now using the property of complex Sadik transform of derivative of a function, we have

$$S_{\varphi}^{c}\left[\frac{d}{dx}erf(2\sqrt{x})\right]=\frac{1}{4^{\gamma+1}}.iz^{\gamma}\left[\frac{-2z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+4}}\right] - \frac{G(0)}{z^{\gamma}} = \frac{1}{4^{\gamma+1}}.iz^{\gamma}\left[\frac{-2z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+4}}\right] - \frac{0}{z^{\gamma}} = \frac{1}{4^{\gamma+1}}.iz^{\gamma}\left[\frac{-2z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+4}}\right],$$

Then $S_{\varphi}^{c}\left[\frac{d}{dx}erf(2\sqrt{x})\right]=\frac{1}{4^{\gamma+1}}.\frac{-2i}{z^{\varphi}\sqrt{z^{\varphi}+4}}$. (16)

By the definition of complex Sadik transform, we have:

$$S_{\varphi}^{c}\left[\frac{d}{dx}erf(2\sqrt{x})\right] = \frac{1}{z^{\gamma}}\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\frac{d}{dx}erf(2\sqrt{x})\right]dx. \quad (17)$$

Now by equations (16) and (17), we get

$$\frac{1}{z^{\gamma}}\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\frac{d}{dx}erf(2\sqrt{x})\right]dx = \frac{1}{4^{\gamma+1}}.\frac{-2i}{z^{\varphi}\sqrt{z^{\varphi}+4}},$$

$$\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\frac{d}{dx}erf(2\sqrt{x})\right]dx = \frac{1}{4^{\gamma+1}}.\frac{-2iz^{\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+4}},$$

Taking $z^{\varphi} = 2$ in the above equation, we have:

$$\int_0^{\infty} e^{-iz^{\varphi}x}\ \left[\frac{d}{dx}erf(2\sqrt{x})\right]dx = \frac{1}{4^{\gamma+1}}.\frac{-2iz^{\gamma}}{2\sqrt{2+4}} = \frac{1}{4^{\gamma+1}}.\frac{-iz^{\gamma}}{\sqrt{6}}.$$

Then $I = \int_0^{\infty} e^{-2x}\left[\frac{d}{dx}erf(2\sqrt{x})\right]dx = \frac{1}{4^{\gamma+1}}.\frac{-iz^{\gamma}}{\sqrt{6}}$.

7.6 *Evaluate the improper integral*

$$I = \int_0^{\infty} e^{-5x}\left[erf(\sqrt{x}) * erf(\sqrt{x})\right]dx.$$

By the convolution theorem of the complex Sadik transform, we have:

$$S_{\varphi}^{c}\left[erf(\sqrt{x}) * erf(\sqrt{x})\right] = z^{\gamma} F^{c}{}_{1}(z).$$

$$F^{c}{}_{2}(z) = z^{\gamma}\, \frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}} * \frac{-z^{-\gamma}}{z^{\varphi}\sqrt{z^{\varphi}+1}} = z^{\gamma}\, \frac{z^{-2\gamma}}{z^{2\varphi}(z^{\varphi}+1)}, \quad (18)$$

Now by the definition of complex Sadik transform, we have:

$$S_{\varphi}^{c}\left[erf(\sqrt{x}) * erf(\sqrt{x})\right] = \frac{1}{z^{\gamma}} \int_{0}^{\infty} e^{-iz^{\varphi}x} \left[erf(\sqrt{x}) * erf(\sqrt{x})\right] dx. \quad (19)$$

Now by equations (18) and (19), we get

$$\frac{1}{z^{\gamma}} \int_{0}^{\infty} e^{-iz^{\varphi}x} \left[erf(\sqrt{x}) * erf(\sqrt{x})\right] dx = z^{\gamma}\, \frac{z^{-2\gamma}}{z^{2\varphi}(z^{\varphi}+1)},$$

$$\int_{0}^{\infty} e^{-iz^{\varphi}x} \left[erf(\sqrt{x}) * erf(\sqrt{x})\right] dx = \frac{1}{z^{2\varphi}(z^{\varphi}+1)},$$

Taking $z^{\varphi} = 5$ in the above equation, we have:

$$\int_{0}^{\infty} e^{-iz^{\varphi}x} \left[erf(\sqrt{x}) * erf(\sqrt{x})\right] dx = \frac{1}{150}.$$

$$\text{Then } I = \int_{0}^{\infty} e^{-5x} \left[erf(\sqrt{x}) * erf(\sqrt{x})\right] dx = \frac{1}{150}.$$

8. Discussion and Conclusion

In this paper, we succeeded in using the integral transformation complex Sadik transform function of the error to find a solution for the improper integrals that have the error function. We note that the integral transformation complex Sadik transform gives an accurate solution without doing full arithmetic work.

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