

## Soc-T-ABSO submodules and related concepts

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### Abstract

In this paper, we present the idea of Soc-T-ABSO submodule observations and examine how-T-ABSO submodule and Socle relate to one another in terms of qualities and characteristics. Submodules of several kinds are have now been introduced. It is possible to develop new concepts based on Soc-T-ABSO submodules using the implications of this work.

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## 1. Introduction

In this study,  $V$  has the identity and would be a commutative ring, and  $W$  is a unitary  $V$ -module. The idea of prime submodules of modules was researched by Chin-Pi Lu in 1983 [4]. The idea of prime submodules has numerous generalizations, including Muntaha H. Abdul-Raza introduced Quasi – prime modules and Quasi – prime submodules in 1999 and researched [7]. M. Behoodi and H. Koochi developed and researched the idea of weakly prime modules [8]. in 2004. The two ideas of 2-Absorbing and Weakly 2-Absorbing Submodules were generalized in 2011 by Ahmad Yousefian and Fatemeh Soheilnia [3]. Abdulrahman Abdullah researched 2-absorbing modules and a few of their generalizations in 2015 [1]. In 2019, Omar A. Abdulla published the

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Pseudo Quasi-2-Absorbing submodules and Some Related Concepts, which he described as a generalization of the Pseudo2-Absorbing quasi primary submodules [9].

There are two parts to this research. We introduce various concepts in the first section, along with some of their fundamental characteristics. The second section examines the connections, characteristics, and fundamental findings of the Soc – T – ABSO submodule.

## 2. Preliminaries

There are numerous fundamental notions, and this section lists their characteristics.

**Definition 2.1:** [4] A proper submodule  $A$  of a  $V$  – module  $W$  is said to be prime submodule if  $ax \in A$  for some  $a \in V, x \in W$ , implies that  $x \in A$  or  $a \in (A:_V W)$

**Definition 2.2:** [5] A  $V$ -module  $W$  is said to be semi simple if  $W$  is a summation of simple submodule of  $W$ . Besides, the socle of  $W$  is sum of simple submodule of  $W$  and denoted by  $\text{Soc}(W)$  and  $W$  is called semisimple if  $W = \text{Soc}(W)$ . For an arbitrary a submodule  $A$  of a  $V$  – module  $W$ ,  $\text{Soc}(A) = A \cap \text{Soc}(W)$

**Definition 2.3:** [3] A proper submodule  $A$  of a  $V$ -module  $W$  is said to be T – ABSO submodule of  $W$  if whenever  $a, b \in V, x \in W$  such that  $abx \in A$ , then either  $ax \in A$  or  $bx \in A$  or  $ab \in (A:_V W)$ .

**Definition 2.4:** [7], [11] A proper submodule  $A$  of a  $V$ -module  $W$  is said to be quasi-prime submodule if for some  $a, b \in V, x \in W$  such that  $abx \in A$ , then either  $ax \in A$  or  $bx \in A$ .

**Definition 2.5:** [5], [6] A submodule  $A$  of a  $V$ -module  $W$  is said to be essential in  $W$  brief  $A \leq_e W$  if for all non-zero submodule  $B$  of  $W$ ,  $A \cap B \neq 0$ .

**Definition 2.6** [2], [10] A proper ideal  $I$  of a ring  $V$  is said to be T – ABSO ideal if for some  $a, b, c \in V$  with  $abc \in I$ , then either  $ab \in I$  or  $bc \in I$  or  $ac \in I$ .

**Definition 2.7:** [5] Let  $A$  a proper submodule of a  $V$ -module  $W$

- (a) A proper submodule  $B$  of a  $V$ -module  $W$  is said to be an addition complement, shortly adco, of  $A$  in  $W$  if and only if
  - (1)  $A + B = W$ .
  - (2)  $B$  is minimal in  $A + B$ .

(b) A proper submodule  $C$  of a  $V$ -module  $W$  is said to be an intersection complement, shortly inco, of  $A$  in  $W$  if and only if

- (1)  $A \cap C = 0$ .
- (2)  $C$  is maximal in  $A \cap C$ .

**Definition 2.8:** [5] Let  $\Lambda = \{B_i | i \in I\}$  be a set of submodules  $B_i$  of  $W$  then  $\sum_{i \in I} B_i = \langle \cup_{i \in I} B_i \rangle$  is said to be the sum of the submodules  $\{B_i | i \in I\}$ .

**Proposition 2.9:** [5] If  $f: W \rightarrow W'$ ,  $f$  is monomorphism and  $Im(f) \leq_e W'$ , then  $f(Soc(W)) = Soc(W')$  and  $Soc(W) = f^{-1}(Soc(W'))$

**Corollary 2.10:** [5] Let  $C, D$  are submodules of a  $V$ -module  $W$ . Then, we have  $C \oplus D = W$  if and only if  $D$  is an inco and adco of  $C$  in  $W$ .

**Lemma (Modular Law) 2.11:** [5] Let  $A, B$  and  $C$  be submodules of a  $V$ -module  $W$  and  $B \leq C$  then

$$(A + B) \cap C = (A \cap C) + (B \cap C) = (A \cap C) + B.$$

**Definition 2.12:** [5]

- (1) A subset  $D$  of a module  $W$  is called a generating set of  $W$  if and only if  $\langle D \rangle = W$ .
- (2) A  $V$ -module  $W$  is called finitely generated if and only if there exists a finite generating set.

### 3. Soc – T – ABSO SUBMODULES

**Definition 3.1:** A proper submodule  $A$  of a  $V$ -module  $W$  is called Socle-Two-Absorbing (in short is  $oc - T - ABSO$ ) submodule of  $W$  if whenever  $a, b \in V, x \in W$  such that  $abx \in A$ , then either  $ax \in A + Soc(W)$  or  $bx \in A + Soc(W)$  or  $ab \in (A + Soc(W)) :_V W$ .

**Remarks and Example 3.2:**

1. It is evident that every  $T - ABSO$  submodule is  $Soc - T - ABSO$  submodule, but in general the inverse is not true, for example in  $Z_{24}$  as  $Z$ -module, take  $A = \langle \overline{12} \rangle$  is  $Soc - T - ABSO$  submodule of  $Z_{24}$ , but  $A$  is not  $T - ABSO$  submodule of  $Z_{24}$ , since  $2 \cdot 2 \cdot \overline{3} \in A$ , then  $2 \cdot \overline{3} \notin A$  and  $2 \cdot 2 = 4 \notin (A :_Z Z_{24}) = 12Z$ .
2. The intersection of each pair of distinct prime submodules of a  $V$ -module  $W$  is  $Soc - T - ABSO$  submodule.

**Proof:** Let  $A, B$  be two distinct prime submodules of  $W$ . Assume that  $a, b \in V$  and  $x \in W$  such that  $abx \in A \cap B$ . Suppose that  $ax \notin A \cap B + Soc(W)$  and  $x \notin (A \cap B) + Soc(W)$ , then  $ax \notin A \cap B$  and  $bx \notin A \cap B$ . The case  $ax \notin A$  and  $bx \notin A$  this is contradiction since  $A$  is prime submodule, similar if  $ax \notin B$  and  $bx \notin B$  this is contradiction since  $B$  is prime. The case  $ax \notin A$  and

$bx \notin B$  Since  $abx \in A \cap B$  then  $abx \in A$  &  $abx \in B$ . Then  $b \in (A:_{\mathcal{V}} W)$  and  $a \in (B:_{\mathcal{V}} W)$ , hence  $ab \in (A \cap B:_{\mathcal{V}} W)$  and so  $abW \subseteq A \cap B \subseteq A \cap B + Soc(W)$ , hence  $abW \subseteq A \cap B + Soc(W)$ , then  $ab \in (A \cap B + Soc(W):_{\mathcal{V}} W)$ . Thus,  $A \cap B$  is  $Soc - T - ABSO$  submodule of  $W$ .

1. It is evident that every quasi-*prime* submodule is  $Soc - T - ABSO$  submodule, but ingeneral the inverse is not true, see example in part(1) is  $Soc - T - ABSO$  submodule of  $Z_{24}$ , but  $A$  is not quasi-*prime* submodule of  $Z_{24}$ , since  $2 \cdot 2 \cdot \bar{3} \in A$ , then  $2 \cdot \bar{3} \notin A$ .
2. If  $A, B$  be two  $Soc - T - ABSO$  submodules of a  $V$ -module  $W$  and  $A \subset B$  if  $B$  is  $Soc - T - ABSO$  submodule of  $W$  then it is not necessary that  $A$  is a  $Soc - T - ABSO$  submodule, for example if  $W = Z \oplus Z$  as  $Z$ -module,  $A = \langle (4, 0) \rangle$  and  $B = \langle (2, 0) \rangle$ ,  $Soc(Z \oplus Z) = (0, 0)$ ,  $B + Soc(Z \oplus Z) = \langle (2, 0) \rangle$ , and  $(B + Soc(Z \oplus Z)):_Z Z \oplus Z = (0, 0)$  Then if  $a, b \in Z, x \in Z \oplus Z$  with  $abx \in B$  at least one of it ( $ax$  or  $bx$  is even) i. e either  $ax \in B + Soc(W)$  or  $bx \in B + Soc(W)$ . Then  $B$  is  $Soc - T - ABSO$  submodule of  $W$ , but  $A$  is not  $Soc - T - ABSO$  submodule of  $Z$ -module  $Z \oplus Z$  since  $2 \cdot 2 \cdot (3, 0) \in \langle (4, 0) \rangle$  but  $2 \cdot (3, 0) \notin \langle (4, 0) \rangle + Soc(Z \oplus Z)$  and  $2 \cdot 2 = 4 \notin (A + Soc(Z \oplus Z)):_Z Z \oplus Z = (0, 0)$ .
3. If  $A, B$  be two submodules of a  $V$ -module  $W$  and  $A \subset B$  with  $B$  is  $T - ABSO$  submodule of  $W$ , if  $A$  is a  $Soc - T - ABSO$  of  $W$  then  $A$  is a  $Soc - T - ABSO$  submodule of  $B$ .

**Proof:** Let  $a, b \in V, x \in B$ , so  $x \in W$  such that  $abx \in A$ , hence  $abx \in B$ , but  $A$  is  $Soc - T - ABSO$  submodule of  $W$  and  $B$  is  $T - ABSO$  submodule of  $W$ , then  $ax \in B$  or  $bx \in B$  or  $ab \in (B:_{\mathcal{V}} W)$  and  $ax \in A + Soc(W)$  or  $bx \in A + Soc(W)$  or  $abW \subseteq A + Soc(W)$ . Then  $ax \in (A + Soc(W)) \cap B$  or  $bx \in (A + Soc(W)) \cap B$  or  $abW \subseteq (A + Soc(W)) \cap B$ . But  $(A + Soc(W)) \cap B = A + (B \cap Soc(W)) = A + Soc(B)$  by (Lemma 2. 11 and Definition (2. 2)[5]), hence  $ax \in A + Soc(B)$  or  $bx \in A + Soc(B)$  or  $abB \subseteq A + Soc(B)$ . Then either  $ax \in A + Soc(B)$  or  $bx \in A + Soc(B)$  or  $ab \in (A + Soc(B):_{\mathcal{V}} B)$ . Thus  $A$  is  $Soc - T - ABSO$  submodule of  $B$ .

1. The sum of two  $Soc - T - ABSO$  submodules is not necessary  $Soc - T - ABSO$  submodule for example: Let  $A = 2Z, B = 3Z$  each  $A$  and  $B$  is  $Soc - T - ABSO$  submodules in  $Z$ -module  $Z$ , but  $A + B = 2Z + 3Z = Z$  is not  $Soc - T - ABSO$  submodule of  $Z$ .
2. Let  $A$  and  $B$  be two submodules of a  $V$ -module  $W$  such that  $A \simeq B$ . If  $A$  is  $Soc - T - ABSO$  submodule of  $W$ , it does not have  $B$  is  $Soc - T - ABSO$  submodule of  $W$ , for example:  $A = 2Z$  is  $Soc - T - ABSO$  submodule of  $Z$ -module  $Z$  and  $2Z \simeq 30Z$ , but  $30Z$  is not  $Soc - T - ABSO$  submodule, since  $2 \cdot 3 \cdot 5 = 30 \in 30Z$ , but  $2 \cdot 5 \notin 30Z + Soc(Z) = 30Z, 3 \cdot 5 \notin 30Z + Soc(Z) = 30Z$  and  $2 \cdot 3 \notin (30Z + Soc(Z)):_Z Z = 30Z$ .
3. The intersection of two  $Soc - T - ABSO$  submodules, it is not necessary that  $Soc - T - ABSO$  submodule, for example:  $3Z$  and  $4Z$  are  $Soc - T -$

ABSO submodules in  $Z$ -module  $Z$ , but  $3Z \cap 4Z = 12Z$  which is not  $Soc - T - ABSO$  submodule, since  $2 \cdot 2 \cdot 3 = 12 \in 12Z$ , then  $2 \cdot 3 \notin 12Z + Soc(Z) = 12Z$  and  $2 \cdot 2 \notin (12Z + Soc(Z))_Z Z = 12Z$ .

4. Every finitely generated  $R$ -module  $W \neq 0$  has a  $Soc - T - ABSO$  submodule.

Now, we give theorems which are characterization  $Soc - T - ABSO$  submodule as follows:

**Theorem 3.3:** *Let  $A$  be a proper submodule of  $W$  then the following statements are equivalent:*

1.  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .
2. If  $abH \subseteq A$  for some  $a, b \in V$  and a submodule  $H$  of  $W$  implies either  $aH \subseteq A + Soc(W)$  or  $bH \subseteq A + Soc(W)$  or  $ab \in (A + Soc(W))_V W$ .

**Proof:** (1)  $\rightarrow$  (2): let  $A$  is an  $Soc - T - ABSO$  submodule of  $W$  and let  $H$  be a submodule of  $W$ . Such that  $abH \subseteq A$  for some  $a, b \in V$  assume that  $aH \not\subseteq A + Soc(W)$  and  $bH \not\subseteq A + Soc(W)$  and  $ab \notin (A + Soc(W))_V W$  then there exist  $h_1, h_2 \in H$  with  $ah_1 \notin A + Soc(W)$  and  $bh_2 \notin A + Soc(W)$ . Since  $abh_1 \in A$  and  $ab \notin (A + Soc(W))_V W$  and  $ah_1 \notin A + Soc(W)$ , we get  $bh_1 \in A + Soc(W)$ . Also since  $abh_2 \in A$  and  $ab \notin (A + Soc(W))_V W$  and  $bh_2 \notin A + Soc(W)$ , we get  $ah_2 \in A + Soc(W)$ . Now, since  $ab(h_1 + h_2) \in A$  and  $ab \notin (A + Soc(W))_V W$ , we have  $a(h_1 + h_2) \in A + Soc(W)$  or  $b(h_1 + h_2) \in A + Soc(W)$  if  $a(h_1 + h_2) \in A + Soc(W)$  then  $ah_1 + ah_2 \in A + Soc(W)$  since  $ah_2 \in A + Soc(W)$  then we get  $ah_1 \in A + Soc(W)$  which is contradiction! if  $b(h_1 + h_2) \in A + Soc(W)$  then  $bh_1 + bh_2 \in A + Soc(W)$  and since  $bh_1 \in A + Soc(W)$ , we get  $bh_2 \in A + Soc(W)$  which is contradiction! therefore  $ab \in (A + Soc(W))_V W$  or  $aH \subseteq A + Soc(W)$  or  $bH \subseteq A + Soc(W)$ .

(2)  $\rightarrow$  (1) it is evident.

**Theorem 3.4:** *A proper submodule  $A$  of  $V$ -module  $W$  is  $Soc - T - ABSO$  submodule of  $W$  if and only if  $IJH \subseteq A$  for some ideals  $I, J$  of  $V$  and some submodule  $H$  of  $W$  then either  $IH \subseteq A + Soc(W)$  or  $JH \subseteq A + Soc(W)$  or  $IJ \subseteq (A + Soc(W))_V W$ .*

**Proof:**  $\Rightarrow$ ) suppose that  $A$  is  $Soc - T - ABSO$  submodule of  $W$  and let  $I, J$  are ideals of  $V$  and  $H$  is a submodule of  $W$ , let  $i \in I$  and  $j \in J$ ,  $ijH \subseteq A$ . But  $A$  is  $Soc - T - ABSO$  submodule, then by Theorem(3. 4), we have either  $iH \subseteq A + Soc(W)$  or  $jH \subseteq A + Soc(W)$  or  $ijW \subseteq A + Soc(W)$ , so that either  $IH \subseteq A + Soc(W)$  or  $JH \subseteq A + Soc(W)$  or  $IJW \subseteq A + Soc(W)$ .

$\Leftarrow$ ) Let  $ijx \in A$  for some  $i, j \in V, x \in W$ , hence  $\langle i \rangle \langle j \rangle \langle x \rangle \subseteq A$ , then either  $\langle i \rangle \langle x \rangle \subseteq A + Soc(W)$  or  $\langle j \rangle \langle x \rangle \subseteq A + Soc(W)$  or  $\langle i \rangle \langle j \rangle \subseteq (A + Soc(W))_V W$ , so that either  $ix \in A + Soc(W)$  or  $jx \in A + Soc(W)$  or  $ij \in (A + Soc(W))_V W$ . Thus,  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

**Definition 3.5:** A proper ideal  $I$  of a ring  $V$  is called *Soc - T - ABSO* ideal of  $V$  if whenever  $abc \in I$ , where  $a, b, c \in V$  implies that either  $ab \in I + \text{Soc}(V)$  or  $bc \in I + \text{Soc}(V)$  or  $ac \in I + \text{Soc}(V)$ , where  $\text{Soc}(V) =$  sum of all simple ideals of  $V$ .

**Example 3.6:** Consider a ring  $Z$  if  $I = 2Z$  is *Soc - T - ABSO* ideal of  $Z$ .

**Proposition 3.7:** Let  $A$  a proper submodule of a  $V$ -module  $W$ . If  $A$  is *T - ABSO* submodule of  $W$ , then  $(A :_V W)$  is *Soc - T - ABSO* ideal of  $V$ .

**Proof:** Let  $abc \in (A :_V W)$  for some  $a, b, c \in V$ . Then  $abcW \subseteq A$ , hence  $ab(cx) \subseteq A$  for all  $x \in W$ , but  $A$  is *T - ABSO* submodule of  $W$ . Then either  $a(cx) \in A$  or  $b(cx) \in A$  or  $ab \in (A :_V W)$ , hence either  $a(cW) \subseteq A$  or  $b(cW) \subseteq A$  or  $ab \in (A :_V W)$ , therefore either  $ac \in (A :_V W) \subseteq (A :_V W) + \text{Soc}(V)$  or  $bc \in (A :_V W) \subseteq (A :_V W) + \text{Soc}(V)$  or  $ab \in (A :_V W) + \text{Soc}(V)$ . Then  $(A :_V W)$  is *Soc - T - ABSO* ideal of  $V$ .

**Proposition 3.8:** If  $A$  is *Soc-T-ABSO* submodule of a  $V$ -module  $W$ , with  $\text{soc}(W) \subseteq A$  then  $(A + \text{Soc}(W) :_V \langle x \rangle)$  is *Soc - T - ABSO* ideal of  $V$  for all  $x \in W - (A + \text{Soc}(W))$ .

**Proof:** Let  $x \in W - (A + \text{Soc}(W))$ , then  $(A + \text{Soc}(W) :_V \langle x \rangle)$  is a proper ideal of  $V$ . Let  $abc \in (A + \text{Soc}(W) :_V \langle x \rangle)$  for some  $a, b, c \in V$ , hence  $abc \langle x \rangle \subseteq A + \text{Soc}(W)$ . So that  $abc \langle x \rangle \subseteq A$ , but  $A$  is *Soc - T - ABSO* submodule of  $W$ , then  $a \langle x \rangle \subseteq A + \text{Soc}(W)$  or  $bc \langle x \rangle \subseteq A + \text{Soc}(W)$  or  $abc \in (A + \text{Soc}(W) :_W)$ . If  $a \langle x \rangle \subseteq A + \text{Soc}(W)$  and hence  $a \in (A + \text{Soc}(W) :_V \langle x \rangle)$  so we get  $a \in (A + \text{Soc}(W) :_V \langle x \rangle) + \text{Soc}(V)$  if  $bc \langle x \rangle \subseteq A + \text{Soc}(W)$ , hence,  $bc \in (A + \text{Soc}(W) :_V \langle x \rangle)$ , then  $bc \in (A + \text{Soc}(W) :_V \langle x \rangle) + \text{Soc}(V)$ . Thus,  $(A + \text{Soc}(W) :_V \langle x \rangle)$  is *Soc - T - ABSO* ideal of  $V$ .

**Proposition 3.9:** If  $A$  be a proper submodule of a  $V$ -module  $W$  and  $\text{soc}(W) \subseteq A$ , then the following expressions are equivalent :

1.  $A$  is *Soc - T - ABSO* submodule of  $W$ .
2.  $(A + \text{Soc}(W) :_W I)$  is *Soc - T - ABSO* submodule of  $W$  for all ideal  $I$  of  $V$  with  $IW \not\subseteq A + \text{Soc}(W)$ .
3.  $(A + \text{Soc}(W) :_W \langle a \rangle)$  is *Soc-T-ABSO* submodule of  $W$  for all  $a$  of  $V$  with  $aW \not\subseteq A + \text{Soc}(W)$ .

**Proof:** (1)  $\rightarrow$  (2) let  $I$  be ideal of  $V$  with  $IW \not\subseteq A + \text{Soc}(W)$ . Then  $(A + \text{Soc}(W) :_W I)$  is a proper submodule of  $W$ , let  $a, b \in V, x \in W$  such that  $abx \in (A + \text{Soc}(W) :_W I)$ , hence  $ab(Ix) \subseteq A + \text{Soc}(W)$ , then  $ab(Ix) \subseteq A$ , but  $A$  is *Soc - T - ABSO* submodule of  $W$ . Then either  $a(Ix) \subseteq A + \text{Soc}(W)$  or  $b(Ix) \subseteq A + \text{Soc}(W)$  or  $ab \in (A + \text{Soc}(W) :_V W) \subseteq ((A + \text{Soc}(W) :_W I) :_V W)$

and hence  $ax \in (A + \text{Soc}(W):_W I)$  or  $bx \in (A + \text{Soc}(W):_W I)$  or  $ab \in ((A + \text{Soc}(W):_W I):_V W)$ , then  $(A + \text{Soc}(W):_W I)$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$

(2)  $\rightarrow$  (3) it is clear

(3)  $\rightarrow$  (1) if  $a = 1$  then  $(A + \text{Soc}(W):_W < 1 >) = A + \text{Soc}(W)$ , then  $A + \text{Soc}(W)$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$  since  $\text{Soc}(W) \subseteq A$ . Thus  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proposition 3.10:** Let  $A, B$  and  $D$  be submodules of a  $V$ -module  $W$  with  $D \subseteq B \subseteq \text{Soc}(A)$  and if  $B$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $A$ , then  $D$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $A$ .

**Proof:** Let  $abx \in D$  for  $a, b \in V$  and  $x \in A$  since  $D \subseteq B$ , then  $abx \in B$ , but  $B$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $A$ . This means that either  $ax \in B + \text{Soc}(A)$  or  $bx \in B + \text{Soc}(A)$  or  $ab \in (B + \text{Soc}(A):_V A)$ . Then either  $ax \in \text{Soc}(A) \subseteq D + \text{Soc}(A)$  or  $bx \in D + \text{Soc}(A)$  or  $abA \subseteq \text{Soc}(A) \subseteq D + \text{Soc}(A)$  since  $B \subseteq \text{Soc}(A)$ . Thus,  $D$  is a  $\text{Soc} - T - \text{ABSO}$  submodule of  $A$ .

**Proposition 3.11:** Let  $A$  be a proper submodule of a  $V$ -module  $W$  and  $\text{Soc}(W) \subseteq A$ . Then  $A$  is  $T - \text{ABSO}$  submodule of  $W$  if and only if  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proof:**  $\Rightarrow$ ) It is clear

$\Leftarrow$ ) Let  $abx \in A$  for some  $a, b \in V, x \in W$ . Since  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ , then  $ax \in A + \text{Soc}(W)$  or  $bx \in A + \text{Soc}(W)$  or  $ab \in (A + \text{Soc}(W):_V W)$ , but  $\text{Soc}(W) \subseteq A$ , then  $ax \in A$  or  $bx \in A$  or  $ab \in (A:_V W)$ . Thus,  $A$  is  $T - \text{ABSO}$  submodule of  $W$ .

**Theorem 3.12:** Let  $W, W'$  be two  $V$ -modules,  $f: W \rightarrow W'$  be an isomorphism and  $\text{Im}(f) \leq_e W'$ . If  $B$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W'$  then  $f^{-1}(B)$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proof:** Since  $B < W'$ , then  $f^{-1}(B) < W$  since  $f$  is epimorphism. Let  $abx \in f^{-1}(B)$  for some  $a, b \in V, x \in W$  hence,  $abf(x) \in B$ . But  $B$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W'$ , then either  $af(x) \in B + \text{Soc}(W')$  or  $bf(x) \in B + \text{Soc}(W')$  or  $ab \in (B + \text{Soc}(W'):_W W')$  then  $ax \in f^{-1}(B) + \text{Soc}(W)$  or  $bx \in f^{-1}(B) + \text{Soc}(W)$  or  $ab \in (B + \text{Soc}(W'):_W W')$  since  $f^{-1}(\text{Soc}(W')) = \text{Soc}(W)$ , if  $f: W \rightarrow W'$  be an monomorphism and  $\text{Im}(f) \leq_e W'$  by Proposition(2. 9) If  $ab \in (B + \text{Soc}(W'):_W W')$ , so that  $abW' \subseteq B + \text{Soc}(W')$ , but  $f(W) \subseteq W'$ , hence  $abf(W) \subseteq B + \text{Soc}(W')$  and so that  $abW \subseteq f^{-1}(B) + \text{Soc}(W)$ , then  $ab \in (f^{-1}(B) + \text{Soc}(W):_W W)$ . Thus,  $f^{-1}(B)$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Theorem 3.13:** Let  $W, W'$  be two  $V$ -modules,  $f: W \rightarrow W'$  be an epimorphism. If  $H$  is Soc -  $T$  - ABSO submodule of  $W$ , with  $\ker f \subseteq H$ , then  $f(H)$  is Soc- $T$ -ABSO submodule of  $W'$

**Proof:** Let  $aby \in f(H)$ , where  $y \in W', a, b \in V$  and  $y = f(x)$  for some  $x \in W$ , since  $f$  is epimorphism, then  $abf(x) \in f(H)$ , so that  $abf(x) = f(h)$  for some  $h \in H$ . Then we have  $f(abx) - f(h) = 0$ , hence  $abx - h \in \ker f \subseteq H$ . So that  $abx \in H$ , but  $H$  is Soc -  $T$  - ABSO submodule of  $W$ . Then either  $ax \in H + \text{Soc}(W)$  or  $bx \in H + \text{Soc}(W)$  or  $ab \in (H + \text{Soc}(W))_{:V} W$ . If  $ax \in H + \text{Soc}(W)$ , then  $f(ax) \in f(H + \text{Soc}(W))$  and hence  $af(x) \in f(H) + f(\text{Soc}(W))$ . Thus  $ay \in f(H) + \text{Soc}(W')$  (since  $f(\text{Soc}(W)) \subseteq \text{Soc}(W')$ ) similarly if  $bx \in H + \text{Soc}(W)$ , hence  $by \in f(H) + \text{Soc}(W')$ . Finally if  $ab \in (H + \text{Soc}(W))_{:V} W$ , so that  $abW \subseteq H + \text{Soc}(W)$ , then  $f(abW) \subseteq f(H + \text{Soc}(W))$ , hence  $abW' \subseteq f(H) + f(\text{Soc}(W))$ , so  $abW' \subseteq f(H) + \text{Soc}(W')$ . Then  $ab \in (f(H) + \text{Soc}(W'))_{:V} W'$  thus  $f(H)$  is Soc -  $T$  - ABSO submodule of  $W'$

**Corollary 3.14:** Let  $W$  be a  $V$ -module and  $A$  and  $B$  be submodules of  $W$  with  $B \subseteq A$  then  $A$  is Soc -  $T$  - ABSO submodule of  $W$ . Then  $\frac{A}{B}$  is Soc -  $T$  - ABSO submodule of  $\frac{W}{B}$ .

**Proof:** Consider that the canonical homomorphism  $\pi: W \rightarrow \frac{W}{B}$  defined by  $\pi(x) = x + B \forall x \in W$ . Then the proof is clear by Theorem(3. 14)

**Proposition 3.15:** If  $A$  is Soc -  $T$  - ABSO submodule of a  $V$ -module  $W$ , such that  $\text{Soc}(W) \subseteq A$ , then  $(A_{:V} W)$  is Soc -  $T$  - ABSO ideal of  $V$ .

**Proof:** Let  $abc \in (A_{:V} W)$  for some  $a, b, c \in V$  then  $abcW \subseteq A$ , but  $A$  is Soc -  $T$  - ABSO submodule of  $W$ , then  $a(cW) \subseteq A + \text{Soc}(W)$  or  $b(cW) \subseteq A + \text{Soc}(W)$  or  $ab \in (A + \text{Soc}(W))_{:V} W$ . Since  $\text{Soc}(W) \subseteq A$ , we get  $a(cW) \subseteq A$  or  $b(cW) \subseteq A$  or  $abW \subseteq A$ , hence,  $ac \in (A_{:R} W)$  or  $bc \in (A_{:V} W)$  or  $ab \in (A_{:V} W)$ , so that  $ac \in (A_{:V} W) + \text{Soc}(V)$  or  $bc \in (A_{:V} W) + \text{Soc}(V)$  or  $ab \in (A_{:V} W) + \text{Soc}(V)$ . Thus  $(A_{:V} W)$  is Soc -  $T$  - ABSO ideal of  $V$ .

**Proposition 3.16:** Let  $A$  be a proper submodule of a  $V$ -module  $W$ , with  $A = \cap_{i \in \Lambda} B_i$  where  $B_i$  is  $T$  - ABSO submodule of  $W$  for each  $i \in \Lambda$ , then  $A$  is Soc -  $T$  - ABSO submodule of  $W$ .

**Proof:** Suppose that  $abx \in A$  for some  $a, b \in V, x \in W$ , hence  $abx \in \cap_{i \in \Lambda} B_i$ , and so that  $abx \in B_i, \forall i \in \Lambda$ . Since  $B_i$  is  $T$  - ABSO in  $W$ , then either  $ax \in B_i$  or  $bx \in B_i$  or  $ab \in (B_i_{:V} W), \forall i \in \Lambda$ , hence,  $ax \in \cap_{i \in \Lambda} B_i$  or  $bx \in \cap_{i \in \Lambda} B_i$  or  $ab \in (\cap_{i \in \Lambda} B_i)_{:V} W$ , then  $ax \in \cap_{i \in \Lambda} B_i + \text{Soc}(W)$  or  $bx \in \cap_{i \in \Lambda} B_i + \text{Soc}(W)$  or

$ab \in (\cap_{i \in A} B_i + \text{Soc}(W))_{;V} W$ . Thus,  $A = \cap_{i \in A} B_i$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proposition 3.17:** Let  $A$  be a proper submodule of a  $V$ -module  $W'$  and let  $W, W'$  be  $V$ -modules, with  $A + \text{Soc}(W')$  is  $T - \text{ABSO}$  submodule of  $W'$ . Then a proper submodule  $\text{Hom}_V(W, A + \text{soc}(W'))$  of  $\text{Hom}_V(W, W')$  is a  $\text{Soc} - T - \text{ABSO}$  submodule of  $\text{Hom}_V(W, W')$ .

**Proof:** Let  $abf \in \text{Hom}_V(W, A + \text{Soc}(W'))$ , for some  $a, b \in V, f \in \text{Hom}_V(W, W')$ . Then for all  $x \in W, abf(x) \in A + \text{Soc}(W')$ . But  $A + \text{Soc}(W')$  is  $T - \text{ABSO}$  submodule of  $W'$ . Then either  $af(x) \in A + \text{Soc}(W')$  or  $bf(x) \in A + \text{Soc}(W')$  of  $abW' \subseteq A + \text{Soc}(W')$  and hence  $af \in \text{Hom}_V(W, A + \text{Soc}(W')) + \text{Soc}(\text{Hom}_V(W, W'))$ , since  $[\text{Hom}_V(W, A + \text{Soc}(W')) \subseteq \text{Hom}_R(W, A + \text{Soc}(W')) + \text{Soc}(\text{Hom}_V(W, W'))]$  or  $bf \in \text{Hom}_V(W, A + \text{Soc}(W')) + \text{Soc}(\text{Hom}_V(W, W'))$  or  $ab \text{Hom}_V(W, W') \subseteq \text{Hom}_V(W, A + \text{Soc}(W'))$ . Then  $\text{Hom}_V(W, A + \text{Soc}(W'))$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $\text{Hom}_V(W, W')$ .

**Proposition 3.18:** Let  $A$  be  $T - \text{ABSO}$  submodule of an  $V$ -module  $W$ , with  $A \subseteq \text{Soc}(W)$ . Then  $(A_{;W} yz) = W$  or  $(A_{;W} yz) \subseteq (\text{Soc}(W)_{;W} y) \cup (\text{Soc}(W)_{;W} z)$  for all  $y, z \in V$ .

**Proof:** Let  $A$  be  $T - \text{ABSO}$  submodule of  $W$  if  $yz \in (A_{;V} W)$ . Then  $(A_{;W} yz) = W$  for all  $y, z \in V$ , if  $yz \notin (A_{;V} W)$ , then let  $x \in (A_{;W} yz)$ , hence  $yzx \in A$ , but  $A$  is  $T - \text{ABSO}$  submodule, then either  $yx \in A$  or  $zx \in A$ , and so  $yx \in A + \text{Soc}(W)$  or  $zx \in A + \text{Soc}(W)$ , but  $A \subseteq \text{Soc}(W)$ , then  $yx \in \text{Soc}(W)$  or  $zx \in \text{Soc}(W)$ , hence  $x \in (\text{Soc}(W)_{;W} y)$  or  $x \in (\text{Soc}(W)_{;W} z)$ . So, we get  $(A_{;W} yz) \subseteq (\text{Soc}(W)_{;W} y) \cup (\text{Soc}(W)_{;W} z)$  for all  $y, z \in V$ .

**Proposition 3.19:** Let  $A$  be a proper submodule of a  $V$ -module  $W$ , and  $(A + \text{Soc}(W)_{;W} yz) = W$  or  $(A + \text{Soc}(W)_{;W} yz) \subseteq (\text{Soc}(W)_{;W} y) \cup (\text{Soc}(W)_{;W} z)$  for all  $y, z \in V$  then  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proof:** Let  $yzx \in A$  for some  $y, z \in V, x \in W$ , suppose that  $yz \notin (A + \text{Soc}(W)_{;V} W)$ , then  $(A + \text{Soc}(W)_{;W} yz) \neq W$ . Now, let  $x \in (A + \text{Soc}(W)_{;W} yz) \subseteq (\text{Soc}(W)_{;W} y)$  or  $x \in (A + \text{Soc}(W)_{;W} yz) \subseteq (\text{Soc}(W)_{;W} z)$ , then  $yx \in \text{Soc}(W)$  or  $zx \in \text{Soc}(W)$ , hence  $yx \in A + \text{Soc}(W)$  or  $zx \in A + \text{Soc}(W)$ . Thus  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proposition 3.20:** Let  $A$  be a proper submodule of a  $V$ -module  $W$ . If  $A$  is intersection complement and addition complement of  $\text{Soc}(W)$ . Then  $A$  is  $\text{Soc} - T - \text{ABSO}$  submodule of  $W$ .

**Proof:** Let  $abx \in A$  for some  $a, b \in V, x \in W$ . Since  $A$  is intersection complement and addition complement of  $Soc(W)$ , then  $A \oplus Soc(W) = W$ . Hence,  $A + Soc(W) = W$  and  $A \cap Soc(W) = \{0\}$ , implies that  $ax \in A + Soc(W)$  and  $bx \in A + Soc(W)$ . Hence  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

**Proposition 3.21:** Let  $A$  be a proper submodule of a  $V - module W$ . If  $A$  is addition complement of  $Soc(W)$ , then  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

**Proof:** Let  $abx \in A$  for some  $a, b \in V, x \in W$ . Since  $A$  is addition complement of  $Soc(W)$ , then  $A + Soc(W) = W$  and hence  $ax \in A + Soc(W)$  and  $bx \in A + Soc(W)$ . Hence  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

**Proposition 3.22:** Let  $A$  and  $B$  be submodules of a  $V - module W$ . If  $\langle A \cup B \rangle$  is  $T - ABSO$  submodule of  $W$  and  $B \subseteq Soc(W)$ . Then  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

**Proof:** Let  $abx \in A$  for some  $a, b \in R, x \in W$ , hence  $abx \in \langle A \cup B \rangle$ , then either  $ax \in \langle A \cup B \rangle$  or  $bx \in \langle A \cup B \rangle$  or  $abW \subseteq \langle A \cup B \rangle$ . Since  $\langle A \cup B \rangle = A + B$  then either  $ax \in A + B$  or  $bx \in A + B$  or  $abW \subseteq A + B$ , but  $B \subseteq Soc(W)$ . Hence, either  $ax \in A + Soc(W)$  or  $bx \in A + Soc(W)$  or  $abW \subseteq A + Soc(W)$ , so that  $A$  is  $Soc - T - ABSO$  submodule of  $W$ .

## Conclusion

As a new generalization of the  $T-ABSO$  submodule, the  $Soc-T-ABSO$  submodule are introduced. The following are the study's main findings:

- (1) It is evident that every  $T - ABSO$  submodule is  $Soc - T - ABSO$  submodule, but the converse incorrect in general see Remark and Example(3. 3).
- (2) It is evident that every quasi-*prime* submodule is  $Soc - T - ABSO$  submodule, but the converse incorrect in general see Remark and Example(3. 3).
- (3) A proper submodule  $A$  of a  $V - module W$  is  $Soc - T - ABSO$  submodule of  $W$  if and only if  $IJH \subseteq A$  for some ideals  $I, J$  of  $V$  and some submodule  $H$  of  $W$  then either  $IH \subseteq A + Soc(W)$  or  $JH \subseteq A + Soc(W)$  or  $IJ \subseteq (A + Soc(W))_{:V} W$ .
- (4) If a proper submodule  $A$  of a  $V - module W$  is  $T - ABSO$  submodule of  $W$ , then  $(A_{:V} W)$  is  $Soc - T - ABSO$  ideal of  $V$ .
- (5) If  $W$  be a cyclic a  $V - module$  and  $A$  is  $Soc - T - ABSO$  submodule of  $W$ . Then  $(A + Soc(W))_{:V} W$  is  $Soc - T - ABSO$  ideal of  $V$ .
- (6) If  $A, B$  are  $Soc - T - ABSO$  submodules of  $W$  with  $A \subseteq Soc(W)$  and  $B \subseteq Soc(W)$ , then  $A \cap B$  is  $Soc - T - ABSO$  submodule of  $W$ .
- (7) When  $A$  be a proper submodule of  $V - module W'$  such that  $W, W'$  be  $V - modules$ , with  $A + Soc(W')$  is  $T - ABSO$  of  $W'$ . Then a proper

- submodule  $\text{Hom}_R(W, A + \text{soc}(W'))$  of  $\text{Hom}_V(W, W')$  is a Soc – T – ABSO submodule of  $\text{Hom}_V(W, W')$ .
- (8) If  $A$  is intersection complement and addition complement of  $\text{Soc}(W)$ , then  $A$  is Soc – T – ABSO submodule of  $W$ .

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