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Rad_g-lifting modules

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Abstract

In this article, we present a different class of lifting modules as a proper g-lifting generalization. This class termed by Rad_g-lifting in this work which defined as, $\mathcal{W}$ is called Rad_g-lifting if $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$, there is $\mathcal{W} = A \oplus B$ with $A \leq \Upsilon$ and $\Upsilon \cap B \ll_g B$. Thus, a ring R named Rad_g-lifting if R is a Rad_g-lifting as R-module. We determined it is structure. Several of characterizations, properties, and instances of these modules are described.

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1. Introduction

This article is about associative rings with identity, and All modules have been left as unitary. We will go through several of the key definitions that we will require in our work. Let $\mathcal{B}$ a submodule of a module of $\mathcal{W}$, indicated by $\mathcal{B} \leq \mathcal{W}$. Also, we refer to the direct summand $\mathcal{B}$ of $\mathcal{W}$ by $\mathcal{B} \leq^{\oplus} \mathcal{W}$. An essential module $\mathcal{B}$ in $\mathcal{W}$ indicated by $\mathcal{B} \trianglelefteq \mathcal{W}$, is a submodule which satisfying $\mathcal{B} \cap \Upsilon = 0$

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implies $\Upsilon = 0$ for any Υ in $\mathcal{W}$. As dual, if $\mathcal{B} \leq \mathcal{W}$ is named small in $\mathcal{W}$, written as $\mathcal{B} \ll \mathcal{W}$ if, $\mathcal{W} = \mathcal{B} + \Upsilon$ for $\Upsilon \leq \mathcal{W}$ signifies $\Upsilon = \mathcal{W}$ [8]. We refer to $Rad(\mathcal{W})$ as the Jacobson radical of $\mathcal{W}$, as the sum of all small submodules of $\mathcal{W}$. If There is no maximal submodule of $\mathcal{W}$, then we set $Rad(\mathcal{W}) = \mathcal{W}$. Zhou [20] recall that, a submodule Υ of $\mathcal{W}$ is called generalized small, indicated by $\Upsilon \ll_g \mathcal{W}$, if for $\mathcal{B} \trianglelefteq \mathcal{W}$ with $\mathcal{W} = \Upsilon + \mathcal{B}$ signifies $\mathcal{B} = \mathcal{W}$, in fact, A g-small is referred to as an e-small by Zhou and Zhang [7]. If all proper submodules of $\mathcal{W}$ is small (g-small), then $\mathcal{W}$ is called hollow (generalized hollow) see ([16, 11]). Zhou and Zhang [7] gave the definition of Jacobson generalized radical of R-module $\mathcal{W}$ as; $Rad_g(\mathcal{W}) = \bigcap\{\tau \trianglelefteq \mathcal{W} \mid \tau \text{ is maximal in } \mathcal{W}\} = \sum\{\tau \mid \tau \ll_g \mathcal{W}\}$. Let $\Upsilon, \omega \leq \mathcal{W}$. If Υ is minimal in $\mathcal{W} = \Upsilon + \omega$, we called it Υ is a supplement of ω. A module is said to be supplemented if it contains a supplement for each of its submodules [16]. Let $\aleph$ and γ be submodules If $\mathcal{W} = \aleph + \gamma$ and $\mathcal{W} = \aleph + \eta$ with $\eta \trianglelefteq \gamma$ then $\eta = \gamma$, as well, if $\mathcal{W} = \aleph + \gamma$ and $\aleph \cap \gamma \ll_g \gamma$, then γ is called a g-supplement of $\aleph$ in $\mathcal{W}$. A module is called g-supplemented if it contains g-supplement for each submodule [3]. A module $\mathcal{W}$ is known as lifting. if, $\Upsilon \leq \mathcal{W}$, and $\aleph \leq \Upsilon$, there is $\mathcal{W} = \aleph \oplus \mathcal{B}$ and $\Upsilon \cap \mathcal{B} \ll \mathcal{W}$ [6]. Furthermore, A module $\mathcal{W}$ is called g-lifting if for $\Upsilon \leq \mathcal{W}$, there is $\mathcal{W} = \aleph \oplus \mathcal{B}$ and $\Upsilon \cap \mathcal{B} \ll_g \mathcal{W}$ where $\aleph \leq \Upsilon$ [17]. We are going to define the $\mathrm{Rad_g}$-lifting modules and give some equivalent definitions. Next some of the numerous characteristics of this group of modules and their relationships to other types of modules: Noetherian, uniserial modules, … etc, (see [2, 19, 14, 21… 24 and 25]). A non zero $\mathrm{Rad_g}$-lifting module is not inherited by its submodules, but it is inherited by direct summand and g-coclosed. So we give a condition that make under finite direct sums, the family of $\mathrm{Rad_g}$-lifting modules is closed. Later it is proved that factor module of a duo (or distributive) $\mathrm{Rad_g}$-liftingmodule is $\mathrm{Rad_g}$-lifting. We provide some important counter examples to separate $\mathrm{Rad_g}$-lifting property of modules with some other kinds of modules.

2. $\mathbf{Rad_g}$-Lifting modules

First we will present the definition of $\mathrm{Rad_g}$-lifting and give equivalent definitions. As well as many of their properties, examples about them will be discussed. Later we will discuss the relations between our definition and other types of modules.

First, we will consider our following major definition.

Definition 2.1: A module $\mathcal{W}$ is called $\mathrm{Rad_g}$-lifting if for every submodule Υ of $\mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$, there is a decomposition $\mathcal{W} = \aleph \oplus \beta$ such that $\aleph \leq \Upsilon$ and $\Upsilon \cap \beta$ is g-small in β. Thus, a ring R is called $\mathrm{Rad_g}$-lifting if it is $\mathrm{Rad_g}$-lifting as R-module.

Now give equivalent definitions of our main definition "$\mathrm{Rad_g}$-lifting" and prove the equivalency between them.

Proposition 2.2: An assertions below are equivalent.

(1) $\mathcal{W}$ is Rad$_g$-lifting; ε

(2) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, there is a decomposition $\mathcal{W} = \aleph \oplus \beta$ and $\Upsilon \cap \beta \ll_g \mathcal{W}$ where $\aleph \leq \Upsilon$;

(3) for every submodule Υ of $\mathcal{W}$ has $Rad_g(\mathcal{W})$ can be written as $\Upsilon = \aleph \oplus \beta$, where $\aleph \leq^{\oplus} \mathcal{W}$ and $\beta \ll_g \mathcal{W}$;

(4) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, there exists a direct summand $\aleph$ of $\mathcal{W}$ where $\aleph \leq \Upsilon$ and $\Upsilon/\aleph \ll_g \mathcal{W}/\aleph$;

(5) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, Υ has g-supplement $\aleph$ in $\mathcal{W}$ where $\Upsilon \cap \aleph \leq^{\oplus} \Upsilon$;

(6) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, there is $\varepsilon = \varepsilon^2 \in End(\mathcal{W})$ with $\varepsilon\mathcal{W} \leq \Upsilon$ and $(1-\varepsilon)\Upsilon \ll_g (1-\varepsilon)\mathcal{W}$;

(7) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, there exists $\aleph \leq^{\oplus} \mathcal{W}$ and $\beta \ll_g \mathcal{W}$ with $\aleph \leq \Upsilon$ and $\Upsilon = \aleph + \beta$;

(8) for $\Upsilon \leq \mathcal{W}$ has $Rad_g(\mathcal{W})$, there is a submodule $\aleph$ of $\mathcal{W}$ and $\aleph \leq \Upsilon$ with $\mathcal{W} = \aleph \oplus \beta$ and β a g-supplement of Υ in $\mathcal{W}$.

Proof: (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (1) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By (2), there is a decomposition $\mathcal{W} = \aleph \oplus \beta$ where $\aleph \leq \Upsilon$ and $\Upsilon \cap \beta \ll_g \mathcal{W}$. Since $\Upsilon \cap \beta \leq \beta \leq^{\oplus} \mathcal{W}$, thus by [16, Proposition 3.2], $\Upsilon \cap \beta \ll_g \beta$. Hence $\mathcal{W}$ is Rad$_g$-lifting. $\mathcal{K}$

(1) $\Longrightarrow$ (3) Assume $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By (1), there is a decomposition $\mathcal{W} = \aleph \oplus \mathcal{K}$ where $\aleph \leq \Upsilon$ and $\Upsilon \cap \mathcal{K} \ll_g \mathcal{K}$. By the modular law, $\Upsilon = \aleph \oplus (\Upsilon \cap \mathcal{K})$. Putting $\Upsilon \cap \mathcal{K} = \beta$. Thus, $\Upsilon = \aleph \oplus \beta$ such that $\aleph \leq^{\oplus} \mathcal{W}$ and $\beta \ll_g \mathcal{W}$.

(3) $\Longrightarrow$ (4) Assume that $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. We have $\Upsilon = \mathcal{B} \oplus \aleph$ where $\mathcal{B} \leq^{\oplus} \mathcal{W}$ and $\aleph \ll_g \mathcal{W}$, by (2). Define a conical map $\pi: \mathcal{W} \longrightarrow \mathcal{W}/\mathcal{B}$. Since $\aleph \ll_g \mathcal{W}$, we conclude $\pi(\mathcal{B}) \ll_g \mathcal{W}/\mathcal{K}$, i.e, $(\mathcal{B} + \aleph)/\mathcal{B} = \Upsilon/\mathcal{B} \ll_g \mathcal{W}/\mathcal{B}$.

(4) $\Longrightarrow$ (1) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. Then there is $\mathcal{W} = \mathcal{K} \oplus \beta$ with $\mathcal{K} \leq \Upsilon$ and $\Upsilon/\mathcal{K} \ll_g \mathcal{W}/\mathcal{K}$, by (4). We can conclude this from modular law. $\Upsilon = \mathcal{K} \oplus (\Upsilon \cap \aleph)$. Thus, $\Upsilon/\mathcal{K} \cong \Upsilon \cap \aleph$ and $\mathcal{W}/\mathcal{K} \cong \aleph$. It follows that $\Upsilon \cap \aleph \ll_g \aleph$. Hence, (1) holds.

(1) $\Longrightarrow$ (5) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By (1), there exists $\mathcal{W} = \mathcal{K} \oplus \aleph$ with $\mathcal{K} \leq \Upsilon$ and $\Upsilon \cap \aleph \ll_g \aleph$. So, $\mathcal{W} = \Upsilon + \aleph$ and by modular low $\Upsilon = \mathcal{K} \oplus (\aleph \cap \Upsilon)$. Hence Υ has a g-supplement $\aleph$ in $\mathcal{W}$ where $\Upsilon \cap \aleph \leq^{\oplus} \Upsilon$.

(5) $\Longrightarrow$ (6) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By hypothesis, if we assume Υ has a g-supplement $\aleph$ in $\mathcal{W}$ with $\Upsilon \cap \aleph \ll_g \Upsilon$, i.e, $\mathcal{W} = \Upsilon + \aleph$ and $\Upsilon \cap \aleph \ll_g \aleph$. Also, $\Upsilon = (\Upsilon \cap \aleph) \oplus \beta$ for $\beta \leq \Upsilon$. It follows that $\mathcal{W} = \beta \oplus \aleph$. Suppose that $\varepsilon: \mathcal{W} \rightarrow \beta$; $\varepsilon(k+h) = k$ and $(1-\varepsilon): \mathcal{W} \longrightarrow \aleph$; $(1-\varepsilon)(k+h) = h$ are projection maps for all $k + h \in \mathcal{W} = \beta \oplus \aleph$. It is easily to confirm that $\varepsilon = \varepsilon^2 \in End(\mathcal{W})$. Hence, $\varepsilon\mathcal{W} = \beta \leq \Upsilon$ and $(1-\varepsilon)\Upsilon = \Upsilon \cap (1-\varepsilon)\mathcal{W} = \Upsilon \cap \aleph \ll_g \aleph = (1-\varepsilon)\mathcal{W}$.

(6) $\Longrightarrow$ (3) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By (6), there is an $\xi = \xi^2 \in End(\mathcal{W})$ where $\xi\mathcal{W} \leq \Upsilon$ and $(1-\xi)\Upsilon \ll_g (1-\xi)\mathcal{W}$. Observe that $\mathcal{W} = \xi\mathcal{W} \oplus (1-\xi)\mathcal{W}$. By modular law, we get that $\Upsilon = \Upsilon \cap \mathcal{W} = \Upsilon \cap (\xi\mathcal{W} \oplus (1-\xi)\mathcal{W}) = \xi\mathcal{W} \oplus (\Upsilon \cap (1-\xi)\mathcal{W}) = \xi\mathcal{W} \oplus (1-\xi)\Upsilon$, such that $\xi\mathcal{W} \leq^{\oplus} \mathcal{W}$ and $(1-\xi)\Upsilon \ll_g (1-\xi)\mathcal{W}$, so in $\mathcal{W}$.

(1) $\Longrightarrow$ (7) Let $\Upsilon \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \Upsilon$. By (1), there exists the submodule $\mathrm{K} \leq \Upsilon \leq \mathcal{W}$, where $\mathcal{W} = \mathrm{K} \oplus \beta$ and $\Upsilon \cap \beta \ll_g \beta$. Put $\aleph = \Upsilon \cap \beta$, we conclude that $\Upsilon = \mathrm{K} + \aleph$ and $\aleph \ll_g$ B, so in $\mathcal{W}$.

(7) $\Longrightarrow$ (1) Assume $\Upsilon \leq \mathcal{W}$ with $Rad_{\mathrm{g}}(\mathcal{W}) \subseteq \Upsilon$, so by (7), there exists $\aleph \leq^{\oplus} \mathcal{W}$ and $\mathrm{K} \ll_g \mathcal{W}$ where $\aleph \leq \Upsilon$ and $\Upsilon = \aleph + \mathrm{K}$. Thus, $\mathcal{W} = \aleph \oplus \beta$ for some $\beta \leq \mathcal{W}$. Therefore, β is a g-supplement of $\aleph$ in $\mathcal{W}$, hence, by [18, Lemma 2.3] β is a g-supplement of $\Upsilon = \aleph + \mathrm{K}$ in $\mathcal{W}$, thus $\Upsilon \cap \beta \ll_g \beta$.

(1) $\Longrightarrow$ (8) Let $\Upsilon \leq \mathcal{W}$ and $Rad_g(\mathcal{W}) \subseteq \Upsilon$. Since $\mathcal{W}$ is a $\mathrm{Rad_g}$-lifting, thus for $\aleph \leq \Upsilon$ where $\mathcal{W} = \aleph \oplus \mathrm{K}$ and $\Upsilon \cap \mathrm{K} \ll_g \mathrm{K}$ for $\mathrm{K} \leq \mathcal{W}$. Thus $\mathcal{W} = \Upsilon + \mathrm{K}$ that implies K is a g-supplement of Υ in $\mathcal{W}$. (8) $\Longrightarrow$ (1) Clear.

Now give important proposition, which lead us to our dentition directly and then give us interesting related results later.

Proposition 2.3: If $Rad_g(\mathcal{W}) = \mathcal{W}$ for a module $\mathcal{W}$, then $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting.

Proof: Let $\mathrm{K} \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \mathrm{K}$. By assumption, $\mathcal{W} \leq \mathrm{K}$ that implies $\mathcal{W} = \mathrm{K}$. Clearly, $\mathcal{W} = \mathcal{W} \oplus (0)$ for $\mathcal{W} \leq \mathrm{K}$ and $\mathrm{K} \cap (0) = (0)$ is g-small in (0), as required.

Corollary 2.4: *If $\mathcal{W}$ is an arbitrary module If here is no maximal submodule., then $\mathcal{W}$ is Rad_g-lifting.*

Proof: Since $\mathcal{W}$ has no maximal, then $Rad(\mathcal{W}) = \mathcal{W}$. As $Rad(\mathcal{W}) \subseteq Rad_g(\mathcal{W})$, then $Rad_g(\mathcal{W}) = \mathcal{W}$. By Proposition 2.3, We obtained the outcome.

Corollary 2.5: *If $\mathcal{W}$ is a module which all its finitely generated submodules are g-small, then $\mathcal{W}$ is Rad_g-lifting.*

Proof: By [12, Proposition 3.15] $Rad_g(\mathcal{W}) = \mathcal{W}$. Next by Proposition 2.3, $\mathcal{W}$ is a $\mathrm{Rad_g}$-lifting.

As an illustration of Corollary 2.5 see this example, for any prime p and $n \in \mathbb{Z}_+$, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is $\mathrm{Rad_g}$-lifting, in fact, all its finitely generated submodules are g-small.

Now discusses some prevalent concepts and their relations with $\mathrm{Rad_g}$-lifting module. Recall [15] as the sum of all generalized radical submodules of $\mathcal{W}$, $P_g(\mathcal{W})$; $P_g(\mathcal{W}) = \sum\{\mathrm{K} \leq \mathcal{W} \mid Rad_g(\mathrm{K}) = \mathrm{K}\}$.

Corollary 2.6: *$P_g(\mathcal{W})$ is Rad_g-lifting for a module $\mathcal{W}$.*

Proof: It is enough to show that $Rad_g(P_g(\mathcal{W})) = P_g(\mathcal{W})$. Therefore, $P_g(\mathcal{W}) = \sum\{K \leq \mathcal{W} | Rad_g(K) = K\} = \sum_{K \leq \mathcal{W}} Rad_g(K) = Rad_g(\sum\{K \leq \mathcal{W} | Rad_g(K) = K\}) = Rad_g(P_g(\mathcal{W}))$.By Proposition 2.3, $P_g(\mathcal{W})$ is a Rad_g-lifting.

Remark 2.7: By definitions, any g-lifting and hence any lifting is Rad_g-lifting. In general, the reverse is not correct, as follows: the $\mathbb{Z}$-module $\mathbb{Q}$ is Rad_g-lifting, indeed, $Rad_g(\mathbb{Q}) = \mathbb{Q}$ (see Proposition 2.2), $\mathbb{Q}$ is not g-lifting.

Proposition 2.8: Let $\mathcal{W}$ be a Rad_g-lifting and $\mathcal{B}$ an indecomposable submodule of $\mathcal{W}$ has $Rad_g(\mathcal{W})$, then either $\mathcal{B}$ is a direct summand or g-small.

Proof: Suppose $\mathcal{W}$ is a Rad_g-lifting. Let $\mathcal{B} \leq \mathcal{W}$ and indecomposable with $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. By Proposition 2.2, $\mathcal{B}$ can be written as $\mathcal{B} = \aleph \oplus \beta$, where $\aleph \leq^{\oplus} \mathcal{W}$ and $\beta \ll_g \mathcal{W}$. As a result of this, either $\mathcal{B} = \aleph$ or $\mathcal{B} = \beta$. This brings the proof to a close.

Proposition 2.9: Every generalized hollow module is Rad_g-lifting. $\mathcal{M}$

Proof: Let $\mathcal{W}$ be a generalized hollow, and let $\mathcal{M} \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \mathcal{M}$. If $\mathcal{M} = \mathcal{W}$, clearly, $\mathcal{W} = \mathcal{W} \oplus (0)$ where $\mathcal{W} \leq \mathcal{M}$ and $\mathcal{M} \cap (0) \ll_g (0)$. Assume $\mathcal{M} \subset \mathcal{W}$, then $\mathcal{M} \ll_g \mathcal{W}$. Next, $\mathcal{W} = (0) \oplus \mathcal{W}$ where $(0) \leq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{W} = \mathcal{M} \ll_g \mathcal{W}$, so in $\mathcal{M}$. Hence $\mathcal{W}$ is Rad_g-lifting.

Example 2.10: Every semisimple module is clearly Rad_g-lifting. However, in general, the opposite does not have to be true: it is easy to ensure that $\mathbb{Z}$-module $\mathbb{Z}_8$ is generalized hollow, so it is Rad_g-lifting, by Proposition 2.9. While the $\mathbb{Z}$-module $\mathbb{Z}_8$ is not semisimple.

Example 2.11: In general, the reverse of Proposition 2.9 need not be satisfied, to see this: if $\mathbb{Z}_{24}$ as $\mathbb{Z}$-module is Rad_g-lifting, to see this: it is easy to show $Rad_g(\mathbb{Z}_{24}) = 2\mathbb{Z}_{24}$. Since $2\mathbb{Z}_{24}$ is the only proper submodule of $\mathbb{Z}_{24}$ that contains $Rad_g(\mathbb{Z}_{24})$. Thus, $\mathbb{Z}_{24}$ have trivial decomposition $\mathbb{Z}_{24} = (\bar{0}) \oplus \mathbb{Z}_{24}$ such that $(\bar{0}) \leq 2\mathbb{Z}_{24}$ and $2\mathbb{Z}_{24} \cap \mathbb{Z}_{24} = 2\mathbb{Z}_{24}$ is g-small in $\mathbb{Z}_{24}$. While $\mathbb{Z}$-module$\mathbb{Z}_{24}$ is not generalized hollow. Actuality, $3\mathbb{Z}_{24}$ is not g-small in $\mathbb{Z}_{24}$.

3. Direct summands and factor modules

A Rad_g-lifting is not inherited by their submodules, [see, remark 2.7]. So that we will give some conditions to submodules to make Rad_g-lifting inherited. The same thing is said about factor module and homomorphism of Rad_g-lifting.

First we start with direct summands as follows:

Proposition 3.1: A module $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting if and only if every direct summand of $\mathcal{W}$ contains $Rad_g(\mathcal{W})$ is $\mathrm{Rad_g}$-lifting.

Proof: $\Longrightarrow$) Let $\mathcal{B} \leq^{\oplus} \mathcal{W}$ and $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. If $\Upsilon \leq \mathcal{B}$ with $Rad_g(\mathcal{B}) \subseteq \Upsilon$. By [15, Lemma 3.16] we have $Rad_g(\mathcal{W}) = Rad_g(\mathcal{B})$. Thus, $Rad_g(\mathcal{W}) \subseteq \Upsilon \leq \mathcal{W}$. Since $\mathcal{W}$ is a $\mathrm{Rad_g}$-lifting, there exist $\mathcal{W} = \beta \oplus \aleph$ where $\beta \leq \Upsilon$ and $\Upsilon \cap \aleph \ll_g \aleph$. By the modular law, we have that $\mathcal{B} = \mathcal{B} \cap (\beta \oplus \aleph) = \beta \oplus (\mathcal{B} \cap \aleph)$. Also, we have that $\Upsilon \cap (\mathcal{B} \cap \aleph) = \Upsilon \cap \aleph \ll_g \mathcal{W}$. As $\Upsilon \cap (\mathcal{B} \cap \aleph) \leq \mathcal{B} \cap \aleph \leq^{\oplus} \mathcal{W}$, [12, Proposition 3.2] conclude that $\Upsilon \cap (\mathcal{B} \cap \aleph) \ll_g \mathcal{B} \cap \aleph$. Hence $\mathcal{B}$ is $\mathrm{Rad_g}$-lifting.

$\Longleftarrow$) Since, trivially, $\mathcal{W} \leq^{\oplus} \mathcal{W}$ and $Rad_g(\mathcal{W}) \subseteq \mathcal{W}$, thus, by assumption $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting.

Corollary 3.2: *Let $\mathcal{W}$ be a module and $Rad_g(\mathcal{W}) \leq^{\oplus} \mathcal{W}$. If $\mathcal{W}$ is Rad_g-lifting, then $Rad_g(\mathcal{W})$ is Rad_g-lifting also.*

Proof: Clear by Proposition 3.1, since $Rad_g(\mathcal{W}) \subseteq Rad_g(\mathcal{W})$.

Recall [10] that, a submodule $\mathcal{B}$ of $\mathcal{W}$ is a g-coclosed whenever $\beta \leq \mathcal{B} \leq \mathcal{W}$, such that $\mathcal{B}/\beta \ll_g \mathcal{W}/\beta$, then $\mathcal{B} = \beta$. So, the next proposition give important result.

Proposition 3.3: Let $\mathcal{W}$ be a $\mathrm{Rad_g}$-lifting and $\mathcal{B} \leq \mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. If $\mathcal{B}$ is g-coclosed then $\mathcal{B} \leq^{\oplus} \mathcal{W}$.

Proof: Let $\mathcal{B}$ be a g-coclosed of $\mathcal{W}$ with $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. As $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting, by Proposition 2.2, there exists $\omega \leq^{\oplus} \mathcal{W}$ where $\omega \leq \mathcal{B}$ and $\mathcal{B}/\omega \ll_g \mathcal{W}$. Since $\mathcal{B}$ is g-coclosed, $\omega = \mathcal{B}$, that means $\mathcal{B} \leq^{\oplus} \mathcal{W}$.

Corollary 3.4: *Let $\mathcal{W}$ be a Rad_g-lifting, and $\mathcal{B}$ a g-coclosed of $\mathcal{W}$ has $Rad_g(\mathcal{W})$, then $\mathcal{B}$ is Rad_g-lifting.*

Proof: Direct from Propositions 3.3 and 3.1.

As shown below, the direct sum of finite $\mathrm{Rad_g}$-lifting is not $\mathrm{Rad_g}$-lifting in general.

Example 3.5: Put $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$ as $\mathbb{Z}$-module. As seen in Examples 2.10 and 2.11, $\mathbb{Z}_8$ and $\mathbb{Z}_{24}$ are $\mathrm{Rad_g}$-lifting $\mathbb{Z}$-modules. Notice that the formula will be $Rad_g(\mathbb{Z}_8 \oplus \mathbb{Z}_{24}) = Rad_g(\mathbb{Z}_8) \oplus Rad_g(\mathbb{Z}_{24}) = 2\mathbb{Z}_8 \oplus 2\mathbb{Z}_{24}$. Put $\mathcal{B} = \mathbb{Z}_8 \oplus 2\mathbb{Z}_{24} \leq \mathcal{W}$. It follows that $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$ and $\mathcal{B}$ is not entail non zero direct summand of $\mathcal{W}$. Also, simple to demonstrate $\mathcal{B}$ does not g-small in $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$, (as $2\mathbb{Z}_8 \oplus \mathbb{Z}_{24} \trianglelefteq \mathcal{W}$ and $(2\mathbb{Z}_8 \oplus \mathbb{Z}_{24}) + \mathcal{B} = \mathcal{W}$). Thus, $\mathcal{W} = (0) \oplus \mathcal{W}$ is the only decomposition that $(0) \leq \mathcal{B}$ but $\mathcal{B} \cap \mathcal{W} = \mathcal{B} \ll_g \mathcal{W}$. Thus, $\mathcal{W} = \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$ is not a Rad$_g$-lifting.

We now give a condition that make finite direct sums of Rad$_g$-lifting modules is closed.

Theorem 3.6: *Let $\mathcal{W} = \bigoplus_{i=1}^{n} \mathcal{W}_i$ where $\mathcal{W}_i$ are Rad_g-lifting modules for $i = 1, \dots, n$. If $\mathcal{W}$ is a duo module, then $\mathcal{W}$ is Rad_g-lifting.*

Proof: If $n = 2$. Let $\mathcal{B} \leq \mathcal{W}$ and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. Since $\mathcal{B}$ is a fully invariant in $\mathcal{W}$, [1, Lemma 2.1] implies $\mathcal{B} = (\mathcal{W}_1 \cap \mathcal{B}) \oplus (\mathcal{W}_2 \cap \mathcal{B})$. We deduce that $Rad_g(\mathcal{W}_k) \subseteq \mathcal{W}_k \cap \mathcal{B}$ for $k = 1,2$. Since $\mathcal{W}_k$ is Rad$_g$-lifting, then there are decompositions $\mathcal{W}_k = \aleph_k \oplus \tau_k$, with $\aleph_k \leq \mathcal{W}_k \cap \mathcal{B}$ and $(\mathcal{W}_k \cap \mathcal{B}) \cap \tau_k = \mathcal{B} \cap \tau_k \ll_g \tau_k$. Thus, $\mathcal{W} = (\aleph_1 \oplus \aleph_2) \oplus (\tau_1 \oplus \tau_2)$ such that $\aleph_1 \oplus \aleph_2 \leq (\mathcal{W}_1 \cap \mathcal{B}) \oplus (\mathcal{W}_2 \cap \mathcal{B}) = \mathcal{B}$ and $\mathcal{B} \cap (\tau_1 \oplus \tau_2) = (\mathcal{B} \cap \tau_1) \oplus (\mathcal{B} \cap \tau_2)$ is g-small in $\tau_1 \oplus \tau_2$, by [7, Proposition 2.5(3)]. Hence, $\mathcal{W}$ is Rad$_g$-lifting.

Theorem 3.7: *Let $\mathcal{W} = \bigoplus_{i=1}^{n} \mathcal{W}_i$ where $\mathcal{W}_i$ are Rad_g-lifting for $i = 1, \dots, n$. If $\mathcal{W}$ is a distributive, then $\mathcal{W}$ is Rad_g-lifting.*

Proof: Analogous to proof Theorem 3.6.

The factor module of Rad$_g$-lifting will be discussed as in the following.

Theorem 3.8: *Let $\mathcal{W}$ be a Rad_g-lifting, for $\chi \leq \mathcal{W}$ that satisfy one of the following:*

(1) If for any $\gamma \leq^{\oplus} \mathcal{W}$, $(\gamma + \chi)/\chi \leq^{\oplus} \mathcal{W}/\chi$.
(2) If χ is a distributive in $\mathcal{W}$.
(3) If χ is a fully invariant in $\mathcal{W}$.
(4) If $\mathcal{W} = \chi \oplus \gamma$ such thatχ γ has $Rad_g(\mathcal{W})$.

Then $\mathcal{W}/\chi$ is Rad$_g$-lifting.

Proof:

(1) Suppose $\chi \leq \gamma \leq \mathcal{W}$ with $Rad_g(\mathcal{W}/\chi) \subseteq \gamma/\chi$. Define the canonical map by $\pi: \mathcal{W} \to \mathcal{W}/\chi$. By $Rad_g(\mathcal{W}) \subseteq \mathcal{W}$, we conclude that $\pi(Rad_g(\mathcal{W})) \subseteq Rad_g(\mathcal{W}/\chi)$, that is, $(Rad_g(\mathcal{W}) + \chi)/\chi \subseteq Rad_g(\mathcal{W}/\chi)$. Therefore, $(Rad_g(\mathcal{W}) + \chi)/\chi \subseteq \gamma/\chi$, and hence $Rad_g(\mathcal{W}) \subseteq \gamma$. Since $\mathcal{W}$ is a Rad$_g$-lifting, then there exists $\aleph \leq^{\oplus} \mathcal{W}$ where $\aleph \leq \gamma$ and $\gamma/\aleph \ll_g \mathcal{W}/\aleph$, by Proposition 2.2(3). By (1), $(\aleph + \chi)/\chi \leq^{\oplus} \mathcal{W}/\chi$. Clearly, $(\chi + \aleph)/\chi \leq \gamma/\chi$. Consider a projection map $\rho: \frac{\mathcal{W}}{\aleph} \to \frac{\mathcal{W}/\aleph}{(\chi+\aleph)/\aleph}$. Since $\gamma/\aleph \ll_g \mathcal{W}/\aleph$ then $\frac{\gamma}{\chi+\aleph} \ll_g \frac{\mathcal{W}}{\chi+\aleph}$. Hence, $\mathcal{W}/\chi$ is a Rad$_g$-lifting.

(2) Assume that $W = K \oplus \gamma$ for some $\gamma \leq W$. Claim $(K + \chi)/\chi \leq^{\oplus} W/\chi$. Now $W/\chi = ((K + \chi)/\chi) + ((\gamma + \chi)/\chi)$. Now, as χ is a distributive in W, then $(K + \chi) \cap (\gamma + \chi) = (\chi \cap \gamma) + K = K$. So $((K + \chi)/\chi) \cap ((\gamma + \chi)/\chi) = 0$, so by (1) we get the required.

(3) Let $\aleph \leq^{\oplus} W$, then $W = \aleph \oplus \grave{\aleph}$ for some $\grave{\aleph} \leq W$. As χ is a fully invariant in W, then $W/\chi = ((\aleph + \chi)/\chi) \oplus ((\aleph' + \chi)/\chi)$, by [4, Lemma 3.3], i.e, $(\aleph + \chi)/\chi \leq^{\oplus} W/\chi$. Hence W/χ is a $\mathrm{Rad_g}$-lifting module, by (1).

(4) Since $\gamma \leq^{\oplus} W$ has $Rad_g(W)$, thus by Proposition 3.1, γ is a $\mathrm{Rad_g}$-lifting, as $W/\chi \cong \gamma$, Thus W/χ is a $\mathrm{Rad_g}$-lifting.

According to previous conditions in Theorem 3.8 we give some related corollaries.

Corollary 3.9: *If W is Rad_g-lifting, then $W/P_g(W)$ is a Rad_g-lifting also.*

Proof: By [15, Lemma 3.12], $P_g(W)$ is a fully invariant in W, and so by Theorem 3.8(3), $W/P_g(W)$ is $\mathrm{Rad_g}$-lifting.

Corollary 3.10: *The factor module of a distributive (or, duo) Rad_g-lifting module is also Rad_g-lifting.*

Proof: Clear.

Now we will show that the "$\mathrm{Rad_g}$-lifting" has been transmitted under an action of a homomorphism with some conditions as see in the next corollaries.

Corollary 3.11: *Let W be a distributive (or, duo) and Rad_g-lifting. If $f: W \to \grave{W}$ is a homomorphism then $f(W)$ is Rad_g-lifting.*

Proof: Let W be a distributive and $\mathrm{Rad_g}$-lifting module. Consider $f: W \to \grave{W}$ be any homomorphism. By 1st isomorphism theorem, there exists $U \leq W$ such that $W/\gamma \cong f(W)$. By Corollary 3.10, W/γ is a $\mathrm{Rad_g}$-lifting. Hence $f(W)$ is $\mathrm{Rad_g}$-lifting.

Corollary 3.13: *Let W be a distributive, and $\gamma \leq^{\oplus} W$. Then W is Rad_g-lifting if and only if γ and W/γ are both Rad_g-liftings.*

Proof: $\Longrightarrow$) let W be a distributive and $\gamma \leq^{\oplus} W$, so $W = \gamma \oplus \mathcal{B}$ for a submodule $\mathcal{B}$ of W. By Corollary 3.10, W/γ is $\mathrm{Rad_g}$-lifting. However, $\gamma \cong W/\mathcal{B}$, again by Corollary 3.10, γ is a $\mathrm{Rad_g}$-lifting.

$\Longleftarrow$) Since $W \cong \gamma \oplus (W/\gamma)$, the result is comes by Theorem 3.7.

Corollary 3.13: *Let $\mathcal{W}$ be a weak duo, $\mathcal{B} \leq^{\oplus} \mathcal{W}$. If $\mathcal{W}$ is a Rad_g-lifting, then $\mathcal{B}$ and $\mathcal{W}/\mathcal{B}$ are Rad_g-lifting modules.*

Proof: Let $\mathcal{W}$ be a weak duo and $\mathcal{B} \leq^{\oplus} \mathcal{W}$, then $\mathcal{W} = \mathcal{B} \oplus \mathcal{K}$ for some $\mathcal{K} \leq \mathcal{W}$, and $\mathcal{B}$, $\mathcal{K}$ are fully invariant submodules. By Theorem 3.8(3), $\mathcal{W}/\mathcal{B}$ is a Rad$_g$-lifting, also $\mathcal{B} \cong \mathcal{W}/\mathcal{B}$ is a Rad$_g$-lifting module.

Corollary 3.14: *Let $\mathcal{W}$ be a duo module, and $\mathcal{B} \leq^{\oplus} \mathcal{W}$. Then $\mathcal{W}$ is a Rad_g-lifting if and only if $\mathcal{B}$ and $\mathcal{W}/\mathcal{B}$ are Rad_g-lifting modules.*

Proof: $\Longrightarrow$) Since every duo module is weak duo, then the result is follows by Corollary 3.13.

$\Longleftarrow$) Since $\mathcal{W} \cong \gamma \oplus (\mathcal{W}/\mathcal{B})$, then the result comes by Theorem 3.6.

Corollary 3.15: *Let $\mathcal{W} = \oplus_{i=1}^{n} \mathcal{W}_i$ be a duo. Then $\mathcal{W}_i$ is Rad_g-lifting for $i \in \{1,2,\ldots,n\}$, if and only if $\mathcal{W}$ is Rad_g-lifting.*

Proof: Directly by Theorem 3.6 and Corollary 3.14.

Corollary 3.16: *Let $f: \mathcal{W} \to \grave{\mathcal{W}}$ be a homomorphism from a Rad_g-lifting module $\mathcal{W}$ into a module $\mathcal{W}$. If $Kerf$ is a distributive (or, fully invariant) submodule of $\mathcal{W}$, then Imf is Rad_g-lifting.*

Proof: Since $\mathcal{W}$ is a Rad$_g$-lifting, by Theorem 3.8(2) $\mathcal{W}/Kerf$ is a Rad$_g$-lifting. By 1st theorem isomorphism, $\mathcal{W}/Kerf \cong Imf$ and so Imf is a Rad$_g$-lifting module.

Corollary 3.17: *Let $f: \mathcal{W} \to \grave{\mathcal{W}}$ be an epimorphism of R-modules $\mathcal{W}$, $\grave{\mathcal{W}}$ and $Kerf$ is a distributive (or, fully invariant) submodule of $\mathcal{W}$. Then $\grave{\mathcal{W}}$ is a Rad_g-lifting module whenever $\mathcal{W}$ is Rad_g-lifting.*

A module is termed uniserial if its submodule lattice is linearly ordered by inclusion [16]. The class of modules $\{\mathcal{W}_i | i \in I\}$ is termed relatively projective if $\mathcal{W}_i$ is $\mathcal{W}_j$-projective for $(i \neq j) \in I$ [9].

Now we will connect these above concepts with our definition and prove many of interesting results.

Proposition 3.18: Every uniserial is Rad$_g$-lifting.

Proof: Let $\mathcal{W}$ be a uniserial, so by [13, 2.17], every submodule of $\mathcal{W}$ is a hollow, hence, $\mathcal{W}$ is hollow and so generalized hollow. Hence, by Proposition 2.9, $\mathcal{W}$ is a Rad$_g$-lifting.

An easy applying example, for prime p and a positive integer n, the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is uniserial, so it is $\mathrm{Rad_g}$-lifting. While $\mathbb{Z}_6$ as $\mathbb{Z}$-module is $\mathrm{Rad_g}$-lifting. But it is not uniserial.

Corollary 3.19: *If $\mathcal{W}$ is a Noetherian uniserial module. Then any factor module of $\mathcal{W}$ is Rad_g-lifting.*

Proof: If $\mathcal{W}$ is a Noetherian and uniserial, then any submodule of $\mathcal{W}$ is fully invariant by [13, 2.18]. Hence, every factor module of $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting, by Theorem 3.8 and Proposition 3.19.

Corollary 3.20: *The factor module of uniserial duo module is also Rad_g-lifting.*

Proof: By Proposition 3.18.

The next example explains the reverse of a series of above results is not be correct, in general.

Example 3.21: Let $R = \mathbb{Z}$ and $\mathcal{W} = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$, where p is prime. It is clear that $\mathbb{Z}/p\mathbb{Z}$ is $\mathrm{Rad_g}$-lifting as a $\mathbb{Z}$-module, because it is simple as a $\mathbb{Z}$-module. While the $\mathbb{Z}$-module $\mathbb{Z}$ does not $\mathrm{Rad_g}$-lifting.

Theorem 3.22: *If $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ where $\mathcal{W}_1$ and $\mathcal{W}_2$ are Rad_g-lifting such that $\mathcal{W}_1$ is quasi-projective and $\mathcal{W}_2$-projective, then $\mathcal{W}$ is Rad_g-lifting.*

Proof: Let $\mathcal{B} \leq \mathcal{W}$ and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ with $Rad_g(\mathcal{W}) \subseteq \mathcal{B}$. It follows that $\mathcal{W}_1 \cap (\mathcal{B} + \mathcal{W}_2) \leq \mathcal{W}_1$ and $Rad_g(\mathcal{W}_1) \subseteq \mathcal{W}_1 \cap (\mathcal{B} + \mathcal{W}_2)$. Since $\mathcal{W}_1$ is $\mathrm{Rad_g}$-lifting, then there is a decompositions $\mathcal{W}_i = \aleph_1 \oplus \aleph_2$, $\aleph_1 \leq \mathcal{W}_1 \cap (\mathcal{B} + \mathcal{W}_2)$ and $\aleph_2 \cap (\mathcal{B} + \mathcal{W}_2) \ll_{\mathrm{g}} \aleph_2$. Since $\aleph_1 \oplus \aleph_2 \oplus \mathcal{W}_2 \subseteq (\mathcal{B} + \mathcal{W}_2) \oplus \aleph_2 \oplus \mathcal{W}_2 = \mathcal{B} + (\aleph_2 \oplus \mathcal{W}_2)$ then $\mathcal{W} = \mathcal{B} + (\aleph_2 \oplus \mathcal{W}_2)$. Since $\mathcal{W}_1$ is quasi-projective and $\mathcal{W}_2$-projective, then $\aleph_1$ is $\aleph_2 \oplus \mathcal{W}_2$-projective. By [16, 41.14], there is $\mathcal{B}_1 \leq \mathcal{B}$ where $\mathcal{W} = \mathcal{B}_1 \oplus (\aleph_2 \oplus \mathcal{W}_2)$. To confirm that $\mathcal{B} \cap (\gamma + \aleph_2) = \gamma \cap (\mathcal{B} + \aleph_2)$ for any $\gamma \leq \mathcal{W}_2$: as $\mathcal{B} \cap (\gamma + \aleph_2) \leq \gamma \cap (\mathcal{B} + \aleph_2) + \aleph_2 \cap (\mathcal{B} + \gamma)$ and $\aleph_2 \cap (\mathcal{B} + \gamma) = 0$, then $(\gamma + \aleph_2) \leq \gamma \cap (\mathcal{B} + \aleph_2)$. In the same way, $\gamma \cap (\mathcal{B} + \aleph_2) \leq \mathcal{B} \cap (\gamma + \aleph_2)$. Thus, $\gamma \cap (\mathcal{B} + \aleph_2) = \mathcal{B} \cap (L + \aleph_2)$. Moreover, we conclude that $\mathcal{W}_2 \cap (\mathcal{B} + \aleph_2) \leq \mathcal{W}_2$ with $Rad_g(\mathcal{W}_2) \subseteq \mathcal{W}_2 \cap (\mathcal{B} + \aleph_2)$ and $\mathcal{W}_2$ is a $\mathrm{Rad_g}$-lifting, for then there is a decomposition $\mathcal{W}_2 = \acute{\aleph}_1 \oplus \acute{\aleph}_2$, $\acute{\aleph}_1 \leq \mathcal{W}_2 \cap (\mathcal{B} + \aleph_2) = \mathcal{B} \cap (\mathcal{W}_2 + \aleph_2) \leq \mathcal{B}$ and $\acute{\aleph}_2 \cap (\mathcal{B} + \aleph_2) \ll_{\mathrm{g}} \acute{\aleph}_2$. Therefore, $\mathcal{W} = \mathcal{B}_1 \oplus (\aleph_2 \oplus \mathcal{W}_2)$ $= (\mathcal{B}_1 \oplus \acute{\aleph}_1) \oplus (\aleph_2 \oplus \acute{\aleph}_2)$, $\mathcal{B}_1 \oplus \acute{\aleph}_1 \leq \mathcal{B}$ and $\mathcal{B} \cap \left(\aleph_2 \oplus \acute{\aleph}_2\right) = \acute{\aleph}_2 \cap (\mathcal{B} + \aleph_2)$ is g-small in $\aleph_2 \oplus \acute{\aleph}_2$. Thus, $\mathcal{W}$ is $\mathrm{Rad_g}$-lifting.

Corollary 3.23: *Let $\mathcal{W}$= $\mathcal{W}_1 \oplus \mathcal{W}_2$ where $\mathcal{W}_1$ is semisimple and $\mathcal{W}_2$ is a Rad_g-lifting, and they are relatively projective with $\mathcal{W}_1$, then $\mathcal{W}$ is a Rad_g-lifting module.*

Proof: Since $\mathcal{W}_1$ is semisimple, hence it is a Rad$_g$-lifting [see, Example 2.10] and so by Theorem 3.22, $\mathcal{W}$ is Rad$_g$-lifting.

Proposition 3.24: If $\mathcal{W}$ is a Rad$_g$-lifting, then $\mathcal{W}/Rad_g(\mathcal{W})$ is semisimple.

Proof: Let $Rad_g(\mathcal{W}) \leq \mathcal{B} \leq \mathcal{W}$. By hypothesis, there is decomposition $\mathcal{W} = \gamma \oplus \acute{\gamma}$ for some $\gamma \leq \mathcal{B}$ and $\mathcal{B} \cap \acute{\gamma} \ll_g \acute{\gamma}$. Thus, $\mathcal{W} = \mathcal{B} + \grave{\gamma}$. It follows that $\frac{\mathcal{W}}{Rad_g(\mathcal{W})} = \frac{\mathcal{B}}{Rad_g(\mathcal{W})} + \frac{(\grave{\gamma}+Rad_g(\mathcal{W}))}{Rad_g(\mathcal{W})}$, and $\left(\frac{\mathcal{B}}{Rad_g(\mathcal{W})}\right) \cap \left(\frac{\grave{\gamma}+Rad_g(\mathcal{W})}{Rad_g(\mathcal{W})}\right) = \frac{\mathcal{B}\cap(\grave{\gamma}+Rad_g(\mathcal{W}))}{Rad_g(\mathcal{W})}$ $= \frac{Rad_g(\mathcal{W})+(\mathcal{B}\cap\grave{\gamma})}{Rad_g(\mathcal{W})} = \frac{Rad_g(\mathcal{W})}{Rad_g(\mathcal{W})}$, i.e, $\mathcal{B}/Rad_g(\mathcal{W}) \leq^{\oplus} \mathcal{W}/Rad_g(\mathcal{W})$. Thus $\mathcal{W}/Rad_g(\mathcal{W})$ is a semisimple.

Corollary 3.25: *Let $\mathcal{W}$ be a Rad_g-lifting with $Rad_g(\mathcal{W}) \leq^{\oplus} \mathcal{W}$, then $\mathcal{W}/P_g(\mathcal{W})$ is semisimple.*

Proof: From [15, Lemma 3.16], $P_g(\mathcal{W}) = Rad_g(\mathcal{W})$. By Proposition 3.24, $\mathcal{W}/P_g(\mathcal{W})$ is semisimple.

Corollary 3.26: *If $\mathcal{W}$ is a Rad_g-lifting, then also $\mathcal{W}/Rad_g(\mathcal{W})$.*

Proof: From Propositions 3.24, $\mathcal{W}/Rad_g(\mathcal{W})$ is semisimple, hence Rad$_g$-lifting.

Proposition 3.27: If $\mathcal{W}$ is a Rad$_g$-lifting with $Rad_g(\mathcal{W}) \neq \mathcal{W}$, then there is a $\mathcal{W} = L_1 \oplus L_2$ such that L_1 is semisimple, $Rad_g(\mathcal{W}) \trianglelefteq L_2$ and $L_2/Rad_g(\mathcal{W})$ is semisimple.

Proof: Since $\mathcal{W}$ is a Rad$_g$-lifting, Proposition 3.24 implies $\mathcal{W}/Rad_g(\mathcal{W})$ is a semisimple module. Because $Rad_g(\mathcal{W}) \subset \mathcal{W}$ and by [5, Proposition 2.1], there is a decomposition $\mathcal{W} = L_1 \oplus L_2$ where L_1 is semisimple, $Rad_g(\mathcal{W}) \trianglelefteq L_2$ and $L_2/Rad_g(\mathcal{W})$ is semisimple.

4. Conclusion

We define a new class of lifting modules with different conditions is called "Rad$_g$-lifting", then gave appropriate definitions. Also, the relationship between our definition and other concept had been discussed. We conclude that many properties of Rad$_g$-lifting module were subject to the same conditions and contradicted with other one.

References

[1] A.C. Ozcan, A. Harmanci, Duo modules, *Glasgow Math. Journal Trust*, 48(3), 533-545, (2006). https://doi.org/10.1017/S0017089506003260.

[2] A.D. Diallo, P.C. Diop, Modules whose t-closed submodules are their homomorphic images, *Journal of Discrete Mathematical Sciences and Cryptography.* 23(6),1297–1314, (2020). https://doi.org/10.1080/09720529.2020.1815339.

[3] B. Kosar, C. Nebiyev, N. Sokmez, G-supplemented modules, *Ukrainian Mathematical Journal*, 67(6), 975-980, (2015).https://doi.org/10.1007/s11253-015-1127-8.

[4] B. Ungor, S. Halicioglu, A. Harmancı, On a class of ʓ-supplemented Modules, *Bull. Malays. Math. Sci. Soc,* 37(3), 703-717, (2014).

[5] C. Lomp, On semilocal modules and rings, *Communications in Algebra,* 27(4),1921-1935,(1999).https://doi.org/10.1080/00927879908826539.

[6] D. Keskin, On lifting modules, Comm. Algebra 28(7), 3427-3440, (2000).

[7] D.X, Zhou, X.R. Zhang, Small-essential submodules and morita duality, *Southeast Asian Bull. Math.* 35, 1051-1062, (2011). https://www.researchgate.net/publication/268072936_Small-essential_submodules_and_Morita_duality.

[8] F. Kasch, Modules and rings module, (1982).

[9] G. Azumaya, F. Mbuntum, K. Varadarajan, On M-projective and M-Injective Modules, *J. Math,* 95, 9-16, (1975). https://doi.org/10.11575/PRISM/22103.

[10] H. K. Marhoon, Some Generalizations of Monoform Modules, M.Sc. Thesis, University of Baghdad, (2014).

[11] I.M-A. Hadi and S. H. Aidi, e-Hollow Modules, *International Journal of Advanced Scientific and Technical Research*, (2015). IJASR - Home (rspublication.com).

[12] I.M.A. Hadi, S.H. Aidi, On e-small submodules, *Ibn Al-Haitham Journal For Pure. Appl. Sci,* 28(3), 214-222, (2015). https://jih.uobaghdad.edu.iq/index.php/j/article/view/154.

[13] J. Calrk, C. Lomp, N. Vananja, R. Wisbauer, Lifting modules supplements and projectivity in module theory, Basel-Boten-Berlin, (2006).

[14] N.A. Hussen, M.Q. Hussain and F. H. Mohammed, *T-semihollow-lifting modules,* Vol. 24(7), 1923-1929, (2021). https://doi.org/10.1080/09720529.2021.1957186.

[15] N.M. Kamil, T.Y. Ghawi, Strongly generalized ⊕-radical supplemented modules, AIP Conference Proceedings, in press.

[16] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, (1991).

[17] T.C. Quynh and P. H. Tin, Some properties of e-supplemented and e-lifting modules, *Vietnam J. Math*, 41(3), 303-312, (2013). https://doi.org/10.1007/s10013-013-0022-6.

[18] T.Y. Ghawi, On a class of g-lifting modules, *Journal of Discrete Mathematical Sciences & Cryptography,* 24(6), 1857-1872, (2021). https://doi.org/10.1080/09720529.2021.1961904.

[19] Z.J. Kadhim, A.H. Dheyab, D.J. Abdulkareem, M.Q.Hussain, Semismall compressible and semismall retractable modules, *Journal of Discrete Mathematical Sciences and Cryptography,* 23(7), 1395-1398, (2020). https://doi.org/10.1080/09720529.2020.1714888.

[20] Y. Zhou, Generalizations of Perfect, Semiperfect and semiregular rings, Algebra Colloge, 7(3), 305-318, (2000). https://www.academia.edu/26834416/Generalizations_of_Perfect_Semiperfect_and_Semiregular_Rings.

[21] F. Afzal, Shah Rukh, D. Afzal, M.R.Farahani, M.Cancan, S. Ediz. A study of non-commutative Von-Neumann regular rings. *Journal of Information and Optimization Sciences.* 42(7), 2021. 1425-1436 https://doi.org/10.1080/02522667.2021.1896650.

[22] M. Nadeem, S. Ahmad, M.K. Siddiqui, M.A. Ali, M.R.Farahani, A.J.M. Khalaf. On some applications related with algebraic structures through different well known graphs. *Journal of Discrete Mathematical Sciences and Cryptography.* 24(2), 2021.451-471. https://doi.org/10.1080/09720529.2021.1885806.

[23] K.M. Krishna, P.S. Johnson. New identity on Parseval p-approximate Schauder frames and applications. *Journal of Interdisciplinary Mathematics,* 24(7), 2021, 1751-1760 https://doi.org/10.1080/09720502.2021.1891698.

[24] S.M. Seyed Joula, A.R. Gilani, E. Arfa, Exact BCK-sequence in BCK-algebra, *Journal of Information and Optimization Sciences*, 41(4), 2020, 1153-1162. https://doi.org/10.1080/02522667.2020.1776940.

[25] S.M. Seyed Joula, A.R. Gilani, E. Arfa, Fuzzy BCK-module on t-norm, *Journal of Information and Optimization Sciences,* 41(4), 2020, 1163-1174. https://doi.org/10.1080/02522667.2020.1781390.

[26] A. Alsinai, H.M.U. Rehman, Y. Manzoor, M. Cancan, Z. Taş, M.R. Farahani, Sharp upper bounds on forgotten and SK indices of cactus graph, *Journal of Discrete Mathematical Sciences and Cryptography.* 2022, 1-22. https://doi.org/10.1080/09720529.2022.2027605.