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On enumeration of labeled connected transitive digraphs

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Abstract

If $T_0^c(n)$ is the number of all labeled connected transitive digraphs defined on a set of n elements, then the formula $T_0^c(n) = \sum_{\overline{p}=(p_1,\ldots,p_k)|=n} (-1)^{n-k} \binom{n}{\overline{p}} W^c(\overline{p})$ is true. The summation is over all sequences $\overline{p} = (p_1, \ldots, p_k) \in \mathbb{N}_1^k$ such that $p_1 + \ldots + p_k = n$. Term $W^c(\overline{p})$ denotes the number of labeled connected transitive digraphs of a special form, depending on the parameter $\overline{p}$.

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1. Introduction

Denote by $V(X)$ the family of all labeled transitive digraphs defined on the set $X \doteq \{1, \ldots, n\}$, $n \in \mathbb{N}_1$. We further assume that the set $V^c(X)$ consists of all connected digraphs of the family $V(X)$. There is a one-to-one correspondence between $V(X)$ and the set $P(X)$ of all partial orders defined on the set X. Moreover, there is a one-to-one correspondence between $V(X)$ and the set $T_0(X)$ of all labeled T_0-topologies defined on

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X. Let $T_0(n) \doteq card\, T_0(X)$. Obviously, $T_0(n) = card\, P(X) = card\, V(X)$. We put $T_0^c(n) \doteq card\, V^c(X)$.

A number of papers have been devoted to the study of the sequence $\{T_0(n)\}$. Let us note early works [6, 10, 12, 20] related to direct calculations, works [7-9, 11, 13, 14] related to computer calculations, modern works [1, 19, 21, 22, 29]. At least three types of formulas are known for calculating of numbers $T_0(n)$. In independent papers [6, 10, 12], a formula

$$T_0(n) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} \binom{n}{\overline{p}} V(\overline{p})$$

was obtained in which the summation is over all sequences $\overline{p} = (p_1, \ldots, p_k) \in \mathbb{N}_1^k$ such that $p_1 + \ldots + p_k = n$ (over all compositions of the number n). The numbers $V(\overline{p}) = V(p_1, \ldots, p_k)$ are the numbers of labeled transitive digraphs of a special form depending on the parameter $\overline{p}$. They were studied in [6-8, 23, 24]. In work [13], the formula

$$T_0(n) = n! \sum_{1\cdot j_1 + 2\cdot j_2 + \ldots + n\cdot j_n = n} \prod_{m=1}^{n} \tfrac{1}{j_m!} \left(\tfrac{1}{m!} T_0^c(m) \right)^{j_m}$$

was proved, in which the summation is carried out over all tuples $(j_1, \ldots, j_n) \in \mathbb{N}_0^n$ such that $1 \cdot j_1 + 2 \cdot j_2 + \ldots + n \cdot j_n = n$. In [26], the formula

$$T_0(n) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} (-1)^{n-k} \binom{n}{\overline{p}} W(\overline{p}) \tag{1}$$

was proved. The numbers $W(\overline{p}) = W(p_1, \ldots, p_k)$ are the numbers of labeled transitive digraphs of a special form (such that $W(\overline{p}) \geq V(\overline{p})$).

In this paper, we have proved the formula (see Theorem 1)

$$T_0^c(n) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} (-1)^{n-k} \binom{n}{\overline{p}} W^c(\overline{p}). \tag{2}$$

The numbers $W^c(\overline{p}) = W^c(p_1, \ldots, p_k)$ are the numbers of labeled connected transitive digraphs of a special form (such that $W(\overline{p}) \geq W^c(\overline{p})$).

See Sec. 2 for definitions of the numbers $W(\overline{p})$, $V(\overline{p})$ and $W^c(\overline{p})$ (Definitions 1, 2 and 3, respectively). Sec. 4 contains separate formulas for these numbers.

Works [15-18] are devoted to the study of topological indices for connected graphs and graph connectivity indices.

Remark 1 : If $\sigma \in V(X)$, then σ is a labeled acyclic digraph (we write $\sigma \in A(X)$). According to [25], an explicit formula is valid, similar to formulas (1) and (2):

$$\operatorname{card} A(X) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} (-1)^{n-k} \binom{n}{\overline{p}} A(\overline{p})$$

(where $A(\overline{p}) \doteq 2^{(n^2-p_1^2-\ldots-p_k^2)/2}$). Its generalization is the explicit formula

$$A_n(x) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} (-1)^{n-k} \binom{n}{\overline{p}} (1+x)^{(n^2-p_1^2-\ldots-p_k^2)/2},$$

where $A_n(x) = \sum_{r=0}^{\infty} A_{nr} x^r$ is a generating function (polynomial), in which A_{nr} denotes the number of labeled acyclic digraphs of order n that have exactly r arcs. According to [26], the estimates are valid: $n!/(p_1!\ldots p_k!) \le W(p_1,\ldots,p_k) \le A(p_1,\ldots,p_k)$.

2. Definitions and auxiliary statements

Let $B \doteq \{0,1\}$ be a Boolean set, X be an arbitrary set. The functions $X^2 \to B$ are called characteristic. Any subset $\sigma \subseteq X^2$, called a binary relation (or simply a relation) on the set X, generates a characteristic function

$$\chi_\sigma : X^2 \to B, \qquad \chi_\sigma(x,y) \doteq \begin{cases} 1, & \text{if } (x,y) \in \sigma, \\ 0, & \text{if } (x,y) \notin \sigma. \end{cases}$$

Further, the function $\chi_\sigma(\cdot,\cdot)$ is denoted by $\sigma(\cdot,\cdot)$. On the other hand, any function $\chi : X^2 \to B$ generates a binary relation $\sigma_\chi \subseteq X^2$ such that $(x,y) \in \sigma_\chi$, if $\chi(x,y) = 1$. The mapping $\sigma \to \sigma(\cdot,\cdot)$ is a bijection between a set of binary relations and a set of characteristic functions. By virtue of this circumstance, we call subsets $\sigma \subseteq X^2$ both relations and functions (sometimes digraphs). In the case of $\operatorname{card} X < \infty$, the characteristic function can be interpreted as a binary matrix (a matrix consisting of zeros and ones). In this paper, we use a functional representation that allows us to obtain meaningful results in the most general form (see, for example, [2-5, 26-28]).

In terms of characteristic functions, the statement is true: the inclusion $\sigma \in V(X)$ takes place if and only if

$$\begin{aligned} \sigma(x,y) &= 1, \quad \text{for all } x \in X, \\ \sigma(x,y)\sigma(y,z) &\le \sigma(x,z), \quad \text{for all } x,y,z \in X, \\ \sigma(x,y)\sigma(y,x) &= \delta_{xy}, \quad \text{for all } x,y \in X \end{aligned}$$

(where δ_{xy} is the Kronecker symbol). Further, we consider that $\operatorname{card} X < \infty$. For any $\sigma \in V(X)$, a support set is defined:

$$S(\sigma) \doteq \{y \in X\colon\ \sigma(x,y) = \delta_{xy}, \text{ for all } x \in X\} \neq \varnothing \tag{3}$$

If $\varnothing \neq Y \subseteq X$, then the restriction $\sigma|_{Y^2}$ belongs to $V(Y)$. The inequality $S(\sigma) \neq \varnothing$ appearing in (3) was proved in [2].

Let $(p_1,\ldots,p_k) \in \mathbb{N}_1^k$, $n \doteq p_1 + \ldots + p_k$, $X \doteq \{1,\ldots,n\}$. The composition $(p_1,\ldots,p_k)$ generates a representation

$$\begin{vmatrix} \sigma^{11} & \sigma^{12} & \cdots & \sigma^{1k} \\ \sigma^{21} & \sigma^{21} & \cdots & \sigma^{2k} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma^{k1} & \sigma^{k2} & \cdots & \sigma^{kk} \end{vmatrix} \tag{4}$$

of the relation $\sigma \in V(X)$ in block form, where at the intersection of the ith horizontal and jth vertical stripes there is a block σ^{ij} with p_i rows and p_j columns. The record of the relation $\sigma \in V(X)$ in the form (4) will be called a block representation of type $(p_1,\ldots,p_k)$.

Definition 1: We denote by $W(p_1,\ldots,p_k)$ "the set of all relations $\sigma \in V(X)$, for which in the block representation (4) of type $(p_1,\ldots,p_k)$

(1) all blocks σ^{ij}, $1 \le j < i \le k$, consist of zeros,

(2) all diagonal blocks σ^{ii}, $i = 1,\ldots,k$, are unit matrices e_{p_i},

and let $W(p_1,\ldots,p_k) \doteq card\, W(p_1,\ldots,p_k)$.

Definition 2 : We denote by $V(p_1,\ldots,p_k)$ the set of all relations $\sigma \in W(p_1,\ldots,p_k)$ such that in the representation (4) for σ in the blocks $\sigma^{i-1.i}$, $i = 2,\ldots,k$, each column contains at least one unit (we call these blocks non-degenerate). Let $V(p_1,\ldots,p_k) \doteq card\, V(p_1,\ldots,p_k)$.

Definition 3: We denote by $W^c(p_1,\ldots,p_k) \doteq W(p_1,\ldots,p_k) \cap V^c(X)$ the set of all connected digraphs in the collection $W(p_1,\ldots,p_k)$. Let $W^c(p_1,\ldots,p_k) \doteq card\, W^c(p_1,\ldots,p_k)$.

Let, further, $k \in \mathbb{N}_1$, $(p_1,\ldots,p_{k+1}) \in \mathbb{N}_1^{k+1}$, $p \doteq p_1 + \ldots + p_k$, $m \doteq p_{k+1}$,

$$X \doteq \{1,\ldots,p+m\}, \qquad M \doteq \{p+1,\ldots,p+m\}.$$

Definition 4: We denote by $W^c(p_1,\ldots,p_k;m)$ the set of all relations $\sigma \in V^c(X)$, for which in the block representation (4) of type $(p_1,\ldots,p_{k+1})$

(1) all blocks σ^{ij}, $1 \le j < i \le k+1$, consist of zeros,

(2) all diagonal blocks σ^{ii}, $i = 1,\ldots,k$, are unit matrices e_{p_i},

(3) the diagonal block $\sigma^{k+1,k+1}$ is a labeled transitive digraph (belongs to the family $V(M)$), and let $W^c(p_1,\ldots,p_k;m) \doteq card\, W^c(p_1,\ldots,p_k;m)$.

Remark 2 : Let us extend Definition 4 to the case $m=0$, setting

$$W^c(p_1,\ldots,p_k;0) \doteq W^c(p_1,\ldots,p_k), \qquad W^c(p_1,\ldots,p_k;0) \doteq W^c(p_1,\ldots,p_k).$$

3. Main result

Proposition 1: If $(p_1,\ldots,p_k,m) \in \mathbb{N}_1^{k+1}$ then

$$W^c(p_1,\ldots,p_k;m) = \sum_{q=1}^{m} (-1)^{q-1} \binom{m}{q} W^c(p_1,\ldots,p_k,q;m-q) \tag{5}$$

Proof **:** Denote $\overline{p} \doteq (p_1,\ldots,p_k)$. In accordance with (3), for any $\sigma \in W^c(\overline{p};m)$, the inequality $S(\sigma|_{M^2}) \neq \varnothing$ is true, therefore, for any non-empty subset $\alpha \subseteq M$, we define the set

$$W^c_\alpha(\overline{p};m) \doteq \{\sigma \in W^c(\overline{p};m):\ \alpha \subseteq S(\sigma|_{M^2})\}.$$

If $\varnothing \neq \alpha \subseteq \beta \subseteq M$, then $W^c_\beta(\overline{p};m) \subseteq W^c_\alpha(\overline{p};m)$, therefore, in accordance with the principle of inclusion-exclusion, the equality takes place:

$$card\, W^c(\overline{p};m) = \sum_{\varnothing \neq \alpha \subseteq M} (-1)^{|\alpha|-1} card\, W^c_\alpha(\overline{p};m).$$

The summation is over all non-empty subsets α of M, and $|\alpha|$ denotes the number of elements in α. Obviously, if non-empty $\alpha,\beta \subseteq M$ are such that $|\alpha| = |\beta|$, then

$$card\, W^c_\alpha(\overline{p};m) = card\, W^c_\beta(\overline{p};m) = card\, W^c_\gamma(\overline{p};m),$$

where $\gamma \doteq \{p+1,\ldots,p+q\}$, $p \doteq p_1+\ldots+p_k$, $q \doteq |\alpha| = |\beta| \in \{1,\ldots,m\}$. It is easy to verify that $W^c_\gamma(\overline{p};m) = W^c(\overline{p},q;m-q)$, therefore"

$$card\, W^c(\overline{p};m) = \sum_{q=1}^{m} (-1)^{q-1} \binom{m}{q} card\, W^c(\overline{p},q;m-q).$$

Proposition 2: For any $n \in \mathbb{N}_1$, the equality is true:

$$T^c_0(n) = \sum_{q=1}^{n} (-1)^{q-1} \binom{n}{q} W^c(q;n-q).$$

Proof **:** (It is carried out by analogy with the proof of Proposition 1.) For any non-empty subset $\alpha \subseteq X \doteq \{1,\dots,n\}$, the set

$$V_\alpha^c(X) \doteq \{\sigma \in V^c(X) :\ \alpha \subseteq S(\sigma)\}$$

is defined. If $\varnothing \neq \alpha \subseteq \beta \subseteq X$, then $V_\beta^c(X) \subseteq V_\alpha^c(X)$, so

$$T_0^c(n) = card\, V^c(X) = \sum_{\varnothing \neq \alpha \subseteq X} (-1)^{|\alpha|-1} card\, V_\alpha^c(X).$$

Finally, the equality $V_\alpha^c(X) = W^c(q; n-q)$ holds, where $q \doteq |\alpha| \in X$.

Proposition 3: If $(p_1,\dots,p_k,m) \in \mathbb{N}_1^{k+1}$ then the equality

$$W^c(p_1,\dots,p_k;m) = \sum_{\overline{q}=(q_1,\dots q_r)|=m} (-1)^{m-r} \binom{m}{\overline{q}} W^c(p_1,\dots,p_k,\overline{q}) \tag{6}$$

holds. The summation is carried out over all sequences $\overline{q} = (q_1,\dots,q_r) \in \mathbb{N}_1^r$ such that $q_1 + \dots + q_r = m$.

Proof **:** (Carried out by induction on m.) Denote $\overline{p} \doteq (p_1,\dots,p_k)$. If $m=1$ then formula (6) turns into equality $W^c(\overline{p};1) = W^c(\overline{p},1)$ (it suffices to compare Definitions 3 and 4). It forms the basis of induction. Let $m>1$, and assume that equality (6) is true for all $\ell < m$ (instead of m, the symbol ℓ should be written) and all $\overline{p}$. Let us show that it is also true for $\ell = m$ and all $\overline{p}$.

Applying the induction hypothesis to the terms on the right side of equality (5), we obtain that

$$W^c(\overline{p};m) - (-1)^{m-1} W^c(\overline{p},m;0) = \sum_{q=1}^{m-1} (-1)^{q-1} \binom{m}{q} W^c(\overline{p},q;m-q)$$

$$= \sum_{q=1}^{m-1} (-1)^{q-1} \binom{m}{q} \sum_{\overline{q}=(q_1,\dots q_r)|=m-q} (-1)^{m-q-r} \binom{m-q}{\overline{q}} W^c(\overline{p},q,\overline{q})$$

$$= \sum_{q=1}^{m-1} \sum_{q_1+\dots+q_r=m-q} (-1)^{m-1-r} \binom{m}{q} \tfrac{(m-q)!}{q_1!\dots q_r!} W^c(\overline{p},q,q_1,\dots,q_r).$$

Next, we use the convention from Remark 2. Passing from double summation to summation over all variables simultaneously, we obtain the equality

$$W^c(\overline{p};m) - (-1)^{m-1} W^c(\overline{p},m) = \sum_{q+q_1+\dots+q_r=m,\ q<m} (-1)^{m-1-r} \tfrac{m!}{q!q_1!\dots q_r!} W^c(\overline{p},q,q_1,\dots,q_r).$$

We make a change of variables $q_1' = q,\ q_i' = q_{i-1},\ i = 2,\ldots,r' = r+1,$ then

$$W^c(\overline{p};m) - (-1)^{m-1} W^c(\overline{p},m) = \sum_{q_1+\ldots+q_r=m,\ q_1<m} (-1)^{m-r} \tfrac{m!}{q_1!\ldots q_r!} W^c(\overline{p},q_1,\ldots,q_r)$$

(we do not write primes for new variables), which was required.

Theorem 1: *For any* $n \in \mathbb{N}_1$, *the equality is true:*

$$T_0^c(n) = \sum_{\overline{p}=(p_1,\ldots p_k)|=n} (-1)^{n-k} \left(\begin{smallmatrix} n \\ \overline{p} \end{smallmatrix}\right) W^c(\overline{p}).$$

The summation is carried out over all sequences $\overline{p} = (p_1,\ldots,p_k) \in \mathbb{N}_1^k$ *such that* $p_1 + \ldots + p_k = n$.

Proof: For $n = 1$ the assertion is obvious. Let $n > 1$. By virtue of Proposition 2 and 3 and equality $W^c(n;0) = W^c(n)$, the chain of equalities is valid:

$$T_0^c(n) - (-1)^{n-1} W^c(n) = T_0^c(n) - (-1)^{n-1} W^c(n;0) = \sum_{q=1}^{n-1} (-1)^{q-1} \left(\begin{smallmatrix} n \\ q \end{smallmatrix}\right) W^c(q;n-q)$$

$$= \sum_{q=1}^{n-1} (-1)^{q-1} \left(\begin{smallmatrix} n \\ q \end{smallmatrix}\right) \sum_{\overline{q}=(q_1,\ldots q_r)|=n-q} (-1)^{n-q-r} \left(\begin{smallmatrix} n-q \\ \overline{q} \end{smallmatrix}\right) W^c(q,\overline{q})$$

$$= \sum_{q=1}^{n-1} \sum_{q_1+\ldots+q_r=n-q} (-1)^{n-1-r} \left(\begin{smallmatrix} n \\ q \end{smallmatrix}\right) \tfrac{(n-q)!}{q_1!\ldots q_r!} W^c(q,q_1,\ldots,q_r).$$

Passing to the summation over all variables simultaneously, we obtain

$$T_0^c(n) = (-1)^{n-1} W^c(n) + \sum_{q+q_1+\ldots+q_r=n,\ q<n} (-1)^{n-1-r} \tfrac{n!}{q!q_1!\ldots q_r!} W^c(q,q_1,\ldots,q_r),$$

and the change of variables $p_1 = q,\ p_i = q_{i-1},\ i = 2,\ldots,k = r+1,$ leads to the required result.

4. Separate formulas for numbers $W(\overline{p})$, $V(\overline{p})$ and $W^c(\overline{p})$

In [26] for calculating the numbers $W(\overline{p})$, the formula

$$W(p,q_1,\ldots,q_k) = \sum_{\sigma \in W(q_1,\ldots,q_k)} N^p(\sigma)$$

was proposed, where $N(\sigma)$ is the number of open sets of the T_0-topology corresponding to the partial order σ. The formulas given below (beginning with the third one) are obtained as a result of computer calculations based on this formula.

$$W(p)=1$$

$$W(p,q)=2^{pq}$$

$$W(p,1,1)=4^p+3^p$$

$$W(p,2,1)=W(p,1,2)=8^p+2\cdot 6^p+5^p$$

$$W(p,1,1,1)=8^p+3\cdot 6^p+2\cdot 5^p+4^p$$

$$W(p,3,1)=W(p,1,3)=16^p+3\cdot 12^p+3\cdot 10^p+9^p$$

$$W(p,2,2)=16^p+4\cdot 12^p+4\cdot 10^p+2\cdot 9^p+4\cdot 8^p+7^p$$

$$W(p,2,1,1)=W(p,1,1,2)=16^p+5\cdot 12^p+6\cdot 10^p+3\cdot 9^p+6\cdot 8^p+3\cdot 7^p+6^p$$

$$W(p,1,2,1)=16^p+5\cdot 12^p+6\cdot 10^p+4\cdot 9^p+4\cdot 8^p+4\cdot 7^p+6^p$$

$$W(p,1,1,1,1)=16^p+6\cdot 12^p+8\cdot 10^p+5\cdot 9^p+9\cdot 8^p+7\cdot 7^p+3\cdot 6^p+5^p$$

In the article [6], the following formulas were obtained for calculating the numbers $V(\overline{p})$ (we give only a part of the formulas from this work).

$$V(p,q)=(2^p-1)^q$$

$$V(p,1,1)=3^p-2^p$$

$$V(p,2,1)=2\cdot 6^p+5^p-2\cdot 4^p-4\cdot 3^p+3\cdot 2^p$$

$$V(p,1,2)=5^p-4^p$$

$$V(p,1,1,1)=4^p-3^p$$

$$V(p,3,1)=3\cdot 12^p+3\cdot 10^p+9^p-3\cdot 8^p-12\cdot 6^p-6\cdot 5^p+9\cdot 4^p+12\cdot 3^p-7\cdot 2^p$$

$$V(p,2,2)=2\cdot 10^p+2\cdot 9^p+2\cdot 8^p+7^p-8\cdot 6^p-8\cdot 5^p+9\cdot 4^p$$

$V(p,2,1,1) = 2\cdot 8^p + 2\cdot 7^p - 6^p - 2\cdot 5^p - 6\cdot 4^p + 5\cdot 3^p$

$V(p,1,3) = 9^p - 8^p$

$V(p,1,2,1) = 2\cdot 7^p - 6^p - 5^p$

$V(p,1,1,2) = 6^p - 5^p$

$V(p,1,1,1,1) = 5^p - 4^p$

As a result of computer calculations for calculating the numbers $W^c(\overline{p})$, the following formulas were obtained.

$W^c(p,1) = 1$

$W^c(p,2) = 3^p - 2^p$

$W^c(p,1,1) = 3^p$

$W^c(p,3) = 7^p - 3\cdot 4^p + 2\cdot 3^p$

$W^c(p,2,1) = W^c(p,1,2) = 7^p + 2\cdot 5^p - 2\cdot 4^p$

$W^c(p,1,1,1) = 7^p + 3\cdot 5^p - 4^p$

$W^c(p,4) = 15^p - 4\cdot 8^p - 3\cdot 6^p + 12\cdot 5^p - 6\cdot 4^p$

$W^c(p,3,1) = W^c(p,1,3) = 15^p + 3\cdot 11^p + 3\cdot 9^p - 3\cdot 8^p - 9\cdot 6^p + 6\cdot 5^p$

$W^c(p,2,2) = 15^p + 4\cdot 11^p + 4\cdot 9^p - 2\cdot 8^p + 4\cdot 7^p - 10\cdot 6^p + 4\cdot 5^p$

$W^c(p,2,1,1) = W^c(p,1,1,2) = 15^p + 5\cdot 11^p + 6\cdot 9^p - 8^p + 6\cdot 7^p - 10\cdot 6^p + 2\cdot 5^p$

$W^c(p,1,2,1) = 15^p + 5\cdot 11^p + 6\cdot 9^p + 4\cdot 7^p - 9\cdot 6^p + 2\cdot 5^p$

$W^c(p,1,1,1,1) = 15^p + 6\cdot 11^p + 8\cdot 9^p + 8^p + 9\cdot 7^p - 8\cdot 6^p + 5^p$

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