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On certain prime numbers in Lucas sequences

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Abstract

In 1964, Shanks conjectured that there are infinitely many primes of the form $\frac{1}{2}(x^2+1)$. Therefore, the aim of this paper is to introduce a technique for studying whether or not there are infinitely many prime numbers of the form $\frac{1}{2}(x^2+1)$ derived from some Lucas sequences of the first kind $\{U_n(P,Q)\}$ or the second kind $\{V_n(P,Q)\}$, where $P \geq 1$ and $Q = \pm 1$. In addition, as applications we represent the procedure of this technique in case of x is an either integer or Lucas number of the first or the second kind with $x \geq 1$ *and* $1 \leq P \leq 20$.

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1. Introduction

In 1912, Edmund Landau [9] conjectured that there are infinitely many primes of the form $p = x^2 + 1$ for some integer x. For this conjecture there were many attempts to prove it, but no one succeeded till 2020 in which Vega [18] proposed a proof for it. Similar conjecture was also proposed by Shanks in 1961 [13] in which he proposed that for some integer x there are infinitely many prime of $p = x^4 + 1$, and this conjecture has not been proven. Then, in 1964 Shanks [12] conjectured that there are infinite prime numbers of the form $p = \frac{1}{2}(x^2+1)$ for some integer x, which has not yet been proven. There are many of these prime numbers, e.g.

$$17019940501 = \frac{1}{2}(184499^2 + 1),$$

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and for more such numbers, see e.g. [12]. On the other hand, it is also known that there infinitely many primes in the Fibonacci sequence $\{F_n\}$ or Lucas sequence $\{L_n\}$, that are defined by the relations

$$F_0 = 0, F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ for\ n \geq 2 \quad (1)$$

and

$$L_0 = 2, L_1 = 1,\ L_n = L_{n-1} + L_{n-2}\ for\ n \geq 2. \quad (2)$$

In fact, this conjecture has remained open, and several authors have obtained the Fibonacci and Lucas primes under specific values of n. For more details about such primes, see e.g. [5], [11], [14] and [15]. Furthermore, this conjecture is also generalized by Lawrence and Michal [16] in case of the Lucas sequences of the first kind $\{U_n(P,Q)\}$ (or simply,$\{U_n\}$) or the second kind $\{V_n(P,Q)\}$ (or simply, $\{V_n\}$) which are defined by the recurrences:

$$U_0(P,Q) = 0, U_1(P,Q) = 1, U_n(P,Q) = PU_{n-1}(P,Q) - QU_{n-2}(P,Q), \quad (3)$$

$$V_0(P,Q) = 2, V_1(P,Q) = P, V_n(P,Q) = PV_{n-1}(P,Q) - QV_{n-2}(P,Q), \quad (4)$$

for $n \geq 2$ and the parameters P and Q are nonzero relatively prime integers. For simplicity, these sequences are also called Lucas sequences, and their numbers are known as the generalized Lucas numbers. In fact, it is also known that the Lucas sequences of the first and second kind satisfy the identity

$$V_n^2(P,Q) = DU_n^2(P,Q) + 4Q^n, \quad (5)$$

where $D = P^2 - 4Q$. Concerning these sequences, we have their characteristics polynomial is defined by

$$X^2 - PX + Q = 0,$$

where

$$\alpha = \frac{P+\sqrt{D}}{2} \text{ and } \beta = \frac{P-\sqrt{D}}{2}$$

are the latter polynomial's roots. Consequently, these sequences can alternatively be defined by

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, V_n(P,Q) = \alpha^n + \beta^n \text{ for } n \geq 0.$$

These sequences are nondegenerate if α/β is not a root of unity, and degenerate otherwise. Therefore, they are degenerate only with $(P,Q) \in \{(\pm 1,1), (\pm 2,1)\}$, for further information see e.g. [10].

In order to combine the Shanks' conjecture with the Lawrence and Michal conjecture, in this paper we study and answer the following question: Are there also infinitely many generalized Lucas primes of the form $p = \frac{1}{2}(x^2+1)$?

It means that we search for the integer solutions (x,n) with $x \geq 1$ of the Diophantine equation

$$p = \frac{1}{2}(x^2+1), \quad (6)$$

where the prime number $p = U_n(P,Q)$(with $n \geq 0$) *or* $V_n(P,Q)$(with $n \geq 0$) in case of $P \geq 1$ and $Q = \pm 1$. Furthermore, we assume such sequences are nondegenerate in order to simply present our results. By demonstrating that, we

indeed obtain that this equation has a finite number of solutions. Therefore, we do in fact provide a negative response to the above question. As applications, we completely resolve such equations of the form (6) where $P \leq 20$ and x is either an integer or a generalized Lucas number.

In the literature, many authors have studied the solutions of certain Diophantine equations related to linear recurrence sequences. For example, by using the congruence argument techniques with some identities related to Lucas sequences, Keskin and Yosma [8] found the solutions of the equations

$$V_n(P,-1) = 3V_m(P,-1)x^2, V_n(P,-1) = 6V_m(P,-1)x^2,$$

and

$$V_n(P,-1) = 3x^2, V_n(P,-1) = 3V_m(P,-1)x^2,$$

for certain values of the parameter P. Also, Karaatli and Keskin [7] studied the solutions of the equations

$$U_n(P,-1) = 5x^2 \; and \; U_n(P,-1) = 5U_m(P,-1)x^2.$$

In [2], we recently solved the following equations completely:

$$U_n(P,\pm 1) = x^2 + 1, \; V_n(P,\pm 1\} = x^2 + 1,$$

and

$$U_n(P,\pm 1) = x^4 + 1, V_n(P,\pm 1) = x^4 + 1,$$

where $x \in Z^+, x \in \{U_n\} \; or \; x \in \{V_n\}$.

In general, our main strategy is based on the straightforward combination of equation (6) and identity (5) to produce elliptic curves of the form

$$y^2 = ax^4 + bx^2 + c, (7)$$

where $a, b, c \in Z \; and \; \Delta = 16ac(b^2 - 4ac)^2 \neq 0$, with Δ denotes the curve's discriminant. Consequently, it is possible to determine the integral points of such curves using either the Magma program [4] with the algorithm SIntegralLjunggrenPoints () (based on the result of Tzanakis in [17]) or by adopting a strategy outlined by Alekseyev and Tengely in [1]. Indeed, based on the result of Alan Baker [3] (or its modification in [6]) upper bounds for the solutions of the following elliptic curves can be found:

$$y^2 = b_0x^m + b_1x^{m-1} + \ldots + b_m,$$

where $m \geq 3 \; and \; b_0 \neq 0, b_1, \ldots \; b_m \in Z$. Hence, we conclude the finiteness for the number of solutions of equation (6).

It is important to mention that number theory has many applications with other fields of sciences or Mathematics, see e.g. [19-22].

Remark 1: In general, the method we've described may be used to demonstrate the finiteness of the solutions to the equations

$$U_n(P,\pm 1) = \frac{1}{2}(x^2 + 1) \text{ and } V_n(P,\pm 1) = \frac{1}{2}(x^2 + 1).$$

Nevertheless, in this paper we only concentrate on the Shanks' problem assuming that the general terms of the nondegenerate Lucas sequences

$U_n(P,\pm1)$ and $V_n(P,\pm1)$ are prime numbers with $P \geq 1$ (Without losing the generality). Indeed, our argument can be applied for nonzero integer P such that the sequence is nondegenerate.

2. Main results

Theorem 1: *Let $P \geq 1, Q \in \{-1,1\}$ such that $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ are nondegenerate Lucas sequences. If the prime number $p = U_n(P,Q)$ or $V_n(P,Q)$, then equation (6) has a finite number of solutions of the form (x,n).*

Proof: We divide the proof into two cases regarding the Lucas sequences of the first or second kind*:*

Case 1: If $p = U_n(P,Q)$. Here, equation (6) results in

$$U_n(P,Q) = \frac{1}{2}(x^2+1).$$

The identity (5) and previous equation are combined to get the equation

$$y^2 = Dx^4 + 2Dx^2 + (D + 16Q^n), \quad (8)$$

with $y = 2V_n$ and $D = P^2 - 4Q$ such that $P \geq 1$ and $Q \in \{-1,1\}$. We will prove that the last equation represents an elliptical curve. Therefore, we have to show that this equation has a nonzero discriminant. As mentioned earlier that the elliptic curve of the form (7) has the discriminant

$$\Delta = 16ac(b^2-4ac)^2.$$

Hence, the discriminant of equation (8) is given by

$$\Delta_U = 65536D^3Q^{2n}(D+16Q^n).$$

If $Q = 1$, then $D = P^2 - 4 > 0\ as\ (P,Q) \notin \{(\pm1,1),(\pm2,1)\}$ since we assumed that the Lucas sequences are nondegenerate. Also, we possess that $D + 16Q^n = P^2 - 4 + 16 = P^2 + 12 > 0\ as\ P \geq 1$. As a result, we get that $\Delta_U > 0$. On the other hand, if $Q = -1$ the result is that

$$\Delta_U = 65536(P^2+4)^3(P^2+4\pm16).$$

Since for all $P \geq 1$, it is clear that $P^2 + 4 \geq 0, P^2 + 20 \geq 0, P^2 - 12 \neq 0$ (as there exist no integer P whose square equals to 12, therefore $\Delta_U \neq 0$. Hence, equation (8) presents an elliptic curve in case of $p = U_n(P,Q)$, where $n \geq 2$.

Case 2: If $p = V_n(P,Q)$. Equation (6) also becomes

$$V_n(P,Q) = \frac{1}{2}(x^2+1),$$

and by substituting the latter equation in identity (5), we get

$$y^2 = Dx^4 + 2Dx^2 + (D - 16DQ^n), \quad (9)$$

where $y = 2DU_n, P \geq 1\ and\ Q = \pm1$. In fact, the discriminant of equation (9) is defined by

$$\Delta_V = 65536D^6Q^{2n}(1-16Q^n).$$

Because it is assumed that the Lucas sequences of the second kind are nondegenerate, then $D \neq 0$. As it is clear that $Q^{2n} = 1 \; and \; 1 - 16Q^n \neq 0 \; for \; Q = 1, -1,$ hence $\Delta_V \neq 0$ for all $n \geq 0$. Once more, we discover that equation (9) represents an equation for an elliptic curve.

Finally, we draw the conclusion that the elliptic curves (8) and (9) have a finite number of solutions based by the result of Alan Baker [3] and its best improvement by Hajdu and Herendi [6], as was indicated in Section 1. The proof of Theorem 1 is now completed.

3. Applications

Theorem 2: *Suppose that* $\{U_n(P,Q)\}$ *is nondegenerate with* $1 \leq P \leq 20 \; and \; Q = \pm 1$. *If* p *is a prime number such that* $p = U_n(P,Q)$, *then the full list of solutions* $(P,Q,x,n) \; with \; x \geq 1$ *and* $n \geq 2$ *of equation (6) is given by*

$$(P,Q,x,n) \in \{(5,1,3,2),(13,1,5,2),(1,-1,3,5),$$
$$(1,-1,5,7),(2,-1,3,3),(5,-1,3,2),(13,-1,5,2)\}.$$

Corollary 1: *Assume that* $\{U_n\}$ *is nondegenerate. If* $p = U_n(P_1,Q_1) \; and \; x = U_k(P_2,Q_2)$ *such that* $n \geq 2, k \geq 1, 1 \leq P_1, P_2 \leq 20$ *and* $Q_1, Q_2 \in \{-1,1\}$, *then the complete set of solutions to equation (6) is given by*

$$((P_1,Q_1),(P_2,Q_2),n,k)$$
$$\in \{((5,1),(3,1),2,2),((5,1),(1,-1),2,4),((5,1),(3,-1),2,2),((13,1),$$
$$(5,1),2,2),((13,1),(1,-1),2,5),$$
$$((13,1),(2,-1),2,3),((13,1),(5,-1),2,2),((1,-1),(3,1),5,2),$$
$$((1,-1),(1,-1),5,4),((1,-1),(3,-1),5,2),((1,-1),(5,1),7,2),$$
$$((1,-1),(1,-1),7,5),((1,-1),(2,-1),7,3),((1,-1),(5,-1),7,2),$$
$$((2,-1),(3,1),3,2),((2,-1),(1,-1),3,4),((2,-1),(3,-1),3,2),$$
$$((5,-1),(3,1),2,2),((5,-1),(1,-1),2,4),((5,-1),(3,-1),2,2),$$
$$\big((13,-1),(5,1),2,2\big),\big((13,-1),(1,-1),2,5\big),\big((13,-1),(2,-2),2,3\big),$$
$$((13,-1),(5,-1),2,2)\}.$$

Corollary 2: *Let* $\{U_n\}$ *and* $\{V_n\}$ *be nondegenerate. If* $p = U_n(P_1,Q_1) \; and \; x = V_k(P_2,Q_2)$ *such that* $n \geq 2, k \geq 0, 1 \leq P_1, P_2 \leq 20 \; and \; Q_1, Q_2 \in \{-1,1\}$, *then the solutions to equation (6) are as follows:*

$$((P_1,Q_1),(P_2,Q_2),n,k) \in \{((5,1),(3,1),2,1)((5,1),(1,-1),2,2),$$
$$((5,1),(3,-1),2,1),((13,1),(5,1),2,1),((13,1),(5,-1),2,1),$$
$$((1,-1),(3,1),5,1),((1,-1),(1,-1),5,2),((1,-1),(3,-1),5,1),$$
$$((1,-1),(5,1),7,1),((1,-1),(5,-1),7,1),((2,-1),(3,1),3,1),$$
$$((2,-1),(1,-1),3,2),((2,-1),(3,-1),3,1),((5,-1),(3,1),2,1),$$
$$\big((5,-1),(1,-1),2,2\big),\big((5,-1),(3,-1),2,1\big),\big((13,-1),(5,1),2,1\big),$$
$$((13,-1),(5,-1),2,1)\}.$$

Theorem 3: *Suppose that* $\{V_n(P,Q)\}$ *is a nondegenerate with* $n \geq 0, 1 \leq P \leq 20$ *and* $Q = \pm 1$. *If the prime number* $p = V_n(P,Q)$, *then the set of solutions to equation (6) is as follows (assuming that* $x \geq 1$*)*

$$(P,Q,x,n) \in \{(5,1,3,1),(13,1,5,1),(5,-1,3,1),(13,-1,5,1)\}.$$

Corollary 3: *Assume that* $\{U_n\}$ *and* $\{V_n\}$ *are nondegenerate. If* $p = V_n(P_1,Q_1)$ *and* $x = U_k(P_2,Q_2)$ *such that* $n \geq 0, k \geq 1, 1 \leq P_1, P_2 \leq 20$ *and* $Q_1, Q_2 \in \{-1,1\}$, *the complete set of solutions to equation (6) are given by*

$$((P_1,Q_1),(P_2,Q_2),n,k) \in \{((5,1),(3,1),1,2),((5,1),(1,-1),1,4),((5,1),(3,-1),1,2),((13,1),(5,1),1,2),((13,1),(1,-1),1,5),((13,1),(2,-1),1,3),((13,1),(5,-1),1,2),((5,-1),(3,1),1,2),((5,-1),(1,-1),1,4),((5,-1),(3,-1),1,2),((13,-1),(5,1),1,2),((13,-1),(1,-1),1,5),((13,-1),(2,-1),1,3),((13,-1),(5,-1),1,2)\}.$$

Corollary 4: *Let* $\{V_n\}$ *be nondegenerate. If* $p = V_n(P_1,Q_1)$ *and* $x = V_k(P_2,Q_2)$ *such that* $n,k \geq 0, 1 \leq P_1, P_2 \leq 20$ *and* $Q_1, Q_2 \in \{-1,1\}$, *then all the solutions of equation (6) are as follows:*

$$((P_1,Q_1),(P_2,Q_2),n,k) \in \{((5,1),(3,1),1,1),((5,1),(1,-1),1,2),((5,1),(3,-1),1,1),((13,1),(5,1),1,1),((13,1),(5,-1),1,1),((5,-1),(3,1),1,1),((5,-1),(1,-1),1,2),((5,-1),(3,-1),1,1),((13,-1),(5,1),1,1),((13,-1),(5,-1),1,1)\}.$$

4. Proofs

Proof of Theorem 2. Since we are interested in the values x and n satisfying equation (6) at each pair (P,Q) with $1 \leq P \leq 20$ *and* $Q \in \{-1,1\}$ such that the sequence $\{U_n(P,Q)\}$ is nondegenerate containing prime terms, the first step is determining the values of x derived from the integral points (x,y) of the genus 1 curves presented by (8). We provide the specifics of computations in the following table, emphasizing that the triples $[A,B,C]$ representing the coefficients of the elliptic curves $y^2 = Ax^4 + Bx^2 + C$ corresponding to the curves in (8) for each pair (P,Q).

(P,Q)	$[A,B,C]$	$\{x\}$
(3,1)	[5,10,21]	{-1,1}
(4,1)	[12,24,28]	{-1,1}
(5,1)	[21,42,37]	{-3,-1,1,3}
(6,1)	[32,64,48]	{-1,1}
(7,1)	[45,90,61]	{-1,1}
(8,1)	[60,120,76]	{-1,1}

(P,Q)	$[A,B,C]$	$\{x\}$
(2,-1)	[8,16,-8] [8,16,24]	{-3,-1,1,3} {}
(3,-1)	[13,26,-3] [13,26,29]	{-1,1} {}
(4,-1)	[20,40,4] [20,40,36]	{-2,-1,0,1,2} {0}
(5,-1)	[29,58,13] [29,58,45]	{-1,1} {-3,3}
(6,-1)	[40,80,24] [40,80,56]	{-53,-1,1,53} {}
(7,-1)	[53,106,37] [53,106,69]	{-1,1} {}

(9,1)	[77,154,93]	{-1,1}		(8,-1)	[68,136,52] [68,136,84]	{-1,1} {}
(10,1)	[96,192,112]	{-1,1}		(9,-1)	[85,170,69] [85,170,101]	{-1,1} {}
(11,1)	[117,234,133]	{-1,1}		(10,-1)	[104,208,88] [104,208,88]	{-1,1} {}
(12,1)	[140,280,156]	{-1,1}		(11,-1)	[125,250,109] [125,250,141]	{-1,1} {}
(13,1)	[165,330,181]	{-5,-1,1,5}		(12,-1)	[148,296,132] [148,296,164]	{-17,-1,1,17} {}
(14,1)	[192,384,208]	{-1,1}		(13,-1)	[173,346,157] [173,346,189]	{-1,1} {-5,5}
(15,1)	[221,442,237]	{-1,1}		(14,-1)	[200,400,184] [200,400,216]	{-1,1} {}
(16,1)	[252,504,268]	{-1,1}		(15,-1)	[229,458,213] [229,458,245]	{-1,1} {}
(17,1)	[285,570,301]	{-1,1}		(16,-1)	[260,520,244] [260,520,276]	{-1,1} {}
(18,1)	[320,640,336]	{-1,1}		(17,-1)	[293,586,277] [293,586,309]	{-1,1} {}
(19,1)	[357,714,373]	{-1,1}		(18,-1)	[328,656,312] [328,656,309]	{-1,1} {}
(20,1)	[396,792,412]	{-1,1}		(19,-1)	[365,730,349] [365,730,381]	{-1,1} {}
(1,-1)	[5,10,-11] [5,10,21]	{-5,-3,-1,1,3,5} {-1,1}		(20,-1)	[404,808,388] [404,808,420]	{-1,1} {}

Then, we just take into account the values of x in the above table (with $x \geq 1$) for which a prime number $p = \frac{1}{2}(x^2 + 1)$ is a term in the sequence $\{U_n(P,Q)\}$ for each corresponding values of P and Q. Finally, for the obtained values of x satisfying the above conditions, we determine the values of n such that $U_n(P,Q) = p = \frac{1}{2}(x^2 + 1)$. Once more, we summarize the specifics of computations in the table below:

(P,Q)	x	p	n
(5,1)	3	5	2
(13,1)	5	13	2
(1,-1)	3 5	5 13	5 7

(P,Q)	x	p	n
(2,-1)	3	5	3
(5,-1)	3	5	2
(13,-1)	5	13	2

So, the proof of Theorem 2 is completed.

Proof Corollary 1. From Theorem 2, we get that $(P_1, Q_1, n) \in \{(5,1,2), (13,1,2), (1,-1,\{5,7\}), (2,-1,3), (5,-1,2), (13,-1,2)\}$.

Therefore, it remains to get the values of (P_2, Q_2, k) such that the corresponding values of x satisfy the equation

$$U_k(P_2, Q_2) = x,$$

where $k \geq 1, 1 \leq P_2 \leq 20$ and $Q_2 = \pm 1$. For example, if we consider $(P_1, Q_1, n) = (5, 1, 2)$. Here, we have $U_2(3, 1) = p = 5 = \frac{1}{2}(3^2 + 1)$ where we look for the values of P_2, Q_2 and k such that $U_k(P_2, Q_2) = 3$ holds with $1 \leq P_2 \leq 20 \; and \; Q_2 = \pm 1$. This clearly implies that $(P_2, Q_2, k) = (3, 1, 2)$, $(1, -1, 4)$ or $(3, -1, 2)$.

In a similar way, the remaining cas-- are treated similarly, and the results are as follows:

(P_1, Q_1)	(P_2, Q_2)	p	n	k
(13,1)	(5,1)	13	2	2
(13,1)	(1,-1)	13	2	5
(13,1)	(2,-1)	13	2	3
(13,1)	(5,-1)	13	2	2
(1,-1)	(3,1)	5	5	2
(1,-1)	(1,-1)	5	5	4
(1,-1)	(3,-1)	5	5	2
(1,-1)	(5,1)	13	7	2
(1,-1)	(1,-1)	13	7	5
(1,-1)	(2,-1)	13	7	3
(1,-1)	(5,-1)	13	7	2

(P_1, Q_1)	(P_2, Q_2)	p	n	k
(2,-1)	(3,1)	5	3	2
(2,-1)	(1,-1)	5	3	4
(2,-1)	(3,-1)	5	3	2
(5,-1)	(3,1)	5	2	2
(5,-1)	(1,-1)	5	2	4
(5,-1)	(3,-1)	5	2	2
(13,-1)	(5,1)	13	2	2
(13,-1)	(1,-1)	13	2	5
(13,-1)	(2,-1)	13	2	3
(13,-1)	(5,-1)	13	2	2
-	-	-	-	-

The above table proves the result of Corollary 1.

Proof Corollary 2. Again, the proof of this corollary follows from the result of Theorem 2 by following the same approach used in the proof of Corollary 1. For example, let's again consider $(P_1, Q_1, x, n) = (5, 1, 3, 2)$. Here, we want to find the values of k, P_2, Q_2 for which

$$5 = p = U_2(5, 1) = \frac{1}{2}(x^2 + 1) = \frac{1}{2}(3^2 + 1) = \frac{1}{2}(V_k(P_2, Q_2)^2 + 1).$$

Where $1 \leq P_2 \leq 20, Q_2 = \pm 1$ and $k \geq 0$. Thus, the last equation is true for the values $(P_2, Q_2, k) = (3, 1, 1), (1, -1, 2) \; or \; (3, -1, 1)$. Hence, the first three solutions given in Corollary 2 are found, and the others acquired similarly. Therefore, Corollary 2 is completely proved.

Proof of Theorem 3. Since we want to find the values of P, Q, x and n that satisfy the equation

$$V_n(P,Q) = p = \frac{1}{2}(x^2+1),$$

where $n \geq 0, x \geq 1, 1 \leq P \leq 20$ and $Q = \pm 1$. The proof of this theorem is achieved by following the same approach used in the proof of Theorem 2 in order to determine all the values of x by using equation (9). So, we omit the specifics of the proof. Such values with their corresponding values of $P, Q \; and \; n$ are summarized in the following table:

(P,Q)	x	p	n
(5,1)	3	5	1
(13,1)	5	13	1

(P,Q)	x	p	n
(5,-1)	3	5	1
(13,-1)	5	13	1

So, the results in the above table provides the complete set of solutions to the equation (6), and Theorem 3 is completely proved.

Proof Corollary 3. From the result of Theorem 3, it is clear that $(P_1, Q_1, n) \in \{(5,\{1,-1\},1),(13,\{1,-1\},1)\}$. Hence, it remains to find the values of $k \geq 1$ with which the corresponding values of x satisfy

$$U_k(P_2, Q_2) = x,$$

where $1 \leq P_2 \leq 20 \; and \; Q_2 = \pm 1$. Let's consider $(P_1, Q_1, n) = (5,\{1,-1\},1)$ with $x = 3$. Then, we have that

$$5 = V_1(5,\{1,-1\}) = \frac{1}{2}(U_k^2\,(P_2, Q_2) + 1),$$

that is satisfied when $(P_2, Q_2, k) \in \{(3,1,2),(1,-1,5),(3,-1,2)\}$. In case of $(P_1, Q_1, n) = (13,\{1,-1\},1) \; and \; x = 5$, we get that

$$13 = V_1(13,\{1,-1\}) = \frac{1}{2}(U_k^2\,(P_2, Q_2) + 1),$$

which is held whenever $(P_2, Q_2, k) \in \{(5,1,2),(1,-1,5),(2,-1,3),(5,-1,2)\}$.

Hence, the proof of the result of Corollary 3 is achieved.

Proof Corollary 4. This corollary is confirmed by the results related to Theorem 3 by following the same technique in the proof of Corollary 3. As a result, we skip the specifics of the proof.

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