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Nearly Endo-T-ABSO submodule and related concepts

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Abstract

In this paper, we discuss the notions Nearly-Endo prime submod, Nearly- EndoT-Absorbing submod, observations, and the traits and characteristics of Endo-T-ABSO and Jacobson radical in relation to one another. The relationship between EndoT-Absorbing submod, Endo prime submod, Nearly-End-prime submod, Nearly-EndoT-Absorbing submods were discovered. We are now introducing submods of several types. The ramifications of this study can be used to create new Nearly-EndoT-Absorbing submod, Nearly-Endo-prime submod based notions.

Subject Classification: *Primary 16D10, Secondary 16D80.*

Keywords: *T-Absorbing submod, EndoT-ABSO submodule, Nearly-Endo prime submodule, Nearly-EndoT-ABSO submodule.*

1. Introduction

W is a unitary G-module in this paper, as well as G having the identity and being a commutative ring. Minimal (maximal) submods were concepts introduced by Kasch[5, 2]. Study of the idea of a prime submod of modules was done in 1983 by Chin-Pi Lu [4, 13-18]. While H. A. Jubair et al. [7] proposed the concept of Nearly prime, S. O. Dakheel [11] was the first to present the Endo prime submod. The 2-Absorbing submod was conceptualized by A. Y. Yousefian and F. Soheilnia in 2011 [3]. The Endo 2-absorbing submod was added to this concept by Abdulrahman Abdullah in 2015 [1]. a module W's

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intersection with all of its maximum submods, or its sum of all small submods the Jacobson radical of that module, indicated by the symbol J(W). This study consists of two parts. In the first section, we introduce numerous ideas and some of their core attributes. The Nearly EndoT-Absorbing submod's relationships, traits, and core discoveries are examined in the second section.

2. Nearly- EndoT-ABSO Submod

Several theorems, relationships, comments, instances, and statements are offered, together with the Nearly EndoT-ABSO submod idea, further definitions, and the required to establish them.

Definition 2.1: $S < W$ of a G-module W is called Nearly Endo Primesubmod (by briefly N-E-Prime submod of W), if when even $f \in End(W), x \in W$ such that $f(x) \in S$, means that either $x \in S + J(W)$ or $f(W) \subseteq P + J(W)$.

Example 2.2: Let Z_6 as $Z -$ module. $S = (\overline{2})$ is $N - E - Prime$ submod since if $f(x) = (2x), \forall\, x \in Z_6, f \in Endo(Z_6)$ and $J(Z_6) = (\overline{2}) \cap (\overline{3}) = (\overline{0})$, $f(2) = 2(2) = (4) \in P = (\overline{2})$, means that either $2 \in (\overline{2}) + J(Z_6) = (\overline{2})$ or $f(Z_6) \subseteq (\overline{2}) + J(Z_6) = (\overline{2})(4) = 8$.

Definition 2.3: $S < M$ of a G-module M, is called Nearly EndoT-ABSO submod (by briefly N-E-T-ABSO submod of M), if when ever $f, g \in Endo(M), x \in M$ such that $(f \circ g)(x) \in S$, means that either $f(x) \in S + J(M)$ or $g(x) \in (S + J(M))$ or $(f\circ g)(M) \subseteq S + J(M)$.

Example 2.4: Consider Z_{15} as Z-module, P=$(\overline{3}) = \{0, 3, 6, 9, 12\}$ is N-EndoT-Absorbing submod, since if $f, g \in Endo(Z_{15}), m \in Z_{15}, f(x) = 3x, \forall\, x \in Z_{15}$ $g(x) = x\ \forall\, x \in Z_{15}$ and $J(Z_{15}) = (\overline{3}) \cap (\overline{5}) = (\overline{0})$ $(f\circ g)(3) = f(3) = 9 \in P$, means that either $f(3) = 9 \in P + J(Z_{15}) = (\overline{3})$ or $g(3) = 3 \in P + J(Z_{15}) = (\overline{3})$ or $(f\circ g)(Z_{15}) \subseteq P + J(Z_{15}) = (\overline{3})$. So that $(f \circ g)(Z_{15}) \subseteq P + J(Z_{15}) = (\overline{3})$.

Remark and Examples 2.5

1) All Nearly-Endoprime submod is Nearly-EndoT-Absorbing submod, conversely, however incorrectin general, As an illustration: Z_6 as $Z -$ module, $(\overline{2})$ is a submod of Z_6, $(\overline{2})$ is N-EndoT-Absorbing submod, since if $f, g \in End(Z_6), f(x) = x + 1$ and $g(x) = 3, \forall\, x \in Z_6, (f \circ g)(2) = f(g(2)) = 4 \in (\overline{2})$ means that $(f \circ g)(Z_6) \subseteq (\overline{2}) + J(Z_6)$ where $J(Z_6) = (\overline{2}) \cap \overline{(3)} = (\overline{0})$. So that $(f \circ g)(Z_6) \subseteq (\overline{2}) + J(Z_6)$. Butit is no Nearly-E-prime submod of Z_6, $f(3) = 4 \in (\overline{2})$, means that either $3 \notin (\overline{2}) + J(Z_6) = (\overline{2})$ or $f(Z_6) \nsubseteq (\overline{2}) + J(Z_6) = (\overline{2})$.
2) Let $P, S \leq M$of a R-module M and $P < S$. If S is Nearly-EndoT-Absorbing submod of M, then P is not necessary that Nearly-EndoT-

Absorbing submodof M, for example: consider Z_{24} as Z-module, Take $S = (\bar{4}), P = (\overline{12}), \forall\, f, g \in Endo(Z_{24}), f(x) = x - 2, g(x) = 2x, \forall\, x \in Z_{24}$ where $J(Z_{24}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$ $S = ((\bar{4}))$ is Nearly-EndoT-Absorbing submod of $M = Z_{24}$ since $(f \circ g)(7) = f(g(7)) = 12 \in S$ means that $g(7) = 14 \in S + J(Z_{24}) = (\bar{2})$, but $P = (\overline{12})$ is not N-EndoT-Absorbing submod of $W = Z_{24}$ since $(f \circ g)(7) = f(g(7)) = 12 \in P$, then $f(7) = 5 \notin P + J(Z_{24}) = (\bar{6})$, $g(7) = 14 \notin P + J(Z_{24}) = (\bar{6})$ and $(f \circ g)(Z_{24}) \nsubseteq P + J(Z_{24})$ Where $(f \circ g)(6) = 10 \notin P + J(Z_{24}) = (\bar{6})$.

3) All Endo Prime submod of a G-module W is Nearly-Endo Prime submod of W.

Proof: Let S be E-Prime submod of and $f \in End(W), m \in W$ such that $f(m) \in S$, to prove $m \in S + J(W)$ *or* $f(W) \subseteq S + J(W)$. Since S is E-Prime submod, then $m \in S$ or $f(W) \subseteq S$, hence $m \in S + J(W)$ $f(W) \subseteq S + J(W)$ since $S \subseteq S + J(W)$ Thus S is N-Endo-Prime submod of W But the converse of (3) is nottrue generally, for example: Consider Z_{20} as Z-modul, $S = (\bar{4})$ is Nearly-EndoT-Absorbing submod since if $f \in End(Z_{20})$, $f(x) = x - 2, \forall\, x \in Z_{20}$ where $J(Z_{20}) = (\bar{2}) \cap (\bar{5}) = (\overline{10})$, $f(6) = 4 \in S = (\bar{4})$, then either $6 \in S + J(Z_{20}) = (\bar{2})$ or $f(Z_{20}) \subseteq S$, but S is not E-Prime submod, since $f(6) = 4 \in S$, then $6 \notin S = (\bar{4})$ and $f(Z_{20}) \nsubseteq S = (\bar{4})$ where $f(3) = 1 \notin S$.

4) Every EndoT-Absorbing submod of a R-module W is Nearly-EndoT-Absorbing submod of W.

Proof: Suppose P be EndoT-Absorbing submod of a R-module W and $f, g \in End(W), m \in W$ such that $(f \circ g)(m) \in P$, however P is EndoT-Absorbing submod of W, then $f(m) \in P$ or $g(m) \in P$ or $(f \circ g)(W) \subseteq P$, hence $f(m) \in P + J(W)$ or $g(m) \in P + J(W)$ or $(f \circ g)(W) \subseteq P + J(W)$ since P$\subseteq$ P+$J(W)$. Thus P N-EndoT-Absorbing submod. However the converse of (4) incorrect in general, forexample:

Consider Z_{24} asZ − module, $P = (\bar{8}))$ is Nearly-EndoT-Absorbing submod, since if $f, g \in End(Z_{24})$, $f(x) = x - 2, g(x) = x + 1, \forall\, x \in Z_{24}$ where $J(Z_{24}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$ suchthat $(f \circ g)(9) = f(g(9)) = 8 \in P = (\bar{8})$, then $g(9) = 10 \in P + J(Z_{24}) = (\bar{2})$, but $f(9) = 7 \notin P$ or $g(9) = 10 \notin P$ or $(f \circ g)(Z_{24}) \nsubseteq P$ where $(f \circ g)(10) = f(g(10)) = 9 \notin P$. So that P is Nearly-EndoT-Absorbing submod of W.

5) Suppose $P, S \leq M$ of a G-module M, $P < S$. If P is Nearly-EndoT-ABSO submod of M, then P Nearly-EndoT-Absorbing submod of Swith $J(M) \subseteq J(S)$.

Proof: Let $(f \circ g)(m) \in P, \forall\, m \in S$ since $S < M, so\ m \in M, f \in\ EndM$Since P is N-Endo-TABSO submod of M, then either $f(m) \in P + J(M)$ or $g(m) \in P + J(M)$ or $(f \circ g)(M) \subseteq P + J(M)$, since $J(M) \subseteq J(S)$, hence $f(m) \in P + J(S)$ or $g(m) \in P + J(S)$ or $(f \circ g)(W) \subseteq P + J(S)$, but $S\ proper\ submodule\ of\ W$, so that $(f \circ g)(S) \subseteq (f \circ g)(M)$, hence $(f \circ g)(S) \subseteq P + J(S)$. Thus P is N-EndoT-Absorbing submod of S.

6) The intersection of two N-EndoT-Absorbing submod not be N-EndoT-Absorbing submod, for example: consider Z_{12} as Z-module take $P = (\overline{4}), S = (\overline{3})$ are N-EndoT-Absorbing submods of Z_{12}, since $\forall\, f, g \in Endo(Z_{12})$, $f(x) = x - 3, g(x) = x - 2, \forall\, x \in Z_{12}$ where $J(Z_{12}) = (\overline{2}) \cap (\overline{3}) = (\overline{6})$ suchthat $(f \circ \mathrm{g})(5) = f\big(g(5)\big) = 0 \in (\overline{4})$, then $f(5) = 2 \in (\overline{4}) + J(Z_{24}) = (\overline{2})$, Also, $(f \circ \mathrm{g})(5) = f\big(g(5)\big) = 0 \in (\overline{3})$, then $g(5) = 3 \in (\overline{3}) + J(Z_{12}) = (\overline{3})$. But $(\overline{4}) \cap (\overline{3}) = (\overline{0})$ is not N-EndoT-Absorbing submods of Z_{12}, since $(f \circ \mathrm{g})(5) = f\big(g(5)\big) = 0 \in (\overline{0})$, then $f(5) = 2 \notin (\overline{0}) + J(Z_{12}) = (\overline{6})$, $g(5) = 2 \notin (\overline{0}) + J(Z_{12}) = (\overline{6})$ and $(f \circ \mathrm{g})(Z_{12}) \nsubseteq (\overline{0}) + J(Z_{12}) = (\overline{6})$, Where $(f \circ \mathrm{g})(6) = 1 \notin (\overline{0}) + J(Z_{12}) = (\overline{6})$

Definition 2.6: [9]A R-module M is referred to as aScalar module if for all $\mathrm{f} \in \mathrm{End}(\mathrm{M})$, there exists $\mathrm{r} \in \mathrm{R}$ such that $\mathrm{f(m)} = \mathrm{rm}$, for $m \in M$.

Corollary 2.7: [9] *All finitely generated multiplication R-modul M is aScalar module.*

Proposition 2.8: Let $P \leq M$ be Nearly-EndoT-Absorbing submod of a R-module M is scalar module and $J(M) \subseteq P$ iff P is T-ABSO submod of M**.**

Proof: $(\Longrightarrow)$ Let $f, g \in \mathrm{Endo}(M), \forall m \in M$ such that $f(m) = am, g(m) = bm$ $(f \circ \mathrm{g})(M) = abm \subseteq P, since$ M is aScalar modul however P is Nearly-EndoT-Absorbing submod of M, then either $am = f(m) \in P + J(M)\ or\ bm = g(m) \in P + J(M)\ or\ (f \circ \mathrm{g})(M) = abM \subseteq P + J(M)$, hence $f(m) = am \in P$ or $bm = g(m) \in P$ or $abM = (f \circ \mathrm{g})(M) \subseteq P$, such that $ab \in (P :_G M)$. Since $J(M) \subseteq P$. Then P is T-ABSO submod of M.$(\Longleftarrow)$Let $(f \circ \mathrm{g})(m) \in P\ where\ f, g \in \mathrm{Endo}(M), \forall m \in M, since$M is aScalar module, next there are a, $\mathrm{b} \in G$ like that $am = f(m), bm = g(m)$ for each $m \in M$, but P is T-ABSO submod of M, then $(f \circ \mathrm{g})(m) = abm \subseteq P$ means that either $am = f(m) \in P\ or\ bm = g(m) \in P$ or $abM \subseteq P$, Since $J(M) \subseteq P$ hence $f(m) \in P + J(M)$ or $g(m) \in P + J(M)$ or $(f \circ \mathrm{g})(M) \subseteq P + J(M)$. Then P is Nearly-EndoT-Absorbing submod of M.

Remark 2.9: If the scalar module's condition is deleted, the proposition's2.8 opposite occurs which is not generally true, for example: Consider Z_{12} asZ-module, $P = (\overline{6})$ is T-ABSO submod, $2 \cdot 6 \cdot (\overline{1}) = 12 = 0 \in P = (\overline{6})$, means that $6 \cdot (\bar{1}) \in P$ or $2 \cdot 6 = 12 = 0 \in (P:_Z Z_{12}) = 6Z$. So that P is T-ABSO submod of Z_{12}. But P is not Nearly-EndoT-Absorbing submod of Z_{12} since if $f, g \in Endo(Z_{12})$, $f(x) = x - 3$, $g(x) = 3x$ such that $(f \circ g)(5) = f(g(5)) = 12 = 0 \in P$, where $J(Z_{12}) = (\bar{2}) \cap (\bar{3}) = (\bar{6})$, then $f(5) = 2 \notin P + J(Z_{12}) = (\bar{6})$ and $g(5) = 15 = 3 \notin P + J(Z_{12}) = (\bar{6})$ and $(f \circ g)(Z_{12}) \nsubseteq P + J(Z_{12})$ Where $(f \circ g)(4) = f(g(4)) = 9 \notin (\bar{6}) = P + J(Z_{12})$

Corollary 2.10: *Let $P \leq M$ of afinitely generated multiplication R-module M, $J(M) \subseteq P$. then P is T-ABSO submod iff P is Nearly-EndoT-Absorbing submod of M.*

Proof: By Proposition2.8 and Corollary2.7 As a result.

Proposition 2.11: Let W be a nonzero modle over a integral domain G. If W torsion module, then W has no N-E-T-ABSO submod

Proof: Suose that W has a T-ABSO submod say P since Wis atorsion module [8], then $\left(\frac{W}{p}\right)$ is a torsion G-module, Also, We exhibit this. $\left(\frac{W}{p}\right)$ is torsion free G-module as follows:

Let $0 \neq (m + P) \in \left(\frac{G}{p}\right)$, suose $r(m + P) = P$ $f$or a few $r \in G$

If r≠ 0, then $rm \in P$, $m \notin P$ since M is divisible, then $\frac{W}{r} = W$ so $m = rm_1$, for some $m \in W$ there fore $rm = r^2 m_1 \in P$, but P is $T - ABSO$, then $rm_1 \in P$ or $r^2 \in (P{:}W)$.

If $rm_1 \in P$, $then$ $m \in P$ $which$ is a Contradiction.

If $r^2 \in (P{:}W)$ then $r^2 w \subseteq P$, but w is divisible, so $r^2 W = W$ there fore $W = P$ whichis Contradiction.

If $r = 0$ and $\left(\frac{W}{P}\right)$ is torsionand torsion freewhich imply that $\left(\frac{W}{P}\right) = 0$, so $W = P$ which is Contradiction. Hence w has no T-ABSO submod so W has no N-E-T-ABSO submod.

Example 2.12: Z_{p^∞} as Z-module satisfies the requirement of Proposition2.11 hence Z_{p^∞} has no N-Endo-T-ABSO $submod$. Also, we can show this as follows:

Frist we show that (0) in not T-ABSO submod of Z_{p^∞}. Let $pp\left(\frac{1}{p2} + Z\right) = Z = 0$ but $p\left(\frac{1}{p2} + Z\right) \neq Z \neq 0$ and $P^2 \notin (0{:}Z_{p^\infty}) = 0$.

Now, for $0 \neq K < Z_{p^\infty}$, let $K = \left(\frac{1}{pi} + Z\right)$ for a few $i \in Z_+$, so $p.p(\frac{1}{pi+2} + Z) \subseteq K$. However $p(\frac{1}{pi+2} + Z) \nsubseteq K$. and $p^z \notin (K{:}Z_{p^\infty}) = 0$. Thus K is not T-ABSO submod of Z_{p^∞}. hence all proper submod of Z_{p^∞}

is not Nearly-EndoT-Absorbing submod. That is Z_{p^∞} has no N-EndoT-Absorbing submod.

Proposition 2.13: Let $f: M \to M$ be a R-hommomorphism. If S is a fully invariant EndoT-Absorbing submod of M, that way $f(M) \nsubseteq S$. Then $f^{-1}(S)$ is Nearly-EndoT-Absorbing submod of M.

Proof: Let $S \leq M$, So $f^{-1}(S)$ is submod of M, $f, g \in$ Endo$(M), \forall m \in M$ such that $(h \circ g)(m) \in f^{-1}(S), then\ f(h0g)(m) \in S\ so\ (f \circ h)(g(m)) \in S. since$ S is EndoT-Absorbing submod of M then either $f(g(m)) \in S\ or\ h(g(m)) \in S\ or\ f(h(M)) \subseteq S.$

Case 1: If $f(g(m)) \in S, f^{-1}\ [f(g(m)) \in S]$ then $g(m) \in f^{-1}(S)\ g(m) \in f^{-1}(S) \subseteq f^{-1}(S) + f^{-1}((J(M))$

Case 2: $h(g(m)) \in S$ but S is EndoT-Absorbing submod of M then $h(m) \in S$ or $g(m) \in S$ or $h(g(m)) \subseteq S$. If $h(m) \in S$ then $f(h(m)) \in f(S) \subseteq S$ since S is fully invariant [10], hence $f(h(m)) \in S, f^{-1}[f(h(m)) \in S]$ so $h(m) \in f^{-1}(S) \subseteq f^{-1}(S) + f^{-1}(J(M))\ then\ h(m) \in f^{-1}(S) + f^{-1}(J(M))$. If $g(m) \in S$ then $f(g(m)) \in f(S) \subseteq S,$ since S is fully invariant $f(g(m)) \in S,$ so $g(m) \in f^{-1}(S) \subseteq f^{-1}(S) + f^{-1}(J(M))\ then\ g(m) \in f^{-1}(S) + f^{-1}(J(M))$. If $h(g(m)) \subseteq S$ then $f(h((g(m)) \in f(S) \subseteq S$ since S is fully invariant $f(h((g(m)) \subseteq S, f^{-1}[f(h(g(m)) \in S]$ so $h(g(m)) \in f^{-1}(S) \subseteq f^{-1}(S) + f^{-1}(J(W))\ then\ h(g(m)) \in f^{-1}(S) + f^{-1}(J(M)).$

Case 3: If $f(h(M) \subseteq S, f^{-1}[f(h(M) \subseteq S\ ]$ then $h(M) \subseteq f^{-1}(S) \subseteq f^{-1}(S) + f^{-1}(J(M))\ so\ h(M) \in f^{-1}(S)$. Thus $f^{-1}(S)$ is a N-EndoT-Absorbing submod of M.

Proposition 2.14: Let $P \leq M$ be of R-module of M, $suppose$ S is afully invariant submod of R-module M and included in P. If $\frac{P}{S}$ is a EndoT-Absorbing submod of $\frac{M}{S}$. then Pis a Nearly-EndoT-Absorbing submod ofM.

Proof: suppose $f, h \in EndoM, x \in M$ such that $foh(x) \in P$. Define $f_1, h_1: \frac{M}{S} \to \frac{M}{S}$ by $f_1(x + S) = f(x) + S, h_1(x + S) = h(x) + S\ for\ each\ x \in M.$ It is clear that f_1, h_1 are well-defined. Now $f \circ h(x + S) = f_1(\ h_1(x + S)) = f_1(x) + S, f_1(h(x) + S) = f \circ h(x) + S \in \frac{P}{S}$. As $\frac{P}{S}$ is EndoT-Absorbing submod of $\frac{M}{S}$. Then either $h_1(x + S) \in \frac{P}{S}\ or\ f_1(x) + S\ or\ f_1(h_1(\frac{M}{S})) \subseteq \frac{P}{S}, h(x) + S \in \frac{P}{S} + J(\frac{M}{S})\ or\ f(x) + S \in \frac{P}{S} + J(\frac{M}{S})\ or\ f(h(M) + s) \subseteq \frac{P}{S} + J(\frac{M}{K})$. There fore $h(x) \in P + J(M)\ or\ f(x) \in P + J(M)\ or\ f(h(M)) \subseteq P + J(M)$. thus P is a Nearly-EndoT-Absorbing submod of M.

Theorem 2.15: Let S ≤ W be afully invariant Nearly-EndoT-Absorbing submod of R-moduleW, suppose $f: W \to W'$ be aepimorphism like that$Ker\, f \subseteq S$. Then $f(S)$ is Nearly-EndoT-Absorbing submod of W' where W ' is W '-projective Module.

$$\begin{tikzcd} & W' \arrow[dl, dashed, "k_1"'] \arrow[d, "g"] \\ W \arrow[r, "f"'] & W' \end{tikzcd} \qquad \begin{tikzcd} & W' \arrow[dl, dashed, "k_2"'] \arrow[d, "h"] \\ W \arrow[r, "f"'] & W' \end{tikzcd}$$

f o k$_1$*= g* *f o k*$_2$*= h*

Now consider the following diagrams

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend right, "k_1 \circ f \quad EndW"'] & W' \arrow[r, "k_1"] & W \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend right, "k_2 \circ f \quad EndW"'] & W' \arrow[r, "k_2"] & W \end{tikzcd}$$

Proof: Suppose $g, h \in EndW', m' \in W'$ like that $(g \circ h)(m') \in f(S)$ Caused by f is epimorphism then $m' = f(m)$ for a few $m \in S$ Caused by W' is W'-projective Module[9], there exist $K_1, K_2: W' \to W$, like that $f \circ K_1 = g$ *and* $f \circ K_2 = h$ $(g \circ h)(m') = (f \circ K_2) \circ (m') \in f(S) = f[(f \circ K_1 \circ K_2)(m')] \in f(S)$.

Then $((K_1 \circ (f \circ K_2)(m') \in S + Kerf$,since$Kerf \subseteq S, and\ m' = f(m), so((K_1 \circ f \circ K_2)(f(m)) \in S$. That is $(K_1 \circ f)(K_2 \circ f)(m)) \in S$, since S is EndoT-Absorbing submod ofW, then either $(K_1 \circ f)(m) \in S + J(W)$ ***or*** $(K_2 \circ f)(m) \in S + J(W)$ ***or*** $(K_1 \circ f) \circ (K_2 \circ f)(W) \subseteq S + J(W)$. If $(K_1 \circ f)(m) \in S + (J(W) \to (K_1(f(m)) \in S + J(W), In\ other\ words$ $K_1(m') \in S + J(W)$, so $f(K_1(m')) \in f(S) + (f(J(W))$ thus $g(m') \in f(S) + f(J(W))$. If $(K_2 \circ f)(m) \in S + J(W), then(K_2(f(m)) \in S + J(W)$, $i.e\ [K_2(m') \in S + J(W)], so\ f(K_2(m')) \in$ $f(S) + f(J(W))$, thus$h(m') \in f(S) + f(J(W))$

If $(K_1 \circ f) \circ (K_2 \circ f)(W) \subseteq S + J(W)$, then $(K_1 \circ f \circ K_2) f(W)) \subseteq S + J(W)$, Since $f(W) = W', (K_1 \circ f \circ K_2)(W') \subseteq S + J(W)$, since $(foK_2) = h, then\ (K_1 \circ h)(W') \subseteq S + J(W)$. So $f[(K_1 \circ h)(W')] \subseteq f(S) + f((J(W)),. i.e\ ((f \circ K_1) \circ h)(W') \subseteq f(S) + f(J(W))$. Since $f \circ K_1 = g$, then $((goh))(W') \subseteq f(S) + f(J(W))$.

Thus $f(S)\ is\ Nearly - Endo - T - ABSO\ submod\ of\ W'$.

Corollary 2.16: Suppose P ≤ M be afully invariant Nearly-EndoT-Absorbing submod of G-module M. If K is submod of M, such that $K \subseteq P, then\ \frac{P}{K}$ is N-EndoT-Absorbing submod of $\frac{M}{K}$, Provided $\frac{M}{K}$, is a M-projective Module.

Proof: $\pi: M \to \frac{M}{K}$, $\pi(P) = \frac{P}{K}$, then by Theorem 2.15, we get result.

Theorem 2.17: Let $P \leq W$ be N-EndoT-Absorbing submod of a R-module W and S is a submod of W, which is W-injective submod such that $J(W) \subseteq J(S)$ and $J(S)$ is distributive submod. Then either $S \subseteq P$ or $S \cap P$ is N-E-TABSO submod of S.

Proof: Suppose that $S \nsubseteq P$, then $S \cap P \subsetneq S$, let $f, g \in EndS, x \in S$ such that $fog(x) \in S \cap P$ $suppose$ $g(x) \notin S \cap P + J(S), then$ $g(x) \notin [S + J(S)] \cap [P + J(S)], since$ $J(S)$ is distributive submod[12], hence $g(x) \notin P + J(S)$, so that $g(x) \notin P + J(W)$, since $J(W) \subseteq J(S)$, to prove $f(x) \in S \cap P + J(S)$ or $f \circ g(S) \subseteq S \cap P + J(S)$, since S is W-injective submod [9, 5], then there exist $h, K: W \to S$ as in the figure :

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "i"] \arrow[d, "f"'] & W \arrow[dl, "h"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & S \arrow[r, "i"] \arrow[d, "g"'] & W \arrow[dl, "K"] \\ & S & \end{tikzcd}$$

Now consider the following diagrams

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[rr, bend right, "h \circ i \quad EndS"'] & W \arrow[r, "h"] & S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "i"] \arrow[rr, bend right, "k \circ i \quad EndS"'] & W \arrow[r, "k"] & S \end{tikzcd}$$

Where I is the inclusion maing and $hoi = f$, $koi = g$ clearly that $\text{h}, \text{k} \in EndS$, But $f \circ g(x) = (f \circ i) \circ (k \circ i)](x) = [(h \circ i \circ k)](x) = (h \circ i)[k(x)] = (h \circ k)(x) \in P$, since P is Nearly-E-TABSO submod of W and $g(x) \notin P + J((W)$ means that $k(x) \notin P + J(W)$. Then either $h(x) \in P + J((W)$ or $h \circ k(W) \in P + J(W)$. If $h(x) \in P + J(W)$, then $h(x) \in S \cap P + J(W)$ since $h(x) \in S$, so $h(x) \in S + J((W)$ but $J(W) \subseteq J(S), hence$ $h(x) \in S \cap P + J(S)$. Thus $f(x) \in S \cap P + J(S)$.

Now, if $(h \circ k)(W) \subseteq P + J(W)$. As $f \circ g(S) = (h \circ i) \circ (k \circ i)](S) = [(h \circ i \circ k)](S) = (h \circ i)[k(S)] = (h \circ k)(S) \subseteq P + J(W), then$ $f \circ g(S) \subseteq P + J(S), since$ $J(W) \subseteq J(S)$. Also, $(f \circ g)(S) \subseteq S$, hence $fog(S) \subseteq S + J(S)$.Thus $(f \circ g)(S) \subseteq \big(P + J(S)\big) \cap \big(S + J(S)\big)$, so that $(f \circ g)(S) \subseteq (P \cap S) + J((S)$, since $J(S)$ is distributive submod. There fore $S \cap P$ is N-Endo-TABSO submod of S.

Conclusions

As a new generalization of the T-ABSO submod, the Nearly-EndoT-Absorbing submod are introduced. The following are the study's main findings:

1) Every N-E-Prime Submod is N-E-T-ABSO submod, but the converse incorrcet in general, see Remarks and Examples 2.5

2) Let P, S be two submods of a G-module W and P< S. If S is N-E-T-ABSO submod of W, then P is not necessary that N-E-T-ABSO submod of W.
3) Every E-Prime submod of a G-module W is N-E-Prime submod.
4) Every E-T-ABSO submod of a G-module W is N-E-T-ABSO submod. But the converse incorrect in general.
5) Let P, S be two submods of a G-module W, and P<S. If P is N-E-T-ABSO submod of W. then P is N-E-T-ABSO submod of S with $J(W) \subseteq J(S)$.
6) The intersection of two N-E-T-ABSO submod not be N-E-T-ABSO submod.
7) Let P be N-E-T-ABSO submod of a G-module W is Scalar module and $J(W) \subseteq P$ if and only if P is T-ABSO submod of W.
8) Let W be a $nonzero\ module$ over a $integral\ domain$. If W torsion module then W $has\ no\ N - Endo - T - ABSO\ submodul$e.
9) Let$f$: W → W be a G-h$omomorphism$. If p is $fully\ invariant$ EndoT-Absorbing submod of W, $uch\ that\ f(W) \nsubseteq P.\ Then\ f^{-1}(p)\ is\ also\ an$ N-E-T-ABSO submod of W.
10) Let P be a fully invariant N-E-T-ABSO submod of G-module W,
Let be $f: W \to W'$ be epimorphism such that $Ker\ f \subseteq P$. Then f(P) is N-E-T-ABSO submod of W' where W ' is W '-projective Module..
11) Let P be a N-E-T-ABSO $submodule$ of a G-m$odule$ W and S is a $submodule$ of W, which is W-$injective$ sub$modul$e such that $J(W) \subseteq J(S)\ and\ J(S)$ is distributive submod. Then either $S \subseteq P\ or\ S \cap P$ is N-E-TABSO submod of S.

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