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Fuzzy stability in the sense of Hyers-Ulam-Rassias of a functional equation on quadratic forms

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Abstract

We obtain a general solution of the 2-variable quadratic functional equation

$$\zeta(y_1 + y_2, y_3 + y_4) + \zeta(y_1\text{-}y_2, y_3\text{-}y_4) = 2\zeta(y_1, y_3) + 2\zeta(y_2, y_4)$$

and discuss the stability of the above functional equation, while the quadratic form $\zeta(a, b) = \alpha a^2 + \beta ab + \gamma c^2$ is a solution of our functional equation.

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1. Introduction

Stability problems in several frameworks were studied through several kinds of functional equations. The concept that today we call Hyers-Ulam-Rassias stability was initiated by a question raised by Ulam [3] in 1940 and the answer given by Hyers [4] in 1941. This stability theory proposes an approximation for the solution of equations of a different nature, for example, differential or functional. In 1978, T. Rassias [5] generalized the results known for more than thirty years and completed what we know today under the name of Hyers-Ulam-Rassias stability. From these seminal papers, the concept called Ulam-like Stability has been developed, and the number of researchers and results has been increasing over time.

Ulam was the pioneer who raised concerns about the stability of group homomorphisms, which paved the way for research to study stability results. By taking into account Cauchy's functional equation, Hyers used Banach spaces to solve this stability result. Aoki [17] extended Hyers work by supposing an infinite Cauchy difference. Work on additive mapping was presented by Rassias [5], and later by Gavruta [18] and Gajda [6]. Jun and Kim [7] proved the stability problem for the cubic functional equation.

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In 1965, Zadeh [1] introduced the concept of fuzzy sets, and later classical theories of science and engineering [2] have been studied by several authors. Readers interested in seeing some recent work on stability can refer to [19-20] and [21].

Recall that a function d: $\Delta \times \Delta \to [0, \infty]$ (Δ is a set) is called a generalized metric on Δ if

$d(y_1, y_2) = 0$ if and only if $y_1 = y_2$;

$d(y_1, y_2) = d(y_2, y_1)$ for all $y_1, y_2 \in \Delta$;

$d(y_1, y_2) \le d(y_3, y_2) + d(y_1, y_3)$ for all $y_1, y_2, y_3 \in \Delta$.

Definition 1 [8, 9]: A fuzzy subset μ of $\Delta \times \mathbb{R}$ is said to be a fuzzy norm on Δ if for any $y_1, y_2 \in \Delta$ and $c \in \mathbb{R}$, $\tau \in \mathbb{R}$, we have

$\mu(y_1, \tau) = 0$ for all non-positive $\tau \in \mathbb{R}^+$,

$\mu(y_1, \tau) = 1$ for all $\tau \in \mathbb{R}^+$ if and only if $y_1 = 0$,

$\mu(cy_1, \tau) = \mu(y, t/|c|)$ for all $\tau \in \mathbb{R}^+$ and $c \neq 0$,

$\mu(y_1 + y_2, \tau + s) \ge \min\{\mu(y_1, \tau), \mu(y_2, s)\}$ for all $s, \tau \in \mathbb{R}$,

$\mu(y_1, \tau)$ is a non-decreasing function on $\mathbb{R}$, and $\sup_{t \in \mathbb{R}} \mu(\Delta, \tau) = 1$.

The pair (Δ, μ) will be called a fuzzy normed space (FNS).

Eyample 1 [10]: Let $(\Delta, ||.||)$ be a normed linear space. Then

$$\mu(y_1, \tau) = \begin{cases} \frac{\tau}{\tau + \|y_1\|}, & \text{if } \tau > 0 \\ 0, & \text{if } \tau \le 0 \end{cases}$$

is a fuzzy norm on Δ.

Eyample 2 [10]: Let $(\Delta, ||.||)$ be a normed linear space. Then

$$\mu(y_2, \tau) = \begin{cases} 1, & \text{if } \tau > \| y_2 \| \\ 0, & \text{if } \tau \le \| y_2 \| \end{cases}$$

is a fuzzy norm on Δ.

Let (Δ, μ) be an FNS. Then, a sequence $y = (y_k)$ is said to be

(I) fuzzy convergent to $L \in \Delta$ if $\lim \mu(y_k - L, \tau) = 1$, for all $\tau > 0$. In this case, we write μ-$\lim y_k = L$.

(II) fuzzy Cauchy sequence if $\lim \mu(y_{k+p} - y_k, \tau) = 1$ for all $\tau > 0$ and $p = 1, 2, ..$

Note that every convergent sequence in (Δ, μ) is Cauchy. The FNS (Δ, μ) is said to be complete if every fuzzy Cauchy sequence is fuzzy convergent and so the FNS (Δ, μ) is a fuzzy Banach space.

In this paper, we will study the equation and analyze the stability, in the Hyers-Ulam-Rassias sense, utilizing Banach's Fixed Point Theorem presented below, in the fuzzy norm space.

2. Fiyed point technique for Hyers-Ulam-Rassias stability

Next, we will study the stability problem in the fuzzy norm space, by means of the Fixed Point Method.

Theorem 1 [10]: (Banach's Contraction Principle). *Let (Δ, d) be a complete generalized metric space and consider a mapping $J : \Delta \to \Delta$ as a strictly contractive mapping, that is, $d(Ay_1, Ay_2) \leq Ld(y_1, y_2)$, $\forall y_1, y_2 \in \Delta$ for some $L < 1$, the Lipschitz constant. So, we have the following:*

(i) *y_0 is the only fiyed point of the mapping J, i.e., $y_0 = A(y_0)$;*

(ii) *The global attractiveness of this fixed point y_0. This means that $\lim A^n y_1 = y_0$, $n \to \infty$, holds with y_1 any starting point, such that $y_1 \in \Delta$;*

(iii) *For all $y_1 \in \Delta$ and $n \in \mathbb{N}$, the following estimation inequalities are fulfilled:*

$$d(A^n y_1, y) \leq L^n \times d(y_1, y_0)$$

$$d(A^n y_1, y_0) \leq \left(\frac{1}{1-L}\right) d(A^n y_1, A^{n+1} y_1)$$

$$d(y_1, y_0) \leq \left(\frac{1}{1-L}\right) d(y_1, Ay_1)$$

Theorem 2 [11]: (An Alternative to the of Fixed Point Theorem). *Under the conditions: (Δ, d) is a complete generalized metric space and $A : \Delta \to \Delta$ is a strictly contractive mapping, with Lipschitz constant L. Then any of the following statements is true:*

$$d(A^n y_1, A^{n+1} y_1) = +\infty, \text{ for all } n \in \mathbb{N}:$$

or

$$d(A^n y_1, A^{n+1} y_1) < +\infty \text{ , for all } n \geq \eta\circ$$

for some natural number $\eta\circ$. Moreover, if the second alternative holds then

(i) *The sequence $(A^n y_1)$ is convergent to a fiyed point y_0 of A;*

(ii) *y_0 is the unique fiyed point of A in the set $Ж = \{y \in \Delta, d(A^{\eta\circ} y_1, y) < +\infty\}$*

(iii) $d(y, y_0) \leq \left(\frac{1}{1-L}\right) d(y, Ay)$, *for all* $y \in Ж$.

Being y_1, any given element of Δ.

Next, we present our main results.

In the following, we consider that Δ is a real vector space and that Λ is a complete normed space. And we consider the following functional equation:

$$\zeta(y_1 + y_2, y_3 + y_4) + \zeta(y_1 - y_2, y_3 - y_4) = 2\zeta(y_1, y_3) + 2\zeta(y_2, y_4) \tag{1}$$

for a mapping $\zeta: \Delta \times \Delta \to \Lambda$. A solution of the above equation (1), is the quadratic form given by $\zeta(a, b) := \alpha a^2 + \beta ab + \gamma c^2$, with $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The stability of the general solution of the functional equation (1) was proved in [13, 14] as follows:

Theorem 3 [13]: *A mapping $\zeta : \Delta \times \Delta \to \Lambda$ satisfies the Equation (1) for all y_1, y_2, y_3, $y_4 \in \Delta$ if and only if there exist three bi-additive mappings ζ_1, ζ_2, ζ_3: $\Delta \times \Delta \to \Lambda$ (ζ_1, ζ_3 symmetric bi-additive and ζ_2 bi-additive) such that $\zeta(a, b) = \zeta_1(a, a) + \zeta_2(a, b) + \zeta_3(b, b)$ for all $a, b \in \Delta$.*

Let φ be a function from $\Delta \times \Delta \times \Delta \times \Delta \to [0, \infty)$. A mapping $\zeta: \Delta \times \Delta \to \Lambda$ is said to be symmetric bi-additive mapping if

$$\mu(\zeta(y_1 + y_2, y_3 + y_4) + \zeta(y_1 - y_2, y_3 - y_4) - 2\zeta(y_1, y_3) - 2\zeta(y_2, y_4), \tau) \geq \varphi(y_1, y_2, y_3, y_4, \tau) \tag{2}$$

for all $y_1, y_2, y_3, y_4 \in \Delta$ and $\tau > 0$.

Next, we will study the stability, for equation (1) in FNS, in the sense of Hyers-Ulam-Rassias (HUR) using the fiyed-point alternative.

Lemma 1: *Let (Δ, μ) be a fuzzy normed space and $\varphi:\Delta \times \Delta \times \Delta \times \Delta \to [0, \infty)$ be a function and*

$$\odot = \{\rho \mid \rho : \Delta \times \Delta \to Y; \rho(0, 0) = 0\}$$

and define

$$d(\rho, \sigma) := d_\mu(\rho, \sigma) = \inf D_\mu$$

where

$$D_\mu = \{\alpha \in \mathbb{R}+ : \mu(\rho(y, z) - \sigma(y, z), \alpha\tau) \geq \varphi(y, y, z, z, \tau), \text{ for all } y, z \in \Delta \text{ and } \tau \in \mathbb{R}+\}, \tag{3}$$

for all $\rho \in \odot$. From this, we have that d is a complete generalized metric on $\odot$.

Proof: Let $\rho, \sigma, \pi, \in \odot$, $d(\rho, \sigma) < k$ and $d(\sigma, \pi) < h$. Then

$$\mu(\rho(y, z) - \sigma(y, z), k\tau) \geq \varphi(y, y, z, z, \tau)$$

and

$$\mu(\pi(y, z) - \sigma(y, z), h\tau) \geq \varphi(y, y, z, z, \tau),$$

for all $y, z \in \Delta$ and $\tau \in \mathbb{R}+$. So

$$\mu(\rho(y, z) - \pi(y, z), (k+h)\tau) \geq \min\{\mu(\rho(y, z) - \sigma(y, z), k\tau),$$

$$\mu(\pi(y, z) - \sigma(y, z), h\tau)\} \geq \varphi(y, y, z, z, \tau),$$

for all $y, z \in \Delta$ and $\tau \in \mathbb{R}^+$. By (3) $d(\rho, \pi) < k + h$ which is the triangle inequality for the distance d. Following the idea of the Lemma 2.1 of [14], we can complete the proof without any difficulty.

Theorem 4: *Under assumptions that Δ is a linear space, that (Λ, μ) is a fuzzy Banach space (FBS) and $\zeta : \Delta \times \Delta \to \Lambda$ for all $y_1, y_2, y_3, y_4 \in \Delta$ and $\tau \in \mathbb{R}+$ such that*

$$\mu(\zeta(y_1 + y_2, y_3 + y_4) + \zeta(y_1 - y_2, y_3 - y_4) - 2\zeta(y_1, y_3) - 2\zeta(y_2, y_4), \tau) \geq \varphi(y_1, y_2, y_3, y_4, \tau). \quad (4)$$

Suppose that a function $\varphi : \Delta \times \Delta \times \Delta \times \Delta \times [0, \infty) \to M_n([0, 1])$ is a matrix valued fuzzy set (MVF set) that satisfies

$$\varphi(2y_1, 2y_2, 2y_3, 2y_4, \alpha\tau) \geq \varphi(y_1, y_2, y_3, y_4, \tau),$$

for all $y_1, y_2, y_3, y_4 \in \Delta$, $\tau \in \mathbb{R}^+$ and the real number $0 < \alpha < 4$. Assume that

$$\lim n \to \infty\ \varphi(2^n y, 2^n y, 2^n z, 2^n w, \tau) = 0. \quad (5)$$

Then there exists a symmetric bi-additive mapping $\psi : \Delta \times \Delta \to \Lambda$ such that

$$\mu(\psi(y_1, y_2) - \sigma(y_1, y_2), \alpha\tau) \geq \varphi(y_1, y_1, y_2, y_2, \tau), \quad (6)$$

for all $y_1, y_2 \in \Delta$ and all $\tau \in \mathbb{R}+$.

***Proof**:* Consider the set $\odot = \{\rho \mid \rho : \Delta \times \Delta \to Y; \rho(0, 0) = 0\}$ and the generalized metric d on $\odot$ given by $d(\rho, \sigma) = \inf\{\alpha \in \mathbb{R}+ : \mu(\rho(y_1, y_2) - \sigma(y_1, y_2), \alpha\tau) \geq \varphi(y_1, y_1, y_2, y_2, \tau)$ for all $y_1, y_2 \in \Delta$ and $\tau \in \mathbb{R}+\}$, for all $\rho, \sigma \in \odot$. See Lemma 1.

Suppose, for all $\rho, \sigma \in \odot$ and for all $\varepsilon > 0$ that $\varepsilon > d(\rho, \sigma)$, we have $\mu(\rho(y_1, y_2) - \sigma(y_1, y_2), \alpha\tau) \geq \varphi(y_1, y_1, y_2, y_2, \tau)$ for all $y_1, y_2 \in \Delta$ and all $\tau \in \mathbb{R}+$.

Now put $y = y_1 = y_2$ and $z = y_3 = y_4$ in (2). In this way we have for all $y, z \in \Delta$ and $t > 0$,

$$\mu(\zeta(2y, 2z) + \zeta(0, 0) - 2\zeta(y, z) - 2\zeta(y, z), \tau) \geq \varphi(y, y, z, z, \tau),$$

$$\mu(\zeta(2y, 2z) - 4\zeta(y, z), \tau) \geq \varphi(y, y, z, z, \tau), \quad (7)$$

$$\mu\left(\frac{\zeta(2x,2z)}{4}-\zeta(x,z),4\tau\right)\geq\varphi(x,x,z,z,\tau) \tag{8}$$

Now we define a mapping $A:\odot\to\odot$ by

$$A\zeta(x,z):=\frac{\zeta(2x,2z)}{4} \tag{9}$$

for all $\zeta\in\odot$ and all y, $z\in\Delta$. and show that A is a strictly contractive mapping of $\odot$. So, observe that, for all ρ, $\sigma\in\odot$, $\varepsilon>0$, there exists a non-negative number $c\in D_\mu$ such that $d(\rho,\sigma)+\varepsilon>c$ or $d(\rho,\sigma)+\varepsilon\in D_\mu$ thus $\mu(\rho(y,z)-\sigma(y,z),\varepsilon\tau)\geq\varphi(y,y,z,z,\tau)$ for all y, $z\in\Delta$ and $t>0$. Replacing ε, ρ, σ by $\varepsilon\alpha$, $A\rho$, $A\sigma$, respectively, we have $\mu(A\rho(y,z)-A\sigma(y,z),\alpha\varepsilon\tau)\geq\varphi(y,y,z,z,\tau)$.

So

$$\mu\left(\frac{\rho(2x,2z)}{4}-\frac{\sigma(2x,2z)}{4},\alpha\varepsilon\tau\right)=\mu(\rho(2x,2z)-\sigma(2x,2z),4\alpha\varepsilon\tau)$$
$$\geq\varphi(2x,2x,2z,2z,4\alpha\tau).$$

It follows from Eq. (9) that

$$\mu(A\rho(y,z)-A\sigma(y,z),\alpha\varepsilon\tau)\geq\varphi(y,y,z,z,4\tau)$$

and by Definition 1, we have

$$\mu(A\rho(y,z)-A\sigma(y,z),\tfrac{\alpha}{4}\varepsilon\tau)\geq\varphi(y,y,z,z,\tau).$$

Hence, by definition of $d(\rho,\sigma)$, we observe that $d(A\rho,A\sigma)<\frac{\alpha}{4}\varepsilon<\frac{\alpha}{4}d(\rho,\sigma)$ (by $d(\rho,\sigma)+\varepsilon>c$ and $\varepsilon\to0^+$) for all real numbers $0<\alpha<4$. This means that A is a strictly contractive mapping with the Lipschitz constant $L=\frac{\alpha}{4}$.

Next, from Eq. (8) we will have

$$\varphi(y,y,z,z,\tau)\leq\mu\left(\frac{\rho(2y,2z)}{4}-\rho(y,z),4\tau\right)=\mu\left(A\rho(y,z)-\rho(y,z),\frac{1}{4}\tau\right)$$

thus by definition of $d(\rho,\sigma)$, this implies that

$$d(A\rho,\rho)<\frac{1}{4}<\infty.$$

From $d(A\rho,\rho)<\frac{1}{4}$ and $d(A\rho,A\sigma)<\frac{\alpha}{4}$, we conclude that $d(AA\rho,A\rho)$ $<\frac{\alpha}{4}\times\frac{1}{4}$ and $d(A^3\rho,A^2\rho)<\left(\frac{\alpha}{4}\right)^2\frac{1}{4},\ldots$ in continue, we have $d(A^{n+1}\rho,A^n\rho)$ $<\left(\frac{\alpha}{4}\right)^n\frac{1}{4}$.

From this fact, we know that for each $\rho\in\odot$ and $n\in\mathbb{N}$, $\{A^n\rho\}$ is a Cauchy sequence in Λ. Taking into account that Λ is complete, there exists $\psi\in\odot$ such that $A^n\rho\to\psi$ as $n\to\infty$ and $\psi(0,0)=0$. Thus there exists $\eta^\circ\in\mathbb{N}$, so that ψ is the unique fixed point of A in the set $\Phi=\{\rho\in\odot, d(A\eta^\circ\rho,\psi)<+\infty\}$.

Now, taking into account the alternative of the fixed point theorem, see Theorems 1 and 2, we can ensure the existence of a fixed point of A. In other words, we have the existence of a mapping $\psi:\Delta\times\Delta\to\Lambda$ such that $\psi(2y,2z)=4\psi(y,z)$, for all $y,z\in\Delta$. Moreover, we have $d(A^n\rho,\psi)\to0$, which implies

$$d(\rho,\psi)\leq\left(\frac{1}{1-L}\right)d(\rho,A\psi)=\frac{1}{4-\alpha}$$

since $L=\frac{\alpha}{4}$ and $d(A\rho,\rho)<\frac{1}{4}$ for $0<\alpha<4$. Then there exists a symmetric bi-additive mapping $\psi:\Delta\times\Delta\to\Lambda$ such that

$$\mu(\psi(y_1,y_2)-\rho(y_1,y_2),\tau)\geq\varphi(y_1,y_1,y_2,y_2,(4-\alpha)\tau),$$

for all $y_1,y_2\in\Delta$ and all $\tau\in\mathbb{R}+$.

Replacing y_1 and y_2 by 2^ny_1 and 2^ny_2 in (7), respectively, we get

$$\mu(\zeta(2^ny_1,2^ny_2)-4\zeta(2^ny_1,2^ny_2),\tau)\geq\varphi(2^ny_1,2^ny_1,2^ny_2,2^ny_2,\tau),$$

for all $y_1,y_2\in\Delta$ and $\tau\in\mathbb{R}+$. Since $\lim n\to\infty$ $\varphi(2^ny_1,2^ny_1,2^ny_2,2^ny_2,2^n\tau)=1$, we observe that A satisfies (6).

We assume that the mapping $\psi:\Delta\times\Delta\to\Lambda$, which satisfies (6), exists and is additive. We will prove the uniqueness of such mapping ψ. Clearly, $\psi(2^ny_1,2^ny_2)=2^n2^n\psi(y_1,y_2)$ and $\sigma(2^ny_1,2^ny_2)=2^n2^n\sigma(y_1,y_2)$, for fix $y_1,y_2\in\Delta$ and for all $n\in\mathbb{N}$. It follows from (5) that

$$\begin{aligned}&\mu(\psi(x_1,x_2)-\varsigma(x_1,x_2),\alpha\tau)\\&=\lim_{n\to\infty}\mu\left(\frac{\psi(2^nx_1,2^nx_2)}{2^n2^n}-\frac{\varsigma(2^nx_1,2^nx_2)}{2^n2^n},\tau\right)\end{aligned}$$

$$\geq \lim_{n\to\infty} \varphi\left(2^n x_1, 2^n x_1, 2^n x_2, 2^n x_2, (4-\alpha)\tau\right)$$
$$= \lim_{n\to\infty} \varphi\left(x_1, x_1, x_2, x_2, \frac{2^n 2^n (4-\alpha)\tau}{\alpha^n}\right).$$

Since $0<\alpha<4$,

$$\lim_{n\to\infty} \varphi\left(x_1, x_1, x_2, x_2, \frac{2^n 2^n (4-\alpha)\tau}{\alpha^n}\right) = \lim_{\tau\to\infty} \varphi(x_1, x_1, x_2, x_2, \tau) = 1.$$

This means that $d(\psi(y_1, y_2), \rho(y_1, y_2)) = 0$ and $\psi = \rho$ is a symmetric bi-additive mapping from $\Delta \times \Delta$ into Λ.

3. Conclusions

In this note, we have studied the stability in the Hyers-Ulam-Rassias sense, making use of Banach's Fixed Point Theorem, in the fuzzy norm space, of (1). We have seen that this method is easy to use and it provides the desired results.

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