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Fuzzy scl–semi 2–ABSO submd and related concepts

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Abstract

The *F-semi*-2-*ABSO* and the *F-scl* are compared in this study to see how they interact. In order to respond to this inquiry, we present a new notion called as fuzzy socle *semi*-2-*ABSO* submd. The findings revealed a number of characteristics that lent credence to the new concept. The connection between *F*-semi-2-*ABSO* and *F-scl*-2-*ABSO* is also discovered using straightforward algebraic techniques. In addition, we investigated the *F-scl-semi*-2-*ABSO* submd structure for *F*-direct sum and homomorphic picture. It also covered how *F-scl-semi*-2-*ABSO* submd relates to other *F*-submd types. The findings from this research are important for the definition of new *F-scl semi*-2-*ABSO* concepts.

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Introduction

A F-subset A of X was described in 1965 by Zadeh [10] as a function from X to the unit interval [0,1]. Ago then, the notion of F-sets has been explored in other domin. This concept has applications in artificial intelligence, computer science, decision theory, logic, control engineering, medicine, expert systems, and other domains. A historical branch of pure mathematics, algebra, inserted the consept of F-sets. Rosenfeld [1] introduced F-subgroups and the F-subgroupoid in 1971. The concepts of F-modls and F-submd were first

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introduced by Ralescu and Negoita in 1975 [2]. F-quotient modls and F-finitely produced submd [3]. Therefore, Saikia an[8] constructed F-essential submd and investigated their properties. This was also examined in this article, all rings with identity are commutative. Modl theory over a commutative ring takes a significant turn thanks to prime submd [4, 15]. The idea of a $prime\ fuzzy\ submodule$ was inserted by Rabi [7] in (2001): Let X be a F-modl of a W-modl G, a F-submd U of X is pointed to as prime fuzzy if $r_b m_t \subseteq U$, together with r_b is F-singl. of W and $m_t \subseteq$ X, leading to one of two $m_t \subseteq U$ or $r_b \subseteq [U: X]$ for every $t, b \in [0, 1]$. Hadi [9] in 2004 proposed the idea of semiprime F-submd: Let $P \subset X$ a F-submd of a F-modl *X* of a W-modl *G. P* is pointed to as a semiprime $\mathrm{F-submd}$ if for each F-singl. r_k of W and $x_v \subseteq X$ such that $r_k^2 x_v \subseteq P$ implies $r_k x_v \subseteq P$. Saad [14], first proposed the concepts of F-scl-prime $submd$ in 2022. A F-Scl-prime (also referred to as a $\mathrm{F-Scl-}$ prime) submd of X if when $r_b m_t \subseteq U$,with r_b is F-singl. of W and $m_t \subseteq$ X, leading to one of two $m_t \subseteq U + F - Scl(X)$ or $r_b \subseteq [U + F - Scl(X): X]$ for every $t, b \in [0, 1]$. Saad [14], also discussed the F-scl-semiprime submd: Let q_s F-singl. of W and m_t is F-singl. of X, then $U \subset X$ of a W-modl G is pointed to as a F-Scl-semi-prime (for short F-Scl-semi prime) submd of X if whenever $q_s^n m_t \subseteq U$ with n∈Z$^+$, implying that $q_s m_t \subseteq U + F - Scl(X)$ for each $t, s \in$ [0; 1]. In (2019), Wafaa[5], introduced the idea of $2 - \mathrm{ABSO}$ F-submd: Let X be a F-modl of a W-modl G, a F-submd U of X is pointed to as $2 - \mathrm{ABSO}$ if $q_s r_b m_t \subseteq U$,with q_s, r_b are F-singl. of W and m_t is F-singl. of X, leading to one of two $q_s m_t \subseteq U$ or $r_b m_t \subseteq U$ or $q_s r_b \subseteq [U: X]$for each $t, b, s \in$ [0; 1]. Wafaa [16] proposed the idea of a semi-2-ABSO F-submd in (2019) as well: Let X be a F-modl of a W-modl G, a F-submd U of X is pointed to as a $\mathrm{semi} - 2 -$ ABSO if $q_s^2 m_t \subseteq U$,with q_s F-singl. of W and m_t is F-singl. of X, implying that either $q_s m_t \subseteq U$ or $q_s^2 \subseteq [U :_R X]$for each $t, s \in [0, 1]$. In 2022, Saad [14] introduced the F-Scl-2-ABSO submd concept: Let $U \subset X$ of a W-modl G is pointed to as a F-scl-2-Absorbing (for abbreviation $\mathrm{F - Scl - 2 - ABSO}$) submd of X if when $r_b s_q m_t \subseteq U$,with r_b, s_q is F-singl. of W and $m_t \subseteq$ X, leading that to one of three $r_b m_t \subseteq U + F - Scl(X)$ or $s_q m_t \subseteq U + F - Scl(X)$ or $r_b \subseteq [U + F - Scl(X): X]$ every $t, b, q \in [0, 1]$. A F-modl X is F-simple if and only If X has only itself and 0_1[13, 16]. If modl X is a summation of simple F-submd, So X is pointed to as a semi-simple. Moreover,a F-Scl-of X is pointed to as sum of simple F-submd of X and denoted by $F - Scl(X)$, that is X is r semi-simple if $X = F - Scl(X)$,[12]**.** Every non-empty F-submd A of a F-modl X of a W-modl G such that $A = (A :_R X)X$ if and only if X is multip. [12, 6]. Let *P* be a F-submd of a F-modl *X* of a W-modl *G*, then *A* is referred to as an irreducible F-submd if for all two F-submd *B* and *K* such that *B*∩*K*=*A* then *B*=*A* or *K*=*A,* another way *A* is defined as reducible,[12].

In this article, the notion of $semi\ 2 - ABSO\ F - submd$ is generalized to $F - Scl - semi - 2 - Absorbing\ submd$. Two sections make up this article. We offer some fundamental definitions and attributes that we will request in the following section in section one. The $F - Scl - semi - 2 - ABSO\ submd's$ many fundamental properties, outputs, and findings are tested in section 2. Note:

o.w., I, F-set, F-submd, F-ideal, F-modl, F-2-ABSO, F-singl., and $F - Scl - 2 - ABSO\ submd$ are abbreviations for $otherwise$, [0,1], F-sets, ideals, submd, modls, and singl..

2. $\mathbf{F - Scl - semi - 2 - ABSO}$ Submds

As a generalization of the F-Scl-2-ABSO submd, we provide the idea of a $\mathrm{F - Scl - semi - 2 - ABSO}$ submd in this section. Additionally, several F-Scl-semi-2-ABSO submd characterizations are offered.

Definition 2.1: Let a $F -\ modl\ X$ of a W-modl G and $P \subset X$, then P is known as $F - Scl - semi - 2 - ABSO\ submd$ of X if $a_n^2 z_m \subseteq X, \forall n, m \in I$, implies either $a_n x_m \subseteq P + F - Soc(X)\ or\ a_n^2 \subseteq (P + F - Soc(X){:}\, X)$.

Furthermore, A proper F-ideal L of W is known as F-Scl-semi-2-ABSO ideal of W if $a_n^2 r_b \subseteq L, \forall n, b \in I$, implies either $a_n r_b \subseteq L + F - Scl(W)$ or $a_n^2 \subseteq L + F - Scl(W)$.

Definition 2.2: A proper F-submd K of a $F - modl\ X$ of a $W - modl\ G$ is called F-Scl-quasi prime submd of X if whenever $r_b s_q m_t \subseteq K$,with r_b, s_q are $F - singl. of\ W$ and $m_t \subseteq$ X, implying that either $r_b m_t \subseteq K + F - Scl(X)$ or $s_q m_t \subseteq K + F - Scl(X)$

Remarks and Examples 2.3

1. For each $F - Scl - prime\ submd$ is F-Scl-semi-2-ABSO submd.

However, the opposite of (1) is not generally true, as in:

Let $X{:}\, Z \to I$ like that $X(y) = \begin{cases} 1 \text{ if } y \in Z \\ 0\ o.w \end{cases}$

X is obviously F-modl of Z as Z-modl

Let $K{:}\, Z \to I$ such that $K(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{3}{4} \text{ if } y \in 16Z - (0) \\ 0\ o.w \end{cases}$

It is evident K is F-submd of X

$$F - Scl(X)(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{1}{4} \text{ if } y \neq 0 \end{cases} K + F - SclX(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{3}{4} \text{ if } y \in 16Z - (0) \\ \frac{1}{4} o.w \end{cases}$$

$$(K + F - SclX(y){:}\, X)(y) = \begin{cases} 1\ if\ y \in 16Z \\ 0\ if\ y \notin 16Z \end{cases}$$

Where $(16Z + Scl(Z){:}\, Z) = (16Z{:}\, Z) = 16Z, K$ is F-Scl-semi-2-ABSO since $4^2_{\frac{1}{2}} \cdot 2_{\frac{1}{3}} = 32_{\frac{1}{3}} \subseteq K$, since $(32) = \frac{3}{4} > \frac{1}{3}$, then $4^2_{\frac{1}{2}} = 16_{\frac{1}{2}} \subseteq (K + F - Scl(X){:}\, X)$, K, however, is not an F-Scl-prime submd since

$8_{\frac{1}{2}} \cdot 2_{\frac{1}{3}} = 16_{\frac{1}{3}} \subseteq K$ since $K(16) = \frac{3}{4} > \frac{1}{3}, 8_{\frac{1}{2}} \nsubseteq K + F - Scl(X)$, and $2_{\frac{1}{3}} \nsubseteq (K + F - Scl(X){:}\, X)$.

2. For each $F - Scl - semi\ prime\ submd$ is $F - Scl - semi - 2 - ABSO\ submd$.
But the reverse, not true in general, as in:
Let $X: Z \to I$ like that $X(y) = \begin{cases} 1 \text{ if } y \in Z \\ 0\ o.w \end{cases}$
X is obviously $F - modl\ of\ Z\ as\ Z - modl$
Let $K: Z \to I$ like that $K(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{3}{4} \text{ if } y \in 4Z - (0) \\ 0\ o.w \end{cases}$
K is clearly the $F - submd\ of\ X$

$$F - Scl(X)(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{1}{4} \text{ if } y \neq 0 \end{cases} K + F - Scl(X)(y) = \begin{cases} 1 \text{if } y = 0 \\ \frac{3}{4} \text{ if } y \in 4Z - (0) \\ \frac{1}{4} o.w \end{cases}$$

$$(K + F - Scl(X){:}\, X)(y) = \begin{cases} 1 \text{ if } y \in 4Z \\ 0 \text{ if } y \notin 4Z \end{cases}$$

where $(4Z + Scl(Z){:}\, Z) = (4Z{:}\, Z) = 4Z, K$ is F-Scl-semi-2-ABSO since $2^2_{\frac{1}{2}} \cdot 1_{\frac{1}{3}} = 4_{\frac{1}{3}} \subseteq K,\ then\ 2^2_{\frac{1}{2}} = 4_{\frac{1}{2}} \subseteq (K + F - Scl(X){:}\, X)$, but K is not F-Scl-semi prime submd, since $2_{\frac{1}{2}} \cdot 1_{\frac{1}{3}} \nsubseteq K + F - Scl(X)\ since\ \big(K + F - Scl(X)\big)(2) = \frac{1}{4} \ngtr \frac{1}{3}$

3. For each $F - Scl - 2 - ABSO\ submd$ is $F - Scl - semi - 2 - ABSO\ submd$.
However, the opposite is not generally true, as in:
Let $X: Z \to I$ like that $X(y) = \begin{cases} 1 \text{ if } y \in Z \\ 0\ o.w \end{cases}$
It is plainly that X is F-modl of Z as Z-modl
Let $K: Z \to I$ like that $K(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{3}{4} \text{ if } y \in 36Z - (0) \\ 0\ o.w \end{cases}$
K is clearly the $F - submd\ of\ X$

$$F - SclX(y) = \begin{cases} 1 \text{ if } y = 0 \\ \frac{1}{4} \text{ if } y \neq 0 \end{cases}$$

$$K+F-SclX(y)=\begin{cases}1 \text{ if } y=0\\ \frac{3}{4} \text{ if } y\in 36Z-(0)\\ \frac{1}{4} o.w\end{cases}$$

$$(K+F-SclX:X)(y)=\begin{cases}1 if\ y\in 36Z\\ 0\ if\ y\notin 36Z\end{cases}$$

Where $(36Z+Scl(Z):Z)=(36Z:Z)=36Z$

K is F-Scl-semi-2-ABSO since $6^2_{\frac{1}{2}}\cdot 2_{\frac{1}{3}}=72_{\frac{1}{3}}\subseteq K$, then $6^2_{\frac{1}{2}}=36_{\frac{1}{2}}\subseteq (K+F-Soc(X):X)$, but K is not F-Scl-2-ABSO, pick $6_{\frac{1}{2}}, 2_{\frac{1}{3}}$ are $F-singl.of\ Z$ and $3_{\frac{1}{3}}\subseteq X$, where $6_{\frac{1}{2}}\cdot 2_{\frac{1}{3}}\cdot 3_{\frac{1}{3}}=36_{\frac{1}{3}}\subseteq K$ ago $K(36)=\frac{3}{4}>\frac{1}{3}$, but $6_{\frac{1}{2}}\cdot 3_{\frac{1}{3}}=18_{\frac{1}{3}}\nsubseteq K+F-Scl(X)$, $2_{\frac{1}{3}}\cdot 3_{\frac{1}{3}}=6_{\frac{1}{3}}\nsubseteq K+F-Scl(X)$, and $6_{\frac{1}{2}}\cdot 2_{\frac{1}{3}}=12_{\frac{1}{3}}\nsubseteq (K+F-SclX:X)$

4. For each $F-Scl-quasi\ prime\ submd$ is $F-Scl-semi-2-ABSO\ submd$.
the reverse not true see example in part (3)

5. For each $\mathrm{F}-\mathrm{semi}-2-\mathrm{ABSO}$ submd is F-Scl-semi 2-ABSO submd.
The example below demonstrates how erroneous the converse is

Let $X:Z\rightarrow I$ like that $X(y)=\begin{cases}1 \text{ if } y\in Z_{36}\\ 0\ o.w\end{cases}$

X is obviously the $F-modl$ of Z_{36} as $Z-modl$

Let $K:Z\rightarrow I$ such that $K(y)=\begin{cases}\frac{1}{2} \text{ if } y\in (\overline{12})\\ 0\ o.w\end{cases}$

K is clearly the $F-submd\ of\ X$

$F-Scl(X)(y)=\begin{cases}1 \text{ if } y\in(\bar{4})\\ \frac{1}{4} \text{ if } y\notin(\bar{4})\end{cases}$ $K+F-Scl(X)(y)=\begin{cases}1 \text{ if } y\in(\bar{4})\\ \frac{1}{4} \text{ if } y\notin(\bar{4})\end{cases}$

$$(K+F-Scl(X):X)(y)=\begin{cases}1 \text{ if } y\in 4Z\\ 0 \text{ if } y\notin 4Z\end{cases}$$

$$(K:X)(y)=\begin{cases}1 \text{ if } y\in 12Z\\ 0 \text{ if } y\notin 12Z\end{cases}$$

Where $\left((\overline{12})+Scl(Z_{36}):Z_{36}\right)=(12Z:Z)=12Z$,K is $F-Scl-semi-2-ABSO\ submd$ since $2^2_{\frac{1}{2}}\cdot 3_{\frac{1}{3}}=12_{\frac{1}{3}}\subseteq K$ since $K(12)=\frac{1}{2}>\frac{1}{3}$, then $2^2_{\frac{1}{2}}=4_{\frac{1}{2}}\subseteq (K+F-Scl(X):X)$, sicne $(K+F-Scl(X):X)(4)=1>\frac{1}{2}$, however K isn't $\mathrm{F}-\mathrm{semi}-2-\mathrm{ABSO}$ submd $since\ 2^2_{\frac{1}{2}}\cdot 3_{\frac{1}{3}}=12_{\frac{1}{3}}\subseteq K$, but $2^2_{\frac{1}{2}}\nsubseteq (K:X)$ since $(K:X)(2^2)=0\ngtr\frac{1}{2}\ and\ 2_{\frac{1}{2}}\cdot 3_{\frac{1}{3}}=6_{\frac{1}{3}}\nsubseteq K\ since\ K(6)=0\ngtr\frac{1}{3}$

Proposition 2.4: Any F-submd of a semi-simple F-modl X is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}\ 2-\mathrm{ABSO}$ submd.

Proof: If P is a F-submd of X, let $h_a^2 x_v \subseteq P$ for $F.singl.\ h_a$ of W and $x_v \subseteq X, where\ a, v \in I$, however X is semi-simple F-modl, for that F-Scl(X)=X, then $x_v \subseteq X = F - Scl(X) \subseteq P + F - Scl(X)$, hence $h_a x_v \subseteq h_a X = h_a F - Scl(X) \subseteq h_a\big(P + F - Scl(X)\big) \subseteq P + F - Scl(X)$. So that P is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ submd of X.

Proposition 2.5: Let P, H be a $F-\ submds$ of a $-\ modl\ X\ of\ a\ W-modl\ G$. Then P is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ submd $if\ and\ only\ if$

$r_s^2 H \subseteq P$ for F singl. r_s of W, implies $r_s H \subseteq P + F - Scl(X)\ or\ r_s^2 \subseteq (P + F - Scl(X): X)$

Proof: $\Rightarrow)\ let\ r_t^2 H \subseteq P$ for a F-singl. r_t of W.

Suppose there exists $x_v \subseteq H$ such that $r_t x_v \nsubseteq P + F - Scl(X)$, since $r_t^2 H \subseteq P$, hence $r_t^2 x_v \subseteq P$, but P is a F-Scl-semi-2-ABSO submd and $r_t x_v \nsubseteq P + F - Scl(X)$. Then $r_t^2 \subseteq [P + F - Scl(X): X]$

$\Leftarrow)$ It is clear.

Proposition 2.5: Let X be a $F-modl$ of a $W-modl\ G$ and $P \subset X$. if P is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ submd of X, then $(P: X)$ is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ ideal**.**

Proof: Let $h_a k_b$ be a F-singl. of W, like that $h_a^2 k_b \subseteq (P: X)$, hence

$h_a^2 k_b X \subseteq P$, then $h_a^2 k_b x_v \subseteq P$ for $x_v \subseteq X$ and assume that $h_a^2 \nsubseteq (P: X)$. Since P is a F-Scl-semi-2-ABSO submd, hence $h_a k_b x_v \subseteq P + F - Scl(X)$. So that $h_a k_b \subseteq (P: X)$. Then $(P: X)$ is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ ideal**.**

We now provide the lemma that will be required in the following proposition.

Lemma 2.6: *Let L and H be a* $F-\ ideals$ *of W. Then L is* $F-Scl-semi-2-ABSO\ ideal\ of\ W\ if\ and\ only\ if\ r_s^2 H \subseteq L\ for\ F\ singl.\ r_s\ of\ W,\ implies\ r_s H \subseteq L + F - Scl(W)\ or\ r_s^2 \subseteq (P + F - Scl(W)$

Proof: $\Rightarrow)\ let\ r_s^2 H \subseteq L$ for a F-singl. r_s of W.Suppose there exists $b_v \subseteq\ H$ such that $r_s b_v \nsubseteq L + F - Scl(W)$, since $r_s^2 H \subseteq L$, hence $r_s^2 b_v \subseteq L$, but L is a $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ ideal, and $r_s b_v \nsubseteq L + F - Scl(W)$. Then $r_s^2 \subseteq L + F - Scl(W)$.Thus, L is $\mathrm{F}-\mathrm{Scl}-\mathrm{semi}-2-\mathrm{ABSO}$ ideal of W

$\Leftarrow)$ It's evident.

The opposite of Proposition 2.5 is true for the F-modl class of multip.:

Proposition 2.7: Let X be a $F-multip.modl$ of a $W-modl\ G$ and $P\subset X$. If $(P{:}X)$ is a $\mathrm{F-semi-2-ABSO}$ ideal, then P is a $\mathrm{F-Scl-semi-2-ABSO}$ submd.

Proof: Let $h_a^2x_v\subseteq P$ for F. singl. h_a of W and $x_v\subseteq X$.

Then $h_a^2<x_v>\subseteq P$. But $<x_v>=LX$ for a few of $F-$ ideal of W since X is a $F-multip.modl$. Then $h_a^2L\subseteq(P{:}X)$. But $(P{:}X)$ is a F-Scl-semi-2-ABSO ideal, then one of two $h_aL\subseteq(P{:}X)+F-Scl(W) or\ h_a^2\subseteq(P{:}X)+F-Scl(W)$by Lemma 2.6. Then $h_aLX\subseteq P+F-Scl(X)\ or\ h_a^2X\subseteq P+F-Scl(X)$Then $h_a<x_v>\subseteq P+F-Scl(X)\ or\ h_a^2\subseteq(P+F-Scl(X){:}X)$

So that, P is a F-Scl-semi-2-ABSO submd.

Theorem 2.8: *Assume that K is an irreducible F-submd of F-modl X of a W-modl G. Consequently, these formulations are equivalent:*

1) K is F-Scl–semi-2-ABSO submd and $(\mathrm{K+F-Scl(X){:}X})$ is F-semiprime ideal;
2) *K is a F-Scl-prime submd;*
3) *If* $F-Scl(X)\subseteq K$, *then K is a* $F-Scl-semiprime\ submd;$
4) *If* $F-Scl(X)\subseteq K$, *then K is a* $F-Scl-quasi\ prime\ submd.$

Proof: (1) $\Longrightarrow$ (2) Let $r_n(r_nx_v)\subseteq K$ for a F-singl. r_n of W and $x_v\subseteq X$. Since K is $F-Scl-semi-2-ABSO\ submd$, then $r_nx_v\subseteq K+F-Scl(X)$ or $r_n^2\subseteq(K+F-Scl(X){:}X)$. If $r_nx_v\subseteq K+F-Scl(X)$, since X a F-semi-simple modl, hence $X=F-Scl(X)$ then $x_v\subseteq X=F-Scl(X)\subseteq K+F-Scl(X)$.Now, if $r_n^2\subseteq(K+F-Scl(X){:}X)$, then $r_n\subseteq(K+F-Scl(X){:}X)$ since $(K+F-Scl(X){:}X)$ is a F-semi prime ideal. Thus K is a F-Scl-prime submd

(1) $\Longrightarrow$ (3) Let $r_n^2x_v\subseteq K$ for F-singl. r_n of W and $x_v\subseteq X$. Ago K is a F-Scl-semi-2-ABSO submd, then $r_nx_v\subseteq K+F-Scl(X)$ or $r_n^2\subseteq(K+F-Scl(X){:}X)$. If $r_nx_v\subseteq K$ the proof is complete. If $r_n^2\subseteq(K+F-Scl(X){:}X)$, then $r_n\subseteq(K+F-Scl(X){:}X)$ since $(K+F-Scl(X){:}X)$ is a F-semiprime ideal. Hence $r_nx_v\subseteq K+F-Scl(X)$. Thus, K is F-Scl-semiprime submd.

(2) ⇒ (3) By [14], we have the outcome

(3) ⇒ (2) let K be F-irreducible submd and K be Scl-semiprime submd such that $F-Scl(X)\subseteq K$, to prove K is Scl-prime submd. Suppose that K is not F-Scl-prime submd, this mean there exists F-singl. r_n of W and $x_v\subseteq X$ such that $r_nx_v\subseteq K$ and $x_v\nsubseteq K+F-Scl(X)$ and $r_n\nsubseteq(K+F-Scl(X){:}X)$, since $r_n\nsubseteq(K+F-Scl(X){:}X)$, then there exist $m_h\subseteq X$ such that $r_nm_h\nsubseteq K+F-Scl(X)$, then $r_nm_h\nsubseteq K$, we claim that $L_1\cap L_2=K$ where $L_1=K+(r_n$ $L_2=K+<x_v>$. Now, let $y_s\subseteq L_1\cap L_2$, this mean $y_s\subseteq K+<r_nm_h>$ and $y_s\subseteq K+<x_v>$, therefore, there exists $f_q,u_z\subseteq K$ and F-singl. a_s,b_l of W such that $y_s=f_q+a_st_nm_h=u_z+b_lx_v$, then $u_z-f_q+b_lx_v=a_sr_nm_h$, hence

$r_n u_z - r_n f_q + r_n b_l x_v = a_s r_n^2 m_h$, so that $a_s r_n^2 m_h \subseteq K$ since $u_z, f_q \subseteq K$ then $r_n u_z - r_n f_q \subseteq K$ and $r_n x_v \subseteq K$, hence $r_n b_l x_v \subseteq K$, but K is F-Scl-semiprime submd, then $a_s r_n m_h \subseteq K + F - Scl(X)$ implies that $a_s r_n m_h \subseteq K$ since $F - Scl(X) \subseteq K$, hence $y_S \subseteq K$, thus $L_1 \cap L_2 \subseteq K$ it is observe that $K \subseteq L_1 \cap L_2$, this mean $K = L_1 \cap L_2$, this contradiction with K is F-irreducible submd. Thus, K is F-Scl-prime submd.

(3) ⇒ (4) let $m_a n_b x_t \subseteq K$ where $m_a n_b$ are $F - singl. of\ W$ and $x_t \subseteq X$ assume that $m_a x_t \nsubseteq \mathrm{K} + \mathrm{F} - \mathrm{Scl}(X)$ and $n_b x_t \nsubseteq K + F - Scl(X)$ let $T = K + F - Scl(X) + (m_a x_t)$ and $U = K + F - Scl(X) + (n_b x_t)$ then T and U are $F - submd\ of\ X$. It is clear that $K + F - Scl(X) \subseteq T$ and $K + F - Scl(X) \subseteq U$ and hence $K + F - Scl(X) \subseteq T \cap U$, let $y_r \subseteq T \cap U$, then $y_r \subseteq T$ and $y_r \subseteq U$, then there exists $i_f + j_h \subseteq K + F - Scl(X) and\ s_q, v_d$ are F-singl. of W such that $i_f + v_d m_a x_t = i_h + S_q n_b x_t$ then:

$n_b i_f + n_b v_d m_a x_t = n_b s_q n_b x_t + j_h n_b, n_b i_f + n_b v_d m_a x_t = n_b j_h + s_q n_b^2 x_t$, $n_b i_f - n_b j_h + v_d m_a n_b x_t = s_q n_b^2 x_t \subseteq K + F - Scl(X)$ sicne($n_b i_f - n_b j_h) \subseteq K + F - Scl(X)$and $v_d m_a n_b x_t \subseteq K + F - Scl(X)$, so that $n_b^2 s_q x_t \subseteq K + F - Scl(X)\ since\ F - Scl(X) \subseteq K$, hence $n_b^2 S_q x_t \subseteq K$, but K is F-Scl-semiprime implies $n_b s_q x_t \subseteq K + F - Scl(X)$. So, $y_r \subseteq K$, that is $T \cap U \subseteq K$, therefore, $K = T \cap U$ which is contradiction, since K is F-irreducible. Hence, either $m_a\, x_t \subseteq K + F - Scl(X)$ or $n_b x_t \subseteq K + F - Scl(X)$. For that, K is a $F -\ Scl - quasi\ prime\ submd$.

(4) ⇒ (1) let $r_n a_s x_v \subseteq K$ for $F - singl.\ r_n, a_s$ of W and $x_v \subseteq X$, then $r_n x_v \subseteq K + F - Scl(X)$ or $a_s x_v \subseteq K + F - Scl(X)$ since K is $F - Scl - quasi - prime\ submd$. So that K is $F - Scl - 2 - ABSO\ submd$. Now, to prove $(K + F - Scl(X): X)$ is F-semiprime ideal, let $r_n^2 \subseteq (K + F - Scl(X): X)$, for F-singl. $r_n\ of\ W$, then $r_n^2 X \subseteq K + F - Scl(X)$, but $F - Scl(X) \subseteq K$, then $r_n^2 X \subseteq K$, hence $r_n^2 x_v \subseteq K$ for all F-singl. $x_v \subseteq X$, then $r_n x_v \subseteq K + F - Scl(X)$ since K is Scl-quasi-prime, hence $r_n X \subseteq K + F - Scl(X)$, so that $r_n \subseteq (K + F - Scl(X): X)$. Thus, $(K + F - Scl(X): X)$ is F-semiprime ideal.

Proposition 2.9: Let X_1, X_2 be F-modls of W-modls G_1, G_2 resp. Let an epimorphism $f: G_1 \longrightarrow G_2$, and K is $\mathrm{F - Scl - semi\ - 2 - ABSO\ submd\ of} X_1$ like that $F - \ker f \subseteq K$. Then $f(K)$ is $\mathrm{F - Scl - semi\ - 2 - ABSO\ submd}$ of X_2.

Proof: Let $r_s^2 z_b \subseteq f(K)$ for $F - singl.\ r_s$ of W, and $z_b \subseteq X_2$, since f is onto, so $z_b = f(x_v)$ for $x_v \subseteq X$ then $r_s^2 f(x_v) = f(s_a)$ for a F-singl. $s_a \subseteq K$. then $r_s^2 x_v - s_a \subseteq F - \ker f \subseteq K$, thus $r_s^2 x_v \subseteq K$. But K is a $F - Scl - semi - 2 - ABSO\ submd$, hence $r_s x_v \subseteq K + F - Scl(X_1)$ or $r_s^2 \subseteq (K + F - Scl(X_1): X_1)$. If $r_s^2 x_v \subseteq K + F - Scl(X_1)$ then $r_s f(x_v) \subseteq f(K) + F - Sclf(X_1)$, hence $r_s z_b \subseteq f(K) + F - Scl(X_2)$ if $r_s^2 \subseteq (K + F - Scl(X_1): X_1)$, then $r_s^2 X_1 \subseteq K + F - Scl(X_1)$, hence $r_s^2 f(X_1) \subseteq f(K) + F - Sclf(X_1)$, $r_s^2 X_2 \subseteq f(K) + F - Scl(X_2)$, since f is onto, thus $r_s^2 \subseteq (f(K) + F - Scl(X_2): X_2)$. Thus, $f(K)$ is $F - Scl - semi - 2 - ABSO\ submd$ of X_2.

Remark 2.10: The following example demonstrates how the preceding proposition cannot be true without the condition *f*, which is an epimorphism:

Let $X_1 : Z \to I$ like that $X_1(g) = \begin{cases} 1 \text{ if } g \in Z \\ 0 \; o.w \end{cases}$

Let $X_2 : Z \to I$ like that $X_2(g) = \begin{cases} 1 \text{ if } g \in Z \\ 0 \; o.w \end{cases}$

It is evident that X_1, X_2 are $F - modl \; of \; Z - modl \; Z$.

Let $K : Z \to I$ such that $K(g) = \begin{cases} \frac{1}{2} \text{ if} g \in 4Z \\ 0 \text{ if} g \notin 4Z \end{cases}$

It is evident that K are $F - submd \; of \; F - modl \; X_1$

$$F - Scl(X_1)(g) = \begin{cases} 1 \text{ if } g \in (0) \\ 0 \text{ if } g \notin (0) \end{cases}$$

$$K + F - Scl(X_1)(g) = \begin{cases} 1 \text{ if } g \in 4Z \\ 0 \text{ if } g \notin 4Z \end{cases}$$

$$(K + F - Scl(X_1) : X_1)(g) = \begin{cases} 1 \text{ if } g \in 4Z \\ 0 \text{ if } g \notin 4Z \end{cases}$$

Let a F-homomorphism $f : X_1 \to X_2$ if $f : Z \to Z$ with $f(n) = 9n$ be homomorphis, but not epimorphism, where $f(4Z) = 36Z$

$$\text{Let } f(K) : Z \to I \text{ such that } K(g) = \begin{cases} \frac{1}{3} \text{ if } g \in 36Z \\ 0 \text{ if } g \notin 36Z \end{cases}$$

It is evident that $f(K)$ are F-submd of F-modl X_2

$$F - Scl(X_2)(g) = \begin{cases} 1 \text{ if } g \in (0) \\ 0 \text{ if } g \notin (0) \end{cases}$$

$$K + F - Scl(X_2)(g) = \begin{cases} 1 \text{ if } g \in 36Z \\ 0 \text{ if } g \notin 36Z \end{cases}$$

$$(K + F - Scl(X_2) : X_2)(g) = \begin{cases} 1 \text{ if } g \in 36Z \\ 0 \text{ if } g \notin 36Z \end{cases}$$

K is $\mathrm{F - Scl - semi - 2 - ABSO}$ submd since $2^2_{\frac{1}{2}} \cdot 1_{\frac{1}{3}} = 4_{\frac{1}{3}} \subseteq K$, $since \; K(4) = \frac{1}{2} > \frac{1}{3}$, $then \; 2^2_{\frac{1}{2}} = 4_{\frac{1}{2}} \subseteq (K + F - Scl(X_1) : X_1)$ since $(K + F - Scl(X_1) : X_1)(4) = 1 > \frac{1}{2}$, however $f(K)$ isn't $\mathrm{F - Scl - semi - 2 -}$ ABSO $submd$ ago $2^2_{\frac{1}{2}} \cdot 9_{\frac{1}{4}} = 36_{\frac{1}{4}} \subseteq f(K)$, $since \; f(K)(36) = \frac{1}{3} > \frac{1}{4}$, then $2_{\frac{1}{2}} \cdot 9_{\frac{1}{4}} = 18_{\frac{1}{4}} \nsubseteq f(K) + F - Scl(X_2)$since $f(K) + F - Scl(X_2)(18) = 0 \ngtr \frac{1}{4}$ and $2^2_{\frac{1}{2}} = 4_{\frac{1}{2}} \nsubseteq (K + F - Scl(X_2) : X_2)$since $(K + F - Scl(X_2) : X_2)(4) = 0 \ngtr \frac{1}{2}$,

Proposition 2.11: Let X_1, X_2 be $F - modls \; of \; W - modls \; G_1, G_2$ resp. Let an epimorphism $f : G_1 \longrightarrow G_2$ and K is a $\mathrm{F - semi - 2 - ABSO}$ submd of X_2, then $f^{-1}(K)$ is a $\mathrm{F - Scl - semi - 2 - ABSO}$ submd of X_1.

Proof: Let $r_s^2 v_x \subseteq f^{-1}(K)$ for a F-singl. r_s of W and $v_x \subseteq X_1$, hence $f(r_s^2 v_x) \subseteq K$ so $r_s^2 f(v_x) \subseteq K$. Since K is a $F - semi - 2 - ABSO\ submd$, then one of two $r_s f(v_x) \subseteq K$ or $r_s^2 \subseteq (K: X_2)$, so that $r_s v_x \subseteq f^{-1}(K)$ or $r_s^2 f^{-1}(X_2) \subseteq f^{-1}(K)$, then $r_s v_x \subseteq f^{-1}(K) + F - Scl(X_1)$ or $r_s^2 f^{-1}(X_2) \subseteq f^{-1}(K) + F - Scl(X_1)$ $r_s v_x \subseteq f^{-1}(K) + F - Scl(X_1)$ or $r_s^2 \subseteq (f^{-1}(K) + F - Scl(X_1): X_1)$. For that, $f^{-1}(K)$ is a F − Scl − semi − 2 − $ABSO\ submd$ of X_1.

Proposition 2.12: Let X_1, X_2 be a $F - modl\ of\ W - modls\ G_1, G_2$ resp., and K_1, K_2 are $F - Scl - semi - 2 - ABSO\ submd$ of X_1, X_2 resp. such that $(K_1 + F - Scl(X_1): X_1) = (K_2 + F - Scl(X_2): X_2)$, then $K = K_1 \oplus K_2$ is a $F - Scl - semi - 2 - ABSO\ submd$ of $X = X_1 \oplus X_2$.

Proof: Let $r_s^2(v_s, h_y) \subseteq K_1 \oplus K_2$, so that $r_s^2 v_x \subseteq K_1$ and $r_s^2, h_y \subseteq K_2$. since K_1, K_2 are F − Scl − semi − 2 − ABSO submds, then $r_s v_x \subseteq K + F - Scl(X_1)$ or $r_s^2 \subseteq (K_1 + F - Scl(X_1): X_1)$ and $r_s, h_y \subseteq K_2 + F - Scl(X_2)$ or $r_s^2 \subseteq (K_2 + F - Scl(X_2): X_2) = (K_1 + F - Scl(X): X_1)$, hence $r_s v_x \subseteq K_1 + F - Scl(X_1)$ and $r_s, h_y \subseteq K_2 + F - Scl(X_2)$ or $r_s^2 \subseteq (K_1 + F - Scl(X_1): X_1)$. Then $r_s(v_x,, h_y) \subseteq K + F - Scl(X_1) \oplus K_2 + F - Scl(X_2)$ or $r_s^2 \subseteq (K + F - Scl(X): X)$. For that, K is a F − Scl − semi − 2 − ABSO submds of X.

3. Discussion and Conclusion

The research concludes that for each $F - Scl - prime.(F - Scl - semi\ prime)$, and F-Scl-2-ABSO(F-semi-2-ABSO, F-Scl-quasi prime) submd see Remarks and Examples(2.3), The following is the study's some findings:

If a $F - modl\ X$ is semi-simple, then any F-submd is a F-Scl-semi-2-ABSO submd of X. If P, H be F − submds of X, then P is F-Scl-semi-2-ABSO submd if and only if $r_s^2 H \subseteq P$ for F. singl. r_s of W, implies $r_s H \subseteq P + F - Scl(X)$ or $r_s^2 \subseteq (P + F - Scl(X): X)$. Let P be a proper F. submd of multip. $F - modl\ X$ of a $W - modl\ G$. Then P is a $F - Scl - semi - 2 - ABSO$ submd of X, if and only if $(P: X)$ is a $F - Scl - semi - 2 - ABSO\ ideal$.

Finally, let X_1, X_2 be a $F - modl\ of\ W - modls\ G_1, G_2$ resp. and K_1, K_2 are $F - Scl - semi - 2 - ABSO\ submd$ of X_1, X_2 resp., such that $(K_1 + F - Scl(X_1): X_1) = (K_2 + F - Scl(X_2): X_2)$.then $K = K_1 \oplus K_2$ is a $F - Scl - semi - 2 - ABSO\ submd$ of $X = X_1 \oplus X_2$.

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