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A novel hybridized neuro-fuzzy model for solving fuzzy singular perturbation problems with initial conditions

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Abstract

The present paper tries to introduce a new process for solving fuzzy singular perturbation problem(SPP, s) with initial condition. This approach depends on on the partially fuzzy neural network to find the numerical solution of the second order of these problems. This system's trial solution is written as a sum of two parts. The first section meets the fuzzy initial condition and does not have any fuzzy free parameters. The second component consists of a partially fuzzy feed-forward neural network. containing fuzzy adjustable parameters (the fuzzy weights). As a result, the starting condition is fulfilled by construction, and the network is trained to solve the differential equations. When compared to other numerical techniques, this technique proves that neural networks generate solutions with high generalizability and accurateness. A number of examples are given to show the proposed plan.

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1. Introduction

SPP, s appear often in requests and must become a percentage of consideration in modern existences. It is well unstated that this matters are reliant on a parameter $0 < \varepsilon \ll 1$ multiplying the highest derivatives, in such a way that the explanation has a multiscale charm, i.e. exist tinny change strata where it works quickly, although it performs reliably and varies Slowly peel far

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

from the layers [1,2], and [3], correspondingly outcome of, giving SPP, s has important tasks that need be overawed in order to get precise numeric solutions [4-6].

The golden mean, practical systems [7], and engineering challenges have all been solved using fuzzy set theory, which is a great implement for representing indecision and interpreting imprecise or particular input in exact representations. The (FDE), in particular, is a crucial problem from a theoretical standpoint [8,9]. Chang and Zadeh [10] were the first to introduce the fuzzy derivative concept, Prade and Dubois came next. [11], who employed the the principle of extension.

Since it is widely used for modeling problems in science and engineering, many methods for solving fuzzy differential equations FDEs have been developed.

Researchers began using ANN in 1990 to solve ODE and PDE such as [12,13, 20-31].

In 2012, researchers began to solve fuzzy equations by using fuzzy neural networks as in [14].

In this work, partial fuzzy neural networks(PFNN) were used to find the numerical solution for (FSPPs) with initial condition we present a numerical method which trusts on the function estimate and consequences in the structure of a answer written in a differentiable, closed analytic form. This process employments FFNN as the simple estimate part, whose the fuzzy parameters are attuned to minimalize an suitable error function.

2. Basic concepts

The concept of fuzzy numbers has several definitions [15,17].

Definition 1: [15] If U is a generically denoted combination of things by U, then a fuzzy set $\mathcal{A}$in U is a usual arranged pairs:

$$\mathcal{A} = \{(\mathrm{U}, \xi_{\mathcal{A}}(\mathrm{u})) : \mathrm{u} \in \mathrm{U}, 0 \leq \xi_{\mathcal{A}}(\mathrm{u}) \leq 1\}$$

Definition 2: [16] "The support of a fuzzy set $\mathcal{A}$, S($\mathcal{A}$),is the crisp set of all u∈ U such that" $\xi_{\mathcal{A}}(\mathrm{u}) > 0$.

Definition 3: [17] "The (crisp) collection of elements that are related to the fuzzy set $\mathcal{A}$ at least to the degree τ is called the τ-level set":

$$\mathcal{A}_\tau = \{\mathrm{u} \in \mathrm{U} : \xi_{\mathcal{A}}(\mathrm{u}) \geq \tau\}$$

$\grave{\mathcal{A}}_\tau = \{\mathrm{u} \in \mathrm{U} : \xi_{\mathcal{A}}(\mathrm{u}) > \tau\}$ is called "strong τ-level set" or "strong τ-cut."

Definition 4: [15] A fuzzy set is $\mathcal{A}$ convex if

$$\mathcal{A}(\beta u_1 + (1-\beta) u_2) \geq min\{\xi_{\mathcal{A}}(u_1), \xi_{\mathcal{A}}(u_2)\}, u_1, u_2 \in U, \beta \in [0,1]$$

Alternatively, a fuzzy set is convex if all τ-level sets are convex.

Remark 1: [16] For arbitrary $\kappa = (\kappa^l, \kappa^u)$, $\upsilon = (\upsilon^l, \upsilon^u)$ and $C \in$ R, then:

1) $(\kappa + \upsilon)^l\ (\tau) = \kappa^l(\ \tau) + \upsilon^l(\ \tau)$
2) $(\kappa + \upsilon)^u\ (\tau) = \kappa^u(\ \tau) + \upsilon^u(\ \tau)$
3) $(C\kappa)^l(\tau) = C\kappa^l(\ \tau)$, $(C\kappa)^u(\tau) = C\kappa^u(\ \tau)$, if $C \geq 0$
4) $(C\kappa)^l(\tau) = C\kappa^u(\ \tau)$, $(C\kappa)^u(\tau) = C\kappa^l(\ \tau)$, if $C < 0$.For all $\tau \in [0,1]$.

Remark 2: [17] The distance between $\kappa = (\kappa^l, \kappa^u)$, and $\upsilon = (\upsilon^l, \upsilon^u)$ is given as:

$$\mathfrak{D}\ (\kappa, \upsilon) = \left[\int_0^1 (\kappa^l(\ \tau) - \ \upsilon^l(\ \tau))^2\ d\tau + \int_0^1 \kappa^u(\ \tau) - \ \upsilon^u(\ \tau)\ d\tau\right]^{\frac{1}{2}}$$

3. Fuzzy Neural Network [18,19]

(ANN) are knowledge computers that can study any random input-output functional mapping. They are debauched machinery that can be applied in software or hardwar in parallel. The Arithmetic difficulty of an ANN is, in fact, polynomial in the number of neurons in the network. Parallelism has the added benefits.

(i.e.) The (ANN) is a beginner's. Electric elements and computer software can both be used to implement it. It's a distributed parallel processor with a lot of connections. It's a data processing system that shares some characteristics with biological neural networks in terms of performance.

Consider the following fuzzy neural network: n input units, m hidden units, and s output units. The input vector is made up of real integers, whereas the target vector, connection weights, and biases are made up of fuzzy numbers.The fuzzy neural network is called partially fuzzy neural network PFNN if some of the changeable ; otherwise, it is called FFNN .

For the sake of this article, a PFNN will be layer. The number of neurons in each layer, n m s, is used to represent the dimension of a completely FNN.

4. Architecture PFNN

To show the structure of the PFNN by converting real inputs $(\varkappa_1, \varkappa_2, \dots, \varkappa_i, \dots, \varkappa_n)$ into fuzzy outputs$([\gamma_1]_\propto, [\gamma_2]_\propto, \dots, [\gamma_k]_\propto, \dots, [\gamma_s]_\propto)$ throughout the m hidden fuzzy neurons $\left([\mathbb{Q}_1]_\propto, [\mathbb{Q}_2]_\propto, \dots [\mathbb{Q}_j]_\propto, \dots\ [\mathbb{Q}_m]_\propto\right)$. Let η and ζ be the real biases for the fuzzy neurons $[\mathbb{Q}_j]_\propto$, $[\gamma_k]_\propto$ resbictively.

$$\text{Input units:}\varkappa = \varkappa_i, i=1,2,3,\dots,n \tag{1}$$

Hidden units:

$$[\ \mathbb{Q}_j]_\propto = \mathcal{T}\left(\left[\mathcal{N}et_j\right]_\propto\right), j = 1,2,3,\dots,m \tag{2}$$

Where

$$\left[\mathcal{N}et_j\right]_\propto = \sum_{i=1}^{n} \varkappa_i \left[\omega_{ji}\right]_\propto + \eta \tag{3}$$

Output units:

$$[\gamma_k]_\propto = \mathcal{T}\ ([\mathcal{N}et_k]_\propto), k = 1, 2, 3,\dots,s \tag{4}$$

Where

$$[\mathcal{N}et_k]_\propto = \sum_{j=1}^{m} [\omega_{kj}]_\propto [\mathbb{Q}_i]_\propto + \zeta \quad (5)$$

When the inputs $\varkappa$ i's are non-negative, the then by using fuzzy number operations.

Input units: $\varkappa_i = \varkappa_i$ (6)

Hidden units:

$$[\mathbb{Q}_j]_\propto = \mathcal{T}\left([\mathcal{N}et_j]_\propto\right) = \left[[\mathbb{Q}_j]_\propto^{\ell}, [\mathbb{Q}_j]_\propto^{u}\right] = \left[\mathcal{T}([\mathcal{N}et_j]_\propto^{\ell}), \mathcal{T}([\mathcal{N}et_j]_\propto^{u})\right] \quad (7)$$

where

$$[\mathcal{N}et_j]_\propto^{\ell} = \sum_{i=1}^{n} \varkappa_i [\omega_{ji}]_\propto^{\ell} + \eta \quad (8)$$

$$[\mathcal{N}et_j]_\propto^{u} = \sum_{i=1}^{n} \varkappa_i [\omega_{ji}]_\propto^{u} + \eta \quad (9)$$

Output units:

$$[\gamma_k]_\propto = \mathcal{T}([\mathcal{N}et_k]_\propto) = [[\mathbb{Q}_k]_\propto^{\ell}, [\mathbb{Q}_k]_\propto^{u}] = [\mathcal{T}([\mathcal{N}et_k]_\propto^{\ell}), \mathcal{T}([\mathcal{N}et_k]_\propto^{u})] \quad (10)$$

Where

$$[\mathcal{N}et_k]_\propto^{\ell} = \sum_{j\in a} [\omega_{kj}]_\propto^{\ell} [\mathbb{Q}_j]_\propto^{\ell} + \sum_{j\in b} [\omega_{kj}]_\propto^{\ell} [\mathbb{Q}_j]_\propto^{u} + \zeta \quad (11)$$

$$[\mathcal{N}et_k]_\propto^{u} = \sum_{j\in c} [\omega_{kj}]_\propto^{u} [\mathbb{Q}_j]_\propto^{u} + \sum_{j\in d} [\omega_{kj}]_\propto^{u} [\mathbb{Q}_j]_\propto^{\ell} + \zeta \quad (12)$$

For $[\mathbb{Q}_j]_\propto^{u} \geq [\mathbb{Q}_j]_\propto^{\ell} \geq 0$, where a=$\left\{j; [\omega_{kj}]_\propto^{\ell} \geq 0\right\}$, b=$\left\{j; [\omega_{kj}]_\propto^{\ell} < 0\right\}$

c = $\left\{j; [\omega_{kj}]_\propto^{u} \geq 0\right\}$, d=$\left\{j; [\omega_{kj}]_\propto^{u} < 0\right\}$

5. Solve FSPP for ODE by FFNN

To solve any FSPP by PFNN. We use the following architecture of input, output and hidden layer.As shown in (Figure 1).

Input units:

$$\varkappa = \varkappa \quad (13)$$

Hidden units:

$$[\mathbb{Q}_j]_\propto = \left[[\mathbb{Q}_j]_\propto^{\boldsymbol{\ell}}, [\mathbb{Q}_j]_\propto^{\boldsymbol{u}}\right] = [\mathcal{T}([\mathcal{N}et_k]_\propto^{\ell}), \mathcal{T}([\mathcal{N}et_k]_\propto^{u})] \quad (14)$$

Where

$$[\mathcal{N}et_j]_\propto^{\ell} = \varkappa [\omega_j]_\propto^{\ell} + \eta \quad (15)$$

$$[\mathcal{N}et_j]_\propto^{u} = \varkappa [\omega_j]_\propto^{u} + \eta \quad (16)$$

Output units:

$$[\gamma]_\propto = [[\gamma]_\propto^{\ell}, [\gamma]_\propto^{u}] \quad (17)$$

Where

$$[\gamma]_\propto^{\ell} = \sum_{j\in a} [\upsilon_j]_\propto^{\ell} [\mathbb{Q}_j]_\propto^{\ell} + \sum_{j\in b} [\upsilon_j]_\propto^{\ell} [\mathbb{Q}_j]_\propto^{u} \quad (18)$$

$$[\gamma]_\propto^{u} = \sum_{j\in c} [\upsilon_j]_\propto^{u} [\mathbb{Q}_j]_\propto^{u} + \sum_{j\in d} [\upsilon_j]_\propto^{u} [\mathbb{Q}_j]_\propto^{\ell} \quad (19)$$

For $\left[\mathbb{Q}_j\right]_{\propto}^{u} \geq \left[\mathbb{Q}_j\right]_{\propto}^{\ell} \geq 0$, where a=$\left\{j;\left[v_j\right]_{\propto}^{\ell} \geq 0\right\}$,b=$\left\{j;\left[v_j\right]_{\propto}^{\ell} < 0\right\}$

c = $\left\{j;\left[v_j\right]_{\propto}^{u} \geq 0\right\}$, d=$\left\{j;\left[v_j\right]_{\propto}^{u} < 0\right\}$.

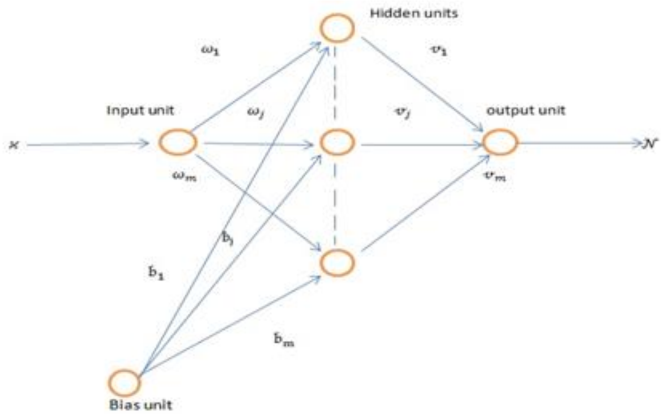

Figure 1
Partially fuzzy feed forward neural network

6. Illustration for solving FSPP for ODE by FFNN

For illustration these steps, we will consider the second order of FSPPs for ODE

$$\varepsilon \frac{\mathrm{d}^2\Upsilon(\varkappa)}{d\varkappa^2} = \mathcal{F}(\varkappa, \Upsilon, \Upsilon', \varepsilon)\ \varkappa \in [\mathfrak{a}, \mathfrak{b}] \tag{20}$$

"with initial condition (IC): $\Upsilon(\mathfrak{a})$=$\mathcal{A}$, $\Upsilon(\mathfrak{a})$=$\mathcal{A}'$ where $\mathcal{A}$" and $\mathcal{A}'$ are a fuzzynumbers in E^1 under the $\propto$ −cut sets and $0 < \varepsilon << 1$.

$$[\mathcal{A}]_{\propto}=[[\mathcal{A}]_{\propto}^{\ell}, [\mathcal{A}]_{\propto}^{u}], \propto\in [0,1]$$

$$[\mathcal{A}']_{\propto}=[[\mathcal{A}']_{\propto}^{\ell}, [\mathcal{A}']_{\propto}^{u}], \propto\in [0,1]$$

A fuzzy "trial solution Υ_τ can be written as":

$$[\Upsilon_\tau(\varkappa, \mathcal{P})]_{\propto}=[\mathcal{A}]_{\propto} + [\mathcal{A}']_{\propto}(\varkappa - \mathfrak{a}) + (\varkappa - \mathfrak{a})^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_{\propto} \tag{21}$$

The error amount is given as follows:

$$\mathrm{E}(\mathcal{P})=\sum_{\mathrm{i}=1}^{\mathrm{g}}\left([\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{\ell}, [\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{u}\right) \tag{22}$$

where

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{\ell}=\left[\left[\frac{\partial^2\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa^2}\right]_{\propto}^{\ell} - [\mathcal{F}\left[\varkappa_i, \Upsilon_\tau(\varkappa_i, \mathcal{P}), \frac{\partial\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa}, \varepsilon\right]_{\propto}^{\ell}\right]^2, \varkappa_i \in [\mathfrak{a}, \mathfrak{b}] \tag{21}$$

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{u}=\left[\left[\frac{\partial^2\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa^2}\right]_{\propto}^{u} - [\mathcal{F}\left[\varkappa_i, \Upsilon_\tau(\varkappa_i, \mathcal{P}), \frac{\partial\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa}, \varepsilon\right]_{\propto}^{u}\right]^2, \varkappa_i \in [\mathfrak{a}, \mathfrak{b}] \tag{22}$$

Where

$$[\Upsilon_\tau(\varkappa, \mathcal{P})]_{\propto}^{\ell}=[\mathcal{A}]_{\propto}^{\ell} + [\mathcal{A}']_{\propto}^{\ell}\ (\varkappa - \mathfrak{a}) + (\varkappa - \mathfrak{a})^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_{\propto}^{\ell} \tag{23}$$

$$[\Upsilon_\tau(\varkappa, \mathcal{P})]_{\propto}^{u}=[\mathcal{A}]_{\propto}^{u} + [\mathcal{A}']_{\propto}^{u}\ (\varkappa - \mathfrak{a}) + (\varkappa - \mathfrak{a})^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_{\propto}^{u} \tag{24}$$

$$\frac{\partial[\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^\ell}{\partial\varkappa}=[\mathcal{A}']_\propto^\ell+(\varkappa-\mathfrak{a})^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa}+2(\varkappa-\mathfrak{a})\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell \quad (25)$$

$$\frac{\partial[\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^u}{\partial\varkappa}=[\mathcal{A}']_\propto^u+(\varkappa-\mathfrak{a})^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa}+2(\varkappa-\mathfrak{a})\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u \quad (26)$$

$$\frac{\partial^2[\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^\ell}{d\varkappa^2}=(\varkappa-\mathfrak{a})^2\,\frac{\partial^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa^2}+2\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell+4(\varkappa-\mathfrak{a})\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa} \quad (27)$$

$$\frac{\partial^2[\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^u}{\partial\varkappa^2}=(\varkappa-\mathfrak{a})^2\,\frac{d^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa^2}+2\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u+4(\varkappa-\mathfrak{a})\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa} \quad (28)$$

It is easy to evaluate the gradient of the performance with respect to the coefficient of the PFNN .

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^\ell=\Big[(\varkappa-\mathfrak{a})^2\,\frac{\partial^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa^2}+2\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell+4(\varkappa-\mathfrak{a})\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa}-[\mathcal{F}(\varkappa_i,[\mathcal{A}]_\propto^\ell+[\mathcal{A}']_\propto^\ell(\varkappa-\mathfrak{a})+(\varkappa-\mathfrak{a})^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell,[\mathcal{A}']_\propto^\ell+(\varkappa-\mathfrak{a})^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial\varkappa}+2(\varkappa-\mathfrak{a})\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell,\varepsilon)\Big]^2,\ \varkappa_i\in[\mathfrak{a},\mathfrak{b}] \quad (29)$$

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^u=\Big[(\varkappa-\mathfrak{a})^2\,\frac{d^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa^2}+2\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u+4(\varkappa-\mathfrak{a})\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa}-\mathcal{F}(\varkappa_i,[\mathcal{A}]_\propto^u+[\mathcal{A}']_\propto^u(\varkappa-\mathfrak{a})+(\varkappa-\mathfrak{a})^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u,[\mathcal{A}']_\propto^u+(\varkappa-\mathfrak{a})^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial\varkappa}+2(\varkappa-\mathfrak{a})\,[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u,\varepsilon)\Big]^2,\ \varkappa_i\in[\mathfrak{a},\mathfrak{b}] \quad (30)$$

7. Numerical illustrations

To show the technicality in these sections, we provide programming examples created in MATLAB 7.11 that exhibit the performance and attributes of the proposed design.

We employed a multi-layer fed forward PFNN with one input unit for solving FSPPs for ODEs, one hidden layer that contained 7 hidden units (neurons) and one linear output unit. Tansig is the sigmoid activation of each hidden unit. Accuracy can be calculated by applying the error law:

$$[\mathrm{E}(\varkappa)]_\alpha^{\mathrm{L}}=|[\gamma_{\mathrm{t}}(\varkappa)]_\alpha^{\mathrm{L}}-[\gamma_{\mathrm{a}}(\varkappa)]_\alpha^{\mathrm{L}}| \quad (31)$$

$$[\mathrm{E}(\varkappa)]_\alpha^{\mathrm{U}}=|[\gamma_{\mathrm{t}}(\varkappa)]_\alpha^{\mathrm{U}}-[\gamma_{\mathrm{a}}(\varkappa)]_\alpha^{\mathrm{U}}| \quad (32)$$

We give figures exhibiting the associated error $[\mathrm{E}(\varkappa)]_\alpha$ both at the same time places that were utilized for preparation and at several additional points within the dominion of each equation to illustrate the properties of the solution supplied by the suggested design.

The latter type of numbers is crucial since they demonstrate the neural solution, s superior interpolation capabilities when compared to other solutions acquired through other approaches.

Example 1: Consider the second order FSPPs (see Figure 2 and Figure 3):

$$\varepsilon\, \gamma''(\varkappa) = 0.24\gamma' - \gamma,\ \varkappa \in [0, 1],$$

with the FIC:

$$[\gamma(0)]_{\propto} = [0.2 \propto -3.2, -2.8 - 0.2 \propto],\ [\gamma'(0)]_{\propto} = [0.2 \propto -1.2, -0.8 - 0.2 \propto],\ 0 \leq \propto \leq 1.$$

And analytic

$$[\gamma_a(x)]^{\ell}_{\propto} = 2e^{3t}sin(4t) - 3e^{3t}cos(4t) + \left(\frac{1}{5} - \frac{1}{5}\propto\right)e^{-3t}sin(4t) - \left(\frac{1}{5} - \frac{1}{5}\propto\right)e^{-3t}cos(4t)$$

$$[\gamma_a(x)]^{u}_{\propto} = 2e^{3t}sin(4t) - 3e^{3t}cos(4t) - \left(\frac{1}{5} - \frac{1}{5}\propto\right)e^{-3t}sin(4t) - \left(\frac{1}{5} - \frac{1}{5}\propto\right)e^{-3t}cos(4t)$$

The fuzzy trial solution

$$[\Upsilon_\tau(\varkappa, \mathcal{P})]_{\propto} = [0.2 \propto -3.2, -2.8 - 0.2 \propto]_{\propto} + [0.2 \propto -1.2, -0.8 - 0.2 \propto]_{\propto}\varkappa + \varkappa^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_{\propto}$$

we have the trial solution

$$[\gamma_\tau(x)]^{\ell}_{\propto} = [0.2 \propto -3.2]^{\ell}_{\propto} + [0.2 \propto -1.2]^{\ell}_{\propto}\varkappa + \varkappa^2\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto}$$

$$[\gamma_\tau(x)]^{u}_{\propto} = [-2.8 - 0.2 \propto]^{u}_{\propto} + [-0.8 - 0.2 \propto]^{u}_{\propto}\varkappa + \varkappa^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto}$$

$$\frac{\partial[\Upsilon_\tau(\varkappa,\mathcal{P})]^{\ell}_{\propto}}{\partial\varkappa} = [0.2 \propto -1.2\,]^{\ell}_{\propto} + \varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa} + 2\varkappa\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto}$$

$$\frac{\partial[\Upsilon_\tau(\varkappa,\mathcal{P})]^{u}_{\propto}}{\partial\varkappa} = [-0.8 - 0.2 \propto]^{u}_{\propto} + \varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa} + 2\varkappa\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto}$$

$$\frac{\partial^2[\Upsilon_\tau(\varkappa,\mathcal{P})]^{\ell}_{\propto}}{d\varkappa^2} = \varkappa^2\ \frac{\partial^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa^2} + 2\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto} + 4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa}$$

$$\frac{\partial^2[\Upsilon_\tau(\varkappa,\mathcal{P})]^{u}_{\propto}}{\partial\varkappa^2} = \varkappa^2\ \frac{d^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa^2} + 2\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto} + 4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa}$$

Then we get:

$$[\mathrm{E}(\mathcal{P})]^{\ell}_{i\propto} = \left[\left[\frac{\partial^2\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa^2}\right]^{\ell}_{\propto} - (\frac{0.24}{\varepsilon}\ [\Upsilon_\tau{}'(\varkappa_i)]^{\ell}_{\propto} - \ [\Upsilon_\tau(\varkappa_i)]^{\ell}_{\propto})\right]^2$$

$$[\mathrm{E}(\mathcal{P})]^{u}_{i\propto} = \left[\left[\frac{\partial^2\Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial\varkappa^2}\right]^{u}_{\propto} - (\frac{0.24}{\varepsilon}\ [\Upsilon_\tau{}'(\varkappa_i)]^{u}_{\propto} - \ [\Upsilon_\tau(\varkappa_i)]^{u}_{\propto})\right]^2$$

By using equation above, we find

$$[\mathrm{E}(\mathcal{P})]^{\ell}_{i\propto} = \Big[\varkappa^2\ \frac{\partial^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa^2} + 2\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto} + 4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa} - [\frac{0.24}{\varepsilon}\Big[[0.2 \propto -1.2\,]^{\ell}_{\propto} + \varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{\ell}_{\propto}}{\partial\varkappa} + 2\varkappa\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto}\Big] - [[0.2 \propto -3.2]^{\ell}_{\propto} + [0.2 \propto -1.2]^{\ell}_{\propto}\varkappa + \varkappa^2\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{\ell}_{\propto}]]\Big]^2$$

$$[\mathrm{E}(\mathcal{P})]^{u}_{i\propto} = \Big[\varkappa^2\ \frac{d^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa^2} + 2\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto} + 4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa} - \Big[\frac{0.24}{\varepsilon}\Big[[-0.8 - 0.2 \propto]^{u}_{\propto} + \varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]^{u}_{\propto}}{\partial\varkappa} + 2\varkappa\,[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto}\Big] - [[-2.8 - 0.2 \propto]^{u}_{\propto} + [-0.8 - 0.2 \propto]^{u}_{\propto}\varkappa + \varkappa^2[\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]^{u}_{\propto}]\Big]\Big]^2$$

Through the tables (Table 1, Table 2, Table 3, Table 4) below, by choosing the perturbed value ε =0.4, the reliability of the technique used can be clarified.

Table 1

$\varepsilon = 0.4$, $\alpha = 0.9$

input	Analytic solution		Solution of FFNN	
x	$[\gamma_a(\chi)]^{\ell}_{\alpha}$	$[\gamma_a(\chi)]^{u}_{\alpha}$	$[\gamma_t(\chi)]^{\ell}_{\alpha}$	$[\gamma_t(\chi)]^{u}_{\alpha}$
0	-3.020000000000000	-2.980000000000000	-3.020000000000000	-2.980000000000000
0.1	-2.698003866217880	-2.659170787411220	-2.698003866268340	-2.659170787554230
0.2	-1.209752223152560	-1.178710058394840	-1.209752223158210	-1.178710058886340
0.3	1.900598440681950	1.921648940001490	1.900598440689650	1.921648949903200
0.4	6.922394533212520	6.934085375472860	6.922394537751900	6.934085366593300
0.5	13.743298130605900	13.747699601413400	13.743298130855300	13.747699677529000
0.6	21.555749040517200	21.555339558573900	21.555749040927600	21.555339504437500
0.7	28.555670688293700	28.552696299090400	28.555670688225100	28.552688543799000
0.8	31.728118399066800	31.724284045400600	31.728118399663200	31.724284006543500
0.9	26.863203151672600	26.859602874774100	26.863203151698200	26.859608543986500
1	8.986184757794860	8.983375878706260	8.986184758843200	8.983375885498340

Table 2

Accurateness, $\varepsilon = 0.4$, $\alpha = 0.9$

$[E(x)]_{\alpha} = \lvert [\gamma_a(\chi)]_{\alpha} - \gamma_t(\chi)]_{\alpha} \rvert$	
$[E(\chi)]^{\ell}_{\alpha}$	$[E(\chi)]^{u}_{\alpha}$
0	0
5.0461E-11	1.43009E-10
5.64704E-12	4.915E-10
7.70006E-12	9.90171E-09
4.53938E-09	8.87956E-09
2.49388E-10	7.61156E-08
4.10434E-10	5.41364E-08
6.86349E-11	7.75529E-06
5.96355E-10	3.88571E-08
2.55653E-11	5.66921E-06
1.04833E-09	6.79208E-09
MSE=2.02721E-18	MSE=8.39045E-12

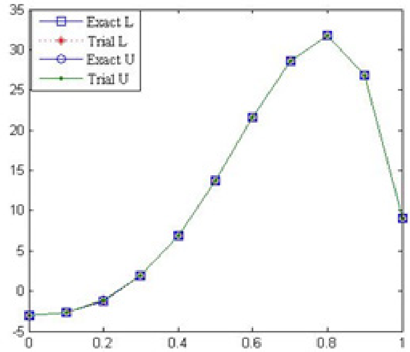

Figure 2
Analytic and neural solution

Table 3
$\varepsilon = 0.4$, x = 0.1

input	"Analytic solution"		"Solution of FFNN "	
α	$[\gamma_a(\chi)]^{\ell}_{\alpha}$	$[\gamma_a(\chi)]^{u}_{\alpha}$	$[\gamma_t(\chi)]^{\ell}_{\alpha}$	$[\gamma_t(\chi)]^{u}_{\alpha}$
0	-2.872752720847840	-2.484421932781260	-2.872752720847960	-2.484421932781440
0.2	-2.833919642041190	-2.523255011587910	-2.833919642041430	-2.523255011589320
0.4	-2.795086563234530	-2.562088090394570	-2.795086563232240	-2.562088090776300
0.6	-2.756253484427870	-2.600921169201230	-2.756253484428750	-2.600921169201760
0.8	-2.717420405621210	-2.639754248007890	-2.717420405621980	-2.639754248007600
1	-2.678587326814550	-2.678587326814550	-2.678587326818870	-2.678587326817650

Table 4
Accurateness $\varepsilon = 0.4$, x = 0.1

$[E(x)]_{\alpha} = \lvert [\gamma_a(\chi)]_{\alpha} - \gamma_t(\chi)]_{\alpha} \rvert$	
$[E(\chi)]^{\ell}_{\alpha}$	$[E(\chi)]^{u}_{\alpha}$
1.15907E-13	1.84297E-13
2.44693E-13	1.40554E-12
2.28617E-12	3.81726E-10
8.82405E-13	5.28022E-13
7.71383E-13	2.90878E-13
4.3201E-12	3.10019E-12
MSE = 2.42878E-20	MSE = 4.2228E-24

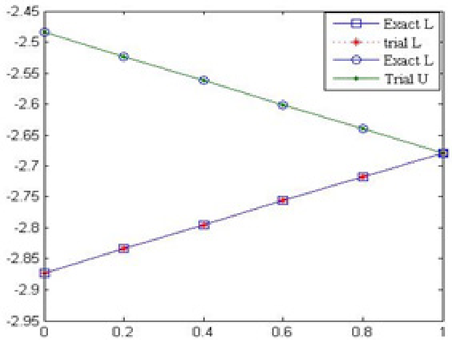

Figure 3

Analytic and neural solution with x = 0.1

Example 2: Let the FSPPs (see Figure 4):

$$\varepsilon\, \gamma''(\varkappa) = -10\gamma'(\varkappa) - 20\gamma(\varkappa) + 50\cos x,\ \varkappa \in [0,1],\ \varepsilon = 0.2$$

with the FIC:

$$[\gamma(0)]_\propto = [4+\propto, 6-\propto],\ [\gamma'(0)]_\propto = [\propto, 2-\propto],\ 0 \le \propto \le 1.$$

The fuzzy analytical solutions:

$$[\gamma_a(x)]_\propto^\ell = \frac{2}{29}\, e^{-t} sin\sqrt{3t}(13 \propto -99)\sqrt{3} + e^{-t} cos\sqrt{3t}\left(\propto - \frac{98}{13}\right) + \left(\frac{150}{13}\right) \cos(t) + \left(\frac{100}{13}\right)\sin(t)$$

$$[\gamma_a(x)]_\propto^u = -\frac{2}{39}\, e^{-t} sin\sqrt{3t}(73 + 13 \propto)\sqrt{3} + e^{-t} cos\sqrt{3t}\left(-\propto - \frac{72}{13}\right) + \left(\frac{150}{13}\right) \cos(t) + \left(\frac{100}{13}\right)\sin(t)$$

The fuzzy trial solutions

$$[\Upsilon_\tau(\varkappa, \mathcal{P})]_\propto = [4+\propto, 6-\propto]_\propto + [\propto, 2-\propto]_\propto \varkappa + \varkappa^2 [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto.$$

Then we get

$$[\gamma_\tau(x)]_\propto^\ell = [4+\propto]_\propto^\ell + [\propto]_\propto^\ell \varkappa + \varkappa^2 [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^\ell$$

$$[\gamma_\tau(x)]_\propto^u = [6-\propto]_\propto^u + [2-\propto]_\propto^u \varkappa + \varkappa^2 [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^u$$

$$\frac{\partial [\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^\ell}{\partial \varkappa} = [\propto]_\propto^\ell + \varkappa^2 \frac{\partial [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial \varkappa} + 2\varkappa\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^\ell$$

$$\frac{\partial [\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^u}{\partial \varkappa} = [2-\propto]_\propto^u + \varkappa^2 \frac{\partial [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial \varkappa} + 2\varkappa\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^u$$

$$\frac{\partial^2 [\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^\ell}{d\varkappa^2} = \varkappa^2\, \frac{\partial^2 [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial \varkappa^2} + 2\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^\ell + 4\varkappa \frac{\partial [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^\ell}{\partial \varkappa}$$

$$\frac{\partial^2 [\Upsilon_\tau(\varkappa,\mathcal{P})]_\propto^u}{\partial \varkappa^2} = \varkappa^2\, \frac{d^2 [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial \varkappa^2} + 2\, [\mathcal{N}(\varkappa, \mathcal{P}, \varepsilon)]_\propto^u + 4\varkappa \frac{\partial [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_\propto^u}{\partial \varkappa}$$

$$[E(\mathcal{P})]_{i\propto}^\ell = \left[\left[\frac{\partial^2 \Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial \varkappa^2}\right]_\propto^\ell - \frac{1}{\varepsilon}(10\, [\Upsilon_\tau'(\varkappa_i)]_\propto^\ell - 20\, [\Upsilon_\tau(\varkappa_i)]_\propto^\ell + 50\cos \varkappa_i)\right]^2$$

$$[E(\mathcal{P})]_{i\propto}^u = \left[\left[\frac{\partial^2 \Upsilon_\tau(\varkappa_i,\mathcal{P})}{\partial \varkappa^2}\right]_\propto^u - \frac{1}{\varepsilon}(10\, [\Upsilon_\tau'(\varkappa_i)]_\propto^u - 20\, [\Upsilon_\tau(\varkappa_i)]_\propto^u + 50\cos \varkappa_i)\right]^2$$

By using equation above, we find

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{\ell}=\Big[\varkappa^2\ \frac{\partial^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{\ell}}{\partial\varkappa^2}+2\ [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{\ell}+4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{\ell}}{\partial\varkappa}-\frac{1}{\varepsilon}[-10\Big[[\propto]_{\propto}^{\ell}+\varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{\ell}}{\partial\varkappa}+2\varkappa\ [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{\ell}\Big]\ +50\cos x-20([4+\propto]_{\propto}^{l}+[\propto]_{\propto}^{l}\varkappa+\varkappa^2\ [\mathrm{N}(\varkappa,\mathrm{P},\varepsilon)]_{\propto}^{l})]\Big]^2$$

$$[\mathrm{E}(\mathcal{P})]_{\mathrm{i}\propto}^{u}=\Bigg[\varkappa^2\ \frac{\mathrm{d}^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u}}{\partial\varkappa^2}+2\ [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u}+4\varkappa\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u}}{\partial\varkappa}-\frac{1}{\varepsilon}\Big[-10\Big[[2-\propto]_{\propto}^{u}+\varkappa^2\frac{\partial[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u}}{\partial\varkappa}+2\varkappa\ [\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u}\Big]\ +50\cos x-20([6-\propto]_{\propto}^{u}+[2-\propto]_{\propto}^{u}\varkappa+\varkappa^2[\mathcal{N}(\varkappa,\mathcal{P},\varepsilon)]_{\propto}^{u})\Big]\Bigg]^2$$

Through the tables below, by choosing the perturbed value ε =0.4, the reliability of the technique used can be clarified (Table 5, Table 6).

Table 5

Example 2, ε = 0.2, α = 0.02

input	Analytic solution		Solution of FFNN	
x	$[\gamma_a(\chi)]_\alpha^\ell$	$[\gamma_a(\chi)]_\alpha^u$	$[\gamma_t(\chi)]_\alpha^\ell$	$[\gamma_t(\chi)]_\alpha^u$
0	4.200000000000000	5.800000000000000	4.200000000000000	5.800000000000000
0.1	2.545438398862380	4.651923684614940	2.545438398862650	4.651923684613450
0.2	3.639304717447050	5.633513977401830	3.639304717443450	5.633513977406740
0.3	4.973356228282860	6.776353366212720	4.973356228286540	6.776353366212320
0.4	6.268609644942650	7.860573635286470	6.268609644946660	7.860573635245320
0.5	7.418546725034660	8.801858881386380	7.418546724357640	8.801858881386410
0.6	8.375272646720140	9.562172407661200	8.375272646725540	9.562172407663360
0.7	9.117762827128240	10.124857350671600	9.117762827125430	10.124857350653100
0.8	9.639316191779490	10.484796873808700	9.639316191775320	10.484796873801600
0.9	9.941593050707960	10.643735427957000	9.941593050766210	10.643735427235600
1	10.031440920103300	10.607783608069800	10.031440920664300	10.607783608033100

Table 6

Accurateness $\varepsilon = 0.2$, $\alpha = 0.02$

$[E(x)]_\alpha = \mid [\gamma_a(\chi)]_\alpha - \gamma_t(\chi)]_\alpha \mid$	
$[E(\chi)]^{\ell}_{\alpha}$	$[E(\chi)]^{u}_{\alpha}$
0	0
2.7045E-13	1.4877E-12
3.59623E-12	4.91251E-12
3.67617E-12	4.00568E-13
4.00835E-12	4.11458E-11
6.77024E-10	2.66454E-14
5.3948E-12	2.15472E-12
2.80664E-12	1.85416E-11
4.17266E-12	7.07345E-12
5.82485E-11	7.21428E-10
5.61025E-10	3.66605E-11
MSE=7.06E-20	MSE=4.76292E-20

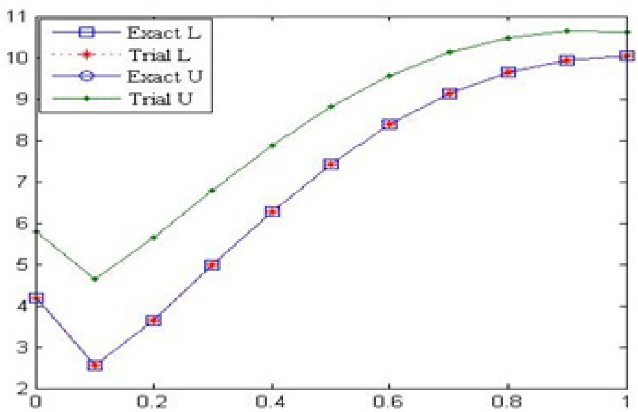

Figure 4

Analytic and neural solution

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