Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 6, pp. 1053–1064
DOI : 10.47974/JIM-1605

A new type of metric space via proximit structure

Dhuha Abdulameer Kadhim *
Department of Mathematics
College of Education for Girls
University of Kufa
Kufa
Iraq

Boushra Y. Hussein †
Department of Mathematics
College of Education
University of Al-Qadisiyah
Iraq

Abstract

The relationship between Dh-functional and the tower, as well as certain fundamental facts and examples, are used to investigate the idea of a new type of proximit space. Theorems relating to proximit space should also be included. Additionally, we demonstrate how proximit and metric space are connected.

Subject Classification: *Primary 30L99, Secondary 47S99, 15A60 .*

Keywords: *Proximit space, Metric space, Dh-functional, Tower, Lo-tower.*

1. Introduction

Lowen [17] showed how approach theory is frequently hidden behind other theories, and how getting it to the foreground and implementing it consistently enhances the theory, and he explained how contractions, continuous functions, compact sets, and convergent sequences are commonly dealt with in topology and analysis, as in many mathematical theories. Lowen [21] has both a completion and a compactification construction in an appropriate subcategory of approach. as well as the notion of contraction by an approach system. R. Lowen and K. Robeys [14,15] established some basic definitions and theorems in approach space which is a topological and metric space extension.

* *E-mail:* dhuha.ebada@uokufa.edu.iq (Corresponding Author)
https://orcid.org/0000-0002-9152-4162
† *E-mail:* boushra.alshebani@qu.edu.iq,
https://orcid.org/ 0000-0003-4909-0126

© Taru Publications

More information about approach space can be found in [13]. M. Sion and W. Haute [25] investigated an alternative notion of approach spaces based on a roach cores. [27] presented the relationship between approach space and metric space which is called a sober metric approach space. In [12,23] authors considered distances. Functional analysis applications of approach theory are discussed in [16,18-20]. A new kind of approach space made by [2]. Furthermore, Abbas and Hussein worked the completion normed approach space in [1]. Additional details are available in [3-11,22,24,26]. This research looks into the regularity of functionals between sets. In section 2, using the concepts of Dh-functional and tower, the definitions of Dh-functional and proximit space are presented. The examples of this idea are studied. In section 3, we receive the most important theorems concerning this object.

2. Preliminaries

All of the subsets of the set $\widetilde{\mathbb{X}}$ is symbolized by $2^{\widetilde{\mathbb{X}}}$.

Definition 2.1: [15] If the function $\delta: \widetilde{\mathbb{X}} \times 2^{\widetilde{\mathbb{X}}} \to [0,\infty]$ mollifies the following conditions, it is said to be a distance on $\widetilde{\mathbb{X}}$ for all $\tilde{a} \in \widetilde{\mathbb{X}}$ and $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(D1) $\delta(\tilde{a}, \{\tilde{a}\}) = 0$,

(D2) $\delta(\tilde{a}, \emptyset) = \infty$,

(D3) $\delta\left(\hat{a}, \hat{\mathcal{A}} \cup \hat{\mathcal{B}}\right) = \min(\delta\left(\hat{a}, \hat{\mathcal{A}}\right), \delta\left(\hat{a}, \hat{\mathcal{B}}\right))$,

(D4) $\forall \varepsilon \in [0,\infty]: \delta(\hat{a}, \hat{\mathcal{A}}) \leq \delta(\hat{a}, \hat{\mathcal{A}}^{(\varepsilon)}) + \varepsilon$ where $\hat{\mathcal{A}}^{(\varepsilon)} = \{\hat{a} \in \widetilde{\mathbb{X}}: \delta(\hat{a}, \hat{\mathcal{A}}) \leq \varepsilon\}$.

Definition 2.2: [15] A collection of functions $t_\varepsilon: 2^{\widetilde{\mathbb{X}}} \to 2^{\widetilde{\mathbb{X}}}$, $\varepsilon \in \Re^+$ is said to be a tower on $\widetilde{\mathbb{X}}$ if the following conditions are fulfilled for every $\varepsilon, \gamma \in \Re^+$:

(To1) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}: \hat{\mathcal{A}} \subset t_\varepsilon\left(\hat{\mathcal{A}}\right)$,

(To2) $t_\varepsilon(\emptyset) = \emptyset$

(To3) $\forall \hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\mathbb{X}}: t_\varepsilon\left(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}\right) = t_\varepsilon\left(\hat{\mathcal{A}}\right) \cup t_\varepsilon\left(\hat{\mathcal{B}}\right)$,

(To4) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}: t_\varepsilon(t_\gamma\left(\hat{\mathcal{A}}\right)) \subset t_{\varepsilon+\gamma}\left(\hat{\mathcal{A}}\right)$,

(To5) $\forall \hat{\mathcal{A}} \in 2^{\widetilde{\mathbb{X}}}: t_\varepsilon\left(\hat{\mathcal{A}}\right) = \bigcap_{\varepsilon<\gamma} t_\gamma\left(\hat{\mathcal{A}}\right)$.

Take note that with (To3) and (To5), we get

$$\forall \hat{\mathcal{A}} \subset \hat{\mathcal{B}} \subset \widetilde{\mathbb{X}}, \forall \alpha \leq \beta \in \Re^+: t_\alpha(\hat{\mathcal{A}}) \subset t_\beta\left(\hat{\mathcal{B}}\right).$$

Definition 2.3: [15] An operator $cl: 2^{\widetilde{\mathbb{X}}} \to 2^{\widetilde{\mathbb{X}}}$ that determines the qualities of a pretopology on a set $\widetilde{\mathbb{X}}$ for all $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$ is determined by an operator that determines the properties of an operator on a set.

1. $\hat{\mathcal{A}} \subset cl\left(\hat{\mathcal{A}}\right)$
2. $cl(\emptyset) = \emptyset$,

3. $cl\left(\hat{\mathcal{A}} \cup \hat{\mathcal{B}}\right) = cl\left(\hat{\mathcal{A}}\right) \cup cl\left(\hat{\mathcal{B}}\right)$.

This operator is said to be a pretopological closure operator. We call a set $\tilde{\mathbb{X}}$ with a pretopology a pretopological space**.**

Proposition 2.4: [15]

1. If $(t_\varepsilon)_{\varepsilon\in\Re^+}$ is tower, then $\forall\varepsilon\in \Re^+, t_\varepsilon$ is a pretopological closureoperator, and (To4) and (To5) are true.
2. If t_ε is a pretopological closure operator, and (To4) and (To5) are fulfilled, then t_0 is a is a topological closure operator, for each $\varepsilon > 0$, t_ε is a pretopological closure operator, and (To4) and (To5) are true.
3. If t_0 is a is a topological closure operator, for every $\varepsilon> 0$, t_ε is a pretopological closureoperator, and (To4) and (To5) are fulfilled, then $(t_\varepsilon)_{\varepsilon\in\Re^+}$ is tower.

3. Proximit Metric Space

The relation between metric and proximit space, Dh-functional, and Lo-tower are presented in this section.

Definition 3.1: We say that $\rho\colon 2^{\tilde{\mathbb{X}}} \times 2^{\tilde{\mathbb{X}}} \longrightarrow [0,\infty]$ is a Dh-functional on a set $\tilde{\mathbb{X}}$ to satisfy the next: for each $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$:

(h1) $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right) = \rho\left(\hat{\mathcal{B}}., \hat{\mathcal{A}}\right)$

(h2) $\hat{\mathcal{A}} = \emptyset$ or $\hat{\mathcal{B}} = \emptyset$ *then* $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}\right) = \infty$

(h3) $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right) = 0$ *then* $\hat{\mathcal{A}} \cap . \hat{\mathcal{B}} \neq \emptyset$

(h4) $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}\right) = \min\{\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right), \rho(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$

(h5) $\rho\left(.\widehat{\mathcal{A}}, \widehat{\mathcal{B}}\right) \leq \rho\left(\widehat{\mathcal{A}^\varepsilon}, \widehat{\mathcal{B}^\eta}\right) + \varepsilon + \eta \forall \varepsilon, \eta \in [0,\infty]$

For every $\hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}, \varepsilon \in [0,\infty]$, we write $t_\varepsilon\left(\hat{\mathcal{A}}\right) := \{\tilde{a} \in \tilde{\mathbb{X}} \mid \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\}$. Therefore the triple $(\tilde{\mathbb{X}}, \rho, t_\varepsilon)$ is called proximit space.

Example 3.2: Let $\tilde{\mathbb{X}} = [0,\infty]$, define $\rho\colon 2^{[0,\infty]} \times 2^{[0,\infty]} \to [0,\infty]$ by

$$\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}\right) = \begin{cases} 0 \hat{\mathcal{A}}, \hat{\mathcal{B}} \; unbounded \\ \infty \hat{\mathcal{A}}, \hat{\mathcal{B}} \; bounded \\ \inf\limits_{\ddot{n}\in\hat{\mathcal{A}}} \inf\limits_{\dot{m}\in\hat{\mathcal{B}}} |\ddot{n} - \dot{m}| \ddot{n}, \dot{m} < \infty \end{cases}$$

Then $(\tilde{\mathbb{X}}, \rho, t_\varepsilon)$ is proximit space.

Solve: Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{[0,\infty]}$

(h1) Let $\ddot{n}, \dot{m} < \infty$

$$\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}.\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|=\inf_{\dot{m}\in\hat{\mathcal{B}}}\inf_{\ddot{n}\in\hat{\mathcal{A}}}|\dot{m}-\ddot{n}|=\rho(\hat{\mathcal{B}},\hat{\mathcal{A}})$$

If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *unbounded*, then $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=0=\rho(\hat{\mathcal{B}},\hat{\mathcal{A}})$.

If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *bounded*, so $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\infty=\rho\left(\hat{\mathcal{B}},\hat{\mathcal{A}}\right)$.

(h2) If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *unbounded*, thus $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\ddot{\mathrm{B}}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}\left|\ddot{\mathrm{B}}-\dot{m}\right|=\infty$

If $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ *bounded*, then $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=0$

(h3) Let $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=0$, so $\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|=0$. Therefore $\ddot{n}=\dot{m}$, thus $\ddot{n}\in\widehat{\mathcal{A}.}\cap\widehat{\mathcal{B}}$. Then $\hat{\hat{\mathcal{A}}}\cap\widehat{\mathcal{B}}\neq\emptyset$.

(h4) Let $\hat{\hat{\mathcal{A}}},\hat{\hat{\mathcal{B}}}$ be bounded

$$\rho\left(\hat{\hat{\mathcal{A}}},\widehat{\hat{\mathcal{B}}}\cup.\hat{\mathcal{C}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}\cup\hat{\mathcal{C}}}|\ddot{n}-\dot{m}|=0=\min\{\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right),\rho(\hat{\mathcal{A}},\hat{\mathcal{C}})\}$$

If $\ddot{n},\dot{m}<\infty$,

$$\begin{aligned}\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\cup\hat{\mathcal{C}}\right)&=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}\cup\hat{\mathcal{C}}}|n-m|\\&=\inf_{\ddot{n}\in\hat{\mathcal{A}}.}\left(\min\left\{\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|,\inf_{\dot{m}\in\hat{\mathcal{C}}}|\ddot{n}-\dot{m}|\right\}\right)\\&=\min\left(\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|,\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{C}}}|\ddot{n}-\dot{m}|\right)\\&=\min\left(\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right),\rho(\hat{\mathcal{A}},\hat{\mathcal{C}})\right)\end{aligned}$$

If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *unbounded*, $\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\cup\hat{\mathcal{C}}\right)=\infty=\min\{\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right),\rho(\hat{\mathcal{A}},\hat{\mathcal{C}})\}$

(h5) If $\ddot{n},\dot{m}<\infty$, $\hat{\mathcal{A}},\hat{\mathcal{B}}\in 2^{[0,\infty]},\varepsilon,\eta\in[0,\infty]$

$$\begin{aligned}\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)&=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|n-m|\leq\inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}}\inf_{\dot{m}\in\hat{\mathcal{B}}^{\varepsilon}}|\ddot{n}-\dot{m}|+\varepsilon+\eta\\&=\rho(\hat{\mathcal{A}}^{\varepsilon},\hat{\mathcal{B}}^{\eta})+\varepsilon+\eta\end{aligned}$$

If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *unbounded*, $\varepsilon,\eta\in[0,\infty]$

$$\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|=0\leq\rho(\hat{\mathcal{A}}^{\varepsilon},\hat{\mathcal{B}}^{\eta})\ +\varepsilon+\eta$$

If $\hat{\mathcal{A}},\hat{\mathcal{B}}$ *bounded*, $\varepsilon,\eta\in[0,\infty]$

$$\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\dot{m}\in\hat{\mathcal{B}}}|\ddot{n}-\dot{m}|=\infty\leq\rho\left(\hat{\mathcal{A}}^{\varepsilon},\hat{\mathcal{B}}^{\eta}\right)+\varepsilon+\eta.$$

Example 3.3: Let $\Re$ be a set of real numbers, then $(\Re^n,\rho,t_\varepsilon)$ is a proximit space.

Solve: Let $\rho\colon 2^{\Re^n}\times 2^{\Re^n}\to[0,\infty]$ be a function defined by

$$\rho\left(\hat{\hat{\mathcal{A}}},\widehat{\hat{\mathcal{B}}^{\wedge}}\right)=\begin{cases}\infty\widehat{\mathcal{A}.}=\emptyset\ or\ \widehat{\mathcal{B}}.=\emptyset\\\inf_{\ddot{n}\in\widehat{\mathcal{A}},}\inf_{\dot{m}\in\widehat{\mathcal{B}},}\mathrm{d}(\ddot{n},\dot{m})\ \widehat{\mathcal{A}.}\neq\emptyset\ and\ \widehat{\mathcal{B}}.\neq\emptyset\end{cases}$$

Let $\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{\mathcal{C}}\in 2^{\Re^n}$ such that $\ddot{n}=(n_1,n_2,\dots,n_n),\dot{m}=(m_1,m_2,\dots,m_n)\in\Re^n$,

(h1) If $\hat{\hat{\mathcal{A}}}\neq\emptyset\ and\hat{\hat{\mathcal{B}}}\neq\emptyset$,

$$\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\hat{\mathcal{B}}}d(\ddot{n},\ddot{m})=\inf_{\ddot{m}\in\hat{\mathcal{B}}}\inf_{\ddot{n}\in\hat{\mathcal{A}}}d(\ddot{m},\ddot{n})=\rho(\hat{\mathcal{B}},\hat{\mathcal{A}})$$

(h2) If $\hat{\mathcal{A}}=\emptyset,\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\emptyset\in\widehat{\mathcal{A}}.}\inf_{\ddot{m}\in\hat{\mathcal{B}}}\mathrm{d}(\emptyset,\ddot{m})=\infty$

If $\hat{\mathcal{B}}=\emptyset,\rho\left(\hat{\mathcal{A}},\hat{\mathcal{B}}\right)=\inf_{\ddot{n}\in\widehat{\mathcal{A}}.}\inf_{\emptyset\in\hat{\mathcal{B}}}\mathrm{d}(\ddot{n},\emptyset)=\infty$

(h3) If $\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\right)=0$, so $\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}}d(\ddot{n}\ddot{n},\ddot{m})=0$. Then $\ddot{n}=\ddot{m}\Longrightarrow\ddot{n}\in\hat{\mathcal{A}}\cap\widehat{\mathcal{B}}$.Hence $\hat{\mathcal{A}}\cap\widehat{\mathcal{B}}\neq\emptyset$.

(h4) $\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\cup\hat{\mathcal{C}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}\cup\hat{\mathcal{C}}}d(\ddot{n}\ddot{n},\ddot{m})=$

$$=\inf_{\ddot{n}\ddot{n}\in\hat{\mathcal{A}}}\left(\min\left\{\inf_{n\ddot{m}\in\widehat{\mathcal{B}}}d(n\ddot{n},\ddot{m}),\inf_{\ddot{m}n\in\hat{\mathcal{C}}}d(\ddot{n}\ddot{n},\ddot{m})\right\}\right)=$$

$$=\min\left\{\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}}d(n,m),\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\hat{\mathcal{C}}}d(n,m)\right\}=\min\left\{\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\right),\rho(\hat{\mathcal{A}},\hat{\mathcal{C}})\right\}$$

(h5) Let $\varepsilon,\eta\in[0,\infty]$

$$\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\right)=\inf_{\ddot{n}\in\hat{\mathcal{A}}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}}d(\ddot{n},\ddot{m})\leq\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\right)$$
$$=\inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}}d(\ddot{n}\ddot{n},\ddot{m})+\varepsilon+\eta$$
$$=\rho\left(\hat{\mathcal{A}}^{\varepsilon},\widehat{\mathcal{B}}^{\eta}\right)+\varepsilon+\eta.$$

Proposition 3.4: (1) Any (and all) of the following structures determine the finest (discrete) proximit space in set $\widetilde{\mathbb{X}}$:

1. Dh-functional: $\rho\colon 2^{\widetilde{\mathbb{X}}}\times 2^{\widetilde{\mathbb{X}}}\to[0,\infty]$ where $\forall\tilde{a}\in\widetilde{\mathbb{X}}$ and $\hat{\mathcal{A}},\widehat{\mathcal{B}}\subset\widetilde{\mathbb{X}}$,

$$\rho\left(\hat{\mathcal{A}},\widehat{\mathcal{B}}\right)=\begin{cases}0\ \tilde{a}\in\hat{\mathcal{A}}\cap\widehat{\mathcal{B}}\\ \infty\tilde{a}\notin\hat{\mathcal{A}}\ or\ \tilde{a}\notin\widehat{\mathcal{B}}\end{cases}$$

2. Tower: $(t_{\varepsilon})_{\varepsilon\in\Re^{+}}$ where $\forall\varepsilon\in\Re^{+}$and $\hat{\mathcal{A}},\widehat{\mathcal{B}}\subset\widetilde{\mathbb{X}}$, $t_{\varepsilon}\left(\hat{\mathcal{A}}\right)=\hat{\mathcal{A}},t_{\varepsilon}\left(\widehat{\mathcal{B}}\right)=\widehat{\mathcal{B}}$.

(2) Using any (or all) of the following structures, determine the coarsest (indiscrete or trivial) proximit space on a set $\widetilde{\mathbb{X}}$:

1. Dh-functional: $\rho\colon 2^{\widetilde{\mathbb{X}}}\times 2^{\widetilde{\mathbb{X}}}\to[0,\infty]$ such that $\forall\hat{\mathcal{A}},\widehat{\mathcal{B}}\subset\widetilde{\mathbb{X}}$,

$$\rho\left(\widehat{\mathcal{A}}.,\widehat{\mathcal{B}}\right)=\begin{cases}0\widehat{\mathcal{A}}.\neq\emptyset\ and\ \widehat{\mathcal{B}}\neq\emptyset\\ \infty\ \hat{\mathcal{A}}=\emptyset\ or\ \widehat{\mathcal{B}}=\emptyset\end{cases}$$

2. Tower: $(t_{\varepsilon})_{\varepsilon\in\Re^{+}}$ where $\forall\varepsilon\in\Re^{+}$ and $\forall\hat{\mathcal{A}}\subset\widetilde{\mathbb{X}}$,

$$t_{\varepsilon}\left(\hat{\mathcal{A}}\right)=\begin{cases}\widetilde{\mathbb{X}}\ \hat{\mathcal{A}}\neq\emptyset\\ \emptyset\ \hat{\mathcal{A}}=\emptyset\end{cases}$$

Proposition 3.5: Every metric space is proximit space**.**

Proof: Given a metric space $(\widetilde{\mathbb{X}},d)$, define $\rho_{d}\colon 2^{\widetilde{\mathbb{X}}}\times 2^{\widetilde{\mathbb{X}}}\to[0,\infty]$ by

$$\rho_{d}\left(\widehat{\mathcal{A}}.,\widehat{\mathcal{B}}\right)=\begin{cases}\infty,\hat{\mathcal{A}}=\emptyset\ or\ \widehat{\mathcal{B}}=\emptyset\\ \inf_{\ddot{n}\in.\hat{\mathcal{A}}}\inf_{\ddot{m}\in\widehat{\mathcal{B}}}d(\ddot{n},\ddot{m}),\widehat{\mathcal{A}}.\neq\emptyset\ and\ \widehat{\mathcal{B}}.\neq\emptyset\end{cases}$$

Then ρ_d is distance on $\widetilde{\mathbb{X}}$ and $(\widetilde{\mathbb{X}}, \rho_d)$ is said to be proximit distance generated by d or proximit metric space. Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}\hat{\mathcal{C}} \in 2^{\widetilde{\mathbb{X}}}$

1. If $\widehat{\mathcal{A}}. \neq \emptyset$ *and* $\widehat{\mathcal{B}}. \neq \emptyset$
$\rho_{\mathrm{d}}(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.) = \inf_{\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\hat{\mathcal{B}}} \mathrm{d}(\ddot{n}, \ddot{m}) = \inf_{\ddot{m}\in\hat{\mathcal{B}}} \inf_{\ddot{n}\in\hat{\mathcal{A}}} \mathrm{d}(\ddot{m}, \ddot{n}) = \rho_{\mathrm{d}}(\widehat{\mathcal{B}}, \hat{\mathcal{A}})$.
If $\hat{\mathcal{A}} = \emptyset$, $\rho_d(\emptyset, \widehat{\mathcal{B}}) = \infty$ and if $\widehat{\mathcal{B}} = \emptyset$, $\rho_d(\hat{\mathcal{A}}, \emptyset) = \infty$.
2. If $\rho_d(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = 0$, so $\inf_{\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}} d(\ddot{n}\ddot{n}, \ddot{m}) = 0$. Then$\ddot{n}\ddot{n} = \ddot{m} \Longrightarrow \ddot{n} \in \hat{\mathcal{A}} \cap \widehat{\mathcal{B}}$. Hence $\hat{\mathcal{A}} \cap \widehat{\mathcal{B}} \neq \emptyset$.
3. If $\widehat{\mathcal{A}}. \neq \emptyset, \widehat{\mathcal{B}} \neq \emptyset$. Then

$$\rho_d(\widehat{\mathcal{A}}., \widehat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \inf_{\ddot{n}\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}\cup\hat{\mathcal{C}}} d(\ddot{n}\ddot{n}, \ddot{m})$$

$$= \inf_{\ddot{n}\ddot{n}\in\hat{\mathcal{A}}} \left(\min\left\{\inf_{\ddot{m}\in\widehat{\mathcal{B}}} d(\ddot{n}\ddot{n}, \ddot{m}), \inf_{\ddot{m}\in\hat{\mathcal{C}}} d(\ddot{n}\ddot{n}, \ddot{m})\right\}\right)$$

$$= \min\left(\inf_{\ddot{n}\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}} d(\ddot{n}\ddot{n}, \ddot{m}), \inf_{\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\hat{\mathcal{C}}} d(\ddot{n}\ddot{n}, \ddot{m})\right).$$

$$= \min\{\rho_d(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho_d(\hat{\mathcal{A}}, \hat{\mathcal{C}})\}$$

If $\hat{\mathcal{A}} = \emptyset$. We get
$\rho_{\mathrm{d}}(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.\cup \widehat{\mathcal{C}}.) = \infty = \min\{\infty, \infty\} = \min\{\rho_d(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}), \rho_d(\hat{\mathcal{A}}., \widehat{\mathcal{C}}.)\}$.
If $\widehat{\mathcal{B}}. = \emptyset$. Then
$\rho_d(\hat{\mathcal{A}}, \widehat{\mathcal{B}} \cup \mathcal{C}) = \infty = \min\{\infty, \infty\} = \min\{\rho_d(\hat{\mathcal{A}}, \widehat{\mathcal{B}}), \rho_d(\hat{\mathcal{A}}, \mathcal{C})\}$.

5. Let $\varepsilon, \eta \in [0, \infty], \widehat{\mathcal{A}}., \widehat{\mathcal{B}}. \in 2^{\mathbb{X}}$. If $\widehat{\mathcal{A}}. \neq \emptyset$ and $\widehat{\mathcal{B}}. \neq \emptyset$

$$\rho_d(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.) = \inf_{\ddot{n}\in\hat{\mathcal{A}}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}} d(n, m) \leq \inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \ddot{m}) + \varepsilon + \eta$$

$$= \rho_d(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta.$$

If $\hat{\mathcal{A}} = \emptyset$, then $\rho_d(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \infty$ and

$$\rho_d(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta = \inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \ddot{m}) + \varepsilon + \eta$$

$$\leq \inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \ddot{m}) + \varepsilon^n + \eta^n = \infty.$$

If $\widehat{\mathcal{B}} = \emptyset$, we get $\rho_d(\hat{\mathcal{A}}^{\varepsilon}, \widehat{\mathcal{B}}^{\eta}) + \varepsilon + \eta = \inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \ddot{m}) + \varepsilon + \eta.$

$$\leq \inf_{\ddot{n}\in\hat{\mathcal{A}}^{\varepsilon}} \inf_{\ddot{m}\in\widehat{\mathcal{B}}^{\eta}} d(\ddot{n}, \ddot{m}) + \varepsilon^n + \eta^n = \infty$$

Theorem 3.6: *If* (t_ε): $\varepsilon \in \Re^+$ *is a tawer on* $\widehat{\mathbb{X}}.$, *then*
$\rho(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.) \coloneqq \min\{\inf.\{\varepsilon \in \Re.^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\widehat{\mathcal{B}}.)\}, \inf\{\gamma \in \Re^+ | \widehat{\mathcal{B}}. \subseteq t_\gamma(\widehat{\mathcal{A}}.)\}\}$
for all $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$ *is a Dh-functional on* $\widetilde{\mathbb{X}}$.

Proof: Let $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \in 2^{\widetilde{\mathbb{X}}}$

(h1) $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) = \min(\inf\{\varepsilon \in \Re^+ : \widehat{\mathcal{A}}. \subseteq t_\varepsilon(\widehat{\mathcal{B}}.)\}, \inf\{\gamma \in \Re^+ : \widehat{\mathcal{B}}. \subseteq t_\gamma(\widehat{\mathcal{A}}.)\})$

$$= \min(\inf\{\gamma \in \Re^+ | \ \widehat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\widehat{\mathcal{B}})\})$$

$$= \rho(\widehat{\mathcal{B}}, \hat{\mathcal{A}})$$

(h2) if $\hat{\mathcal{A}} = \emptyset$, we have

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \min(\inf\{\varepsilon \in \Re^+ | \emptyset \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\emptyset)\})$$
$$= \min(\inf\{\gamma \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{B}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{B}} \subseteq \emptyset\}) = \infty$$

If $\hat{\mathcal{B}} = \emptyset$, we have

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \min(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\emptyset)\}, \inf\{\gamma \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{A}})\})$$
$$= \min(\inf\{\gamma \in \Re^+ | \hat{\mathcal{A}} \subseteq \emptyset\}, \inf\{\varepsilon \in \Re^+ | \emptyset \subseteq t_\gamma(\hat{\mathcal{A}})\}) = \infty.$$

(h3) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, if $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0$. We get

$$\min(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}) = 0,$$

then $(\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}) = 0$. Thus

$$\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}})\} = 0 \;\&\; \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\} = 0.$$

So that $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) = 0$ and $\hat{\mathcal{B}} \subseteq \hat{\mathcal{A}} \subseteq t_\gamma(\hat{\mathcal{A}}) = 0$. Take $\gamma, \varepsilon = 0$, then $\hat{\mathcal{A}} \subseteq t_0(\hat{\mathcal{B}}) \subseteq \hat{\mathcal{B}}$ and $\hat{\mathcal{B}} \subseteq t_0(\hat{\mathcal{A}}) \subseteq \hat{\mathcal{A}}$. So that $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq \emptyset$.

(h4) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$,

$$\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) = \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}} \cup \hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}}. \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) \cup t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{B}}) \cup t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{A}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}}. \subseteq t_\gamma(\hat{\mathcal{A}})\}, \inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\hat{\mathcal{C}})\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$= \min\{\rho(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.), \rho(\widehat{\mathcal{A}}., \hat{\mathcal{C}}.)\}$$

(h5) Let $\widehat{\mathcal{A}}., \widehat{\mathcal{B}}. \in 2^{\tilde{\mathbb{X}}}$, let $\theta, \vartheta \in [0, \infty]$

$$\rho(\widehat{\mathcal{A}}., \widehat{\mathcal{B}}.) = \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\widehat{\mathcal{B}}.)\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}} \subseteq t_\gamma(\hat{\mathcal{A}})\}\}$$
$$\leq \min\{\inf\{\varepsilon \in \Re^+ | \hat{\mathcal{A}} \subseteq t_\varepsilon(\widehat{\mathcal{B}}^\theta)\}, \inf\{\gamma \in \Re^+ | \hat{\mathcal{B}}. \subseteq t_\gamma(\hat{\mathcal{A}}^\vartheta)\}\} + \theta + \vartheta$$
$$= \rho(\hat{\mathcal{A}}^\theta, \hat{\mathcal{B}}^\vartheta) + \theta + \vartheta.$$

Theorem 3.7: *If* $\rho: 2^{\tilde{\mathbb{X}}} \times 2^{\tilde{\mathbb{X}}} \longrightarrow [0, \infty]$ *is an Dh-functional on* $\tilde{\mathbb{X}}$ *for all* $\varepsilon, \gamma \in \Re^+$. *Then the family* t_ε *defined by* $t_\varepsilon(\hat{\mathcal{A}}) = min\{\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{A}}^\gamma\}$ *is a tower on* $\tilde{\mathbb{X}}$.

Proof:

(To1) Let $\widehat{\mathcal{A}}. \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon \in \Re^+$. Suppose that $\tilde{a} \in \hat{\mathcal{A}}$, since $\widehat{\mathcal{A}}. \subseteq \hat{\mathcal{A}}^\varepsilon$ such that $t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{A}}^\varepsilon$. We have $\hat{\mathcal{A}}^\varepsilon \subseteq t_\varepsilon(\hat{\mathcal{A}})$.

(To2) Let $\varepsilon \in \Re^+ \hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$. If $\widehat{\mathcal{A}}. = \emptyset$, we obtain

$$t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{A}}^\varepsilon = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\}.$$

But $\hat{\mathcal{A}} = \emptyset$, then $\rho(\{\tilde{a}\}, \emptyset) = \infty$. Hence $t_\varepsilon(\emptyset) = \emptyset$.

(To3) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon \in \Re^+$.

Assume $\tilde{a} \in t_\varepsilon(\hat{\mathcal{A}}. \cup \hat{\mathcal{B}})$, then $\tilde{a} \in (\hat{\mathcal{A}} \cup \hat{\mathcal{B}})^\varepsilon$. So that

$$(\hat{\mathcal{A}} \cup \hat{\mathcal{B}})^\varepsilon = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\mathfrak{x}\}, \hat{\mathcal{A}} \cup \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}}: \min\{\rho(\{\tilde{a}\}, \hat{\mathcal{A}}), \rho(\{\tilde{a}\}, \hat{\mathcal{B}})\} \leq \varepsilon\}$$

$$= \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon, \rho(\{\tilde{a}\}, \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\} \cup \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{B}}) \leq \varepsilon\}$$
$$= t_\varepsilon(\widehat{\mathcal{A}.}) \cup t_\varepsilon(\widehat{\mathcal{B}})$$

(To4) Let $\hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$, let $\varepsilon, \gamma \in \Re^+$. Assume $\tilde{a} \in t_\varepsilon(t_\gamma(\hat{\mathcal{A}}))$, then $\tilde{a} \in ((\mathcal{A})^{(\gamma)})^{(\varepsilon)}$ such that

$$((\hat{\mathcal{A}})^{(\gamma)})^{(\varepsilon)} = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}^\gamma) \leq \varepsilon\} = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon + \gamma\}$$
$$= t_{\varepsilon+\gamma}(\hat{\mathcal{A}})$$

(To5) Let $\hat{\mathcal{A}} \in 2^{\tilde{\mathbb{X}}}$, $\varepsilon \in \Re^+$. Assume that $\tilde{a} \in t_\varepsilon(\widehat{\mathcal{A}.})$, then $\tilde{a} \in \hat{\mathcal{A}}^\varepsilon$ such that $\hat{\mathcal{A}}^\varepsilon = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\}$. Take $\varepsilon < \gamma$, since $t_\varepsilon(\hat{\mathcal{A}}) = \min\{\hat{\mathcal{A}}^\varepsilon, \hat{\mathcal{A}}^\gamma\}$. Then $\tilde{a} \in \bigcap_{\varepsilon<\gamma} t_\gamma(\hat{\mathcal{A}})$.

Lemma 3.8: *Let $\tilde{\mathbb{X}}$ be a set and ρ: $2^{\tilde{\mathbb{X}}.} \times 2^{\tilde{\mathbb{X}}.} \to [0, \infty]$, then the next statements are equivelent*:

(1) ρ is a Dh-functianal on$\tilde{\mathbb{X}}$,

(2) ρ fulfills (h1)-(h4) and
(h5.1) for all $\hat{\mathcal{A}}, \hat{\mathcal{B}}. \in 2^{\tilde{\mathbb{X}}}, \forall \varepsilon \in [0, \infty]$: $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}.) \leq \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}^{(\varepsilon)}) + \varepsilon$,

(3) ρ fulfills (h1)-(h4) and (h 5.2) $\forall \widehat{\mathcal{A}.}, \widehat{\mathcal{B}.}, \widehat{\mathcal{C}.} \in 2^{\tilde{\mathbb{X}}.}$: $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}.}) \leq \rho(\hat{\mathcal{A}}, \widehat{\mathcal{C}.}) + \sup_{\mathrm{cj} \in \hat{\mathcal{C}}} \rho(\{\mathrm{c}\}, \widehat{\mathcal{B}})$,

(4) ρ satisfies (h1)-(h4) and (h5.3) $\forall A\hat{}, \widehat{\mathcal{B}.}, \hat{\mathcal{C}}, \widehat{\mathcal{D}} \in 2^{\tilde{\mathbb{X}}.}$:
$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}.}) \leq \rho(\widehat{\mathcal{D}.}, \hat{\mathcal{C}}) + \sup_{s \in \hat{\mathcal{C}}} \rho(\{s\}, \widehat{\mathcal{B}}) + \sup_{d \in \widehat{\mathcal{D}}} \rho(\{d\}, \hat{\mathcal{A}}).$$

Proof: (1)⟹(2) From (1), we can deduce that (h1)-(h4) is fulfilled. We demonstrate that (h5.1) is valid.

Since $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}.}) \leq \rho(\hat{\mathcal{A}}^\varepsilon, \widehat{\mathcal{B}}^\eta) + \varepsilon + \eta \forall \eta \in [0, \infty]$, but $\hat{\mathcal{A}} \subseteq \hat{\mathcal{A}}^\varepsilon, \widehat{\mathcal{B}.} \subseteq \widehat{\mathcal{B}}^\eta$.

Thus $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) \leq \rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) + \varepsilon + \eta$, since $\eta \in [0, \infty]$ *and* $\widehat{\mathcal{B}} \coloneqq \widehat{\mathcal{B}}^\varepsilon$. We find $\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}.}) \leq \rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}^\varepsilon) + \varepsilon$.

(2)⟹(1) Suppose that ρ is satisfy (h1)-(h4), we show that (h5).

Let $\hat{\mathcal{A}}, \widehat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}, \varepsilon, \gamma \in [0, \infty]$. Take $\hat{\mathcal{A}} \coloneqq \hat{\mathcal{A}}^\varepsilon$ and $\widehat{\mathcal{B}.} \coloneqq \widehat{\mathcal{B}}^\gamma$, we have
$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) \leq \rho(\mathcal{A}^\varepsilon, \widehat{\mathcal{B}}^\eta) + \varepsilon + \eta$$

(3) ⟹ (4) Let $\hat{\mathcal{A}}, \widehat{\mathcal{B}.}, \hat{\mathcal{C}}, \widehat{\mathcal{D}} \in 2^{\tilde{\mathbb{X}}}$, set $\hat{\mathcal{A}} \coloneqq \widehat{\mathcal{D}}$, we have (4).

(4) ⟹ (3) Given $\hat{\mathcal{A}}, \widehat{\mathcal{B}.}, \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$. Set $\widehat{\mathcal{D}} \subset \hat{\mathcal{A}}$, we obtain (h5.2).

(1) ⟹ (4) Suppose that (1) is satisfy and let $\hat{\mathcal{A}}, \widehat{\mathcal{B}}, \hat{\mathcal{C}}, \widehat{\mathcal{D}}$ belongs to $2^{\tilde{\mathbb{X}}}$ Take
$$\varepsilon = \inf\{\tau \in [0, \infty]: \widehat{\mathcal{D}.} \subset \hat{\mathcal{A}}^{(\tau)}\},$$
and
$$\gamma = \inf\{\tau \in [0, \infty]: \hat{\mathcal{C}} \subset \hat{\mathcal{A}}^{(\tau)}\}$$
Then
$$\rho(\hat{\mathcal{A}}, \widehat{\mathcal{B}}) \leq \rho(\hat{\mathcal{A}}^\varepsilon, \widehat{\mathcal{B}}^\gamma) + \varepsilon + \gamma \leq \rho(\widehat{\mathcal{D}}, \hat{\mathcal{C}}) + \sup_{c \in \hat{\mathcal{C}}} \rho(\{c\}, \widehat{\mathcal{B}}) + \sup_{d \in \widehat{\mathcal{D}}} \rho(\{d\}, \hat{\mathcal{A}}).$$

(4) $\Longrightarrow$ (1) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}, \varepsilon, \gamma \in [0, \infty]$, Take $\hat{\mathcal{D}} := \hat{\mathcal{A}}^{(\varepsilon)}, \hat{\mathcal{C}} := \hat{\mathcal{B}}^{(\gamma)}$

$$\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right) \leq \rho\left(\hat{\mathcal{D}}, \hat{\mathcal{C}}\right) + \sup_{c \in \hat{\mathcal{C}}} \rho\left(\{c\}, \hat{\mathcal{B}}\right) + \sup_{d \in \hat{\mathcal{D}}} \rho\left(\{d\}, \hat{\mathcal{A}}\right)$$

$$\leq \rho\left(\hat{\mathcal{A}}^{\varepsilon}, \hat{\mathcal{B}}^{\gamma}\right) + \varepsilon + \gamma.$$

Definition 3.9: *A function* $t_{\varepsilon}: 2^{\tilde{\mathbb{X}}} \to 2^{\tilde{\mathbb{X}}}$ *is called an Lo-tower on* $\tilde{\mathbb{X}}$ *such that* $\tilde{\mathbb{X}}$ *is a set and for all* $\varepsilon \in \Re^{+}$ *so that the following conditions are fulfilled:*

(Lo-T1) for all $\varepsilon \in \Re^{+}$: $t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \max_{\gamma > \varepsilon} t_{\gamma}(\hat{\mathcal{A}})$,

(Lo-T2) for each $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}, \forall \varepsilon, \gamma \in \Re^{+}$: $t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}} \wedge t_{\gamma}\left(\{\tilde{b}\}\right) = \hat{\mathcal{C}}$ for each $\tilde{b} \in \hat{\mathcal{B}}$, then $t_{\varepsilon+\gamma}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{C}}$.

Theorem 3.10: *If* $\tilde{\mathbb{X}}$ *is a set, then*

(1) *If* ρ *is an Dh-functional on* $\tilde{\mathbb{X}}$*, and for each* $\varepsilon \in \Re^{+}$ *such that* $t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}}$ *if and only if* $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leq \varepsilon$*, then* $(t_{\varepsilon})_{\varepsilon \in R^{+}}$ *isan L-tawer on* $\tilde{\mathbb{X}}$.,

(2) *If* $(t_{\varepsilon})_{\varepsilon \in R^{+}}$ *is Lo-tawer on* $\tilde{\mathbb{X}}$., *so* $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right) := inf\{\varepsilon \in \Re^{+}: t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}}\}$ *is Dh-functianal on* $\tilde{\mathbb{X}}$.

Proof: Let ρ be Dh-functional and let $\varepsilon \in \mathrm{R}^{+}$ such that $t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}}. \Leftrightarrow \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leq \varepsilon$.

(Lo $-$ T1) Let $\tilde{a} \in t_{\varepsilon}\left(\hat{\mathcal{A}}\right)$, $\tilde{a} \in \hat{\mathcal{B}}.$ then $\tilde{a} \in \hat{\mathcal{A}}^{\varepsilon}$ such that

$$\hat{\mathcal{A}}^{\varepsilon} = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\}.$$

Since $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leq \varepsilon$, then for $\varepsilon < \gamma$, $\max\{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon < \gamma\}$. Thus $\tilde{a} \in \max\{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \gamma\}$, then $\tilde{a} \in \max_{\gamma > \varepsilon} t_{\gamma}(\hat{\mathcal{A}})$. So

$$t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \max_{\gamma > \varepsilon} t_{\gamma}(\hat{\mathcal{A}}).$$

(Lo.T22) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}., \hat{\mathcal{C}} \in 2^{\tilde{\mathbb{X}}}$, *let* $\varepsilon, \gamma \in \Re^{+}$ *and* $\tilde{b} \in \hat{\mathcal{B}}.$ such that $t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}}. \wedge t_{\gamma}\left(\{\tilde{b}\}\right) = \hat{\mathcal{C}}$ if andonly if $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}.\right) \leq \varepsilon \wedge \rho\left(\{\tilde{b}\}, \hat{\mathcal{C}}\right) \leq \gamma$.

Suppose that $\tilde{a} \in t_{\varepsilon+\gamma}\left(\hat{\mathcal{A}}\right)$ implies $\tilde{a} \in \mathcal{A}^{\varepsilon+\gamma} = \{\tilde{a} \in \tilde{\mathbb{X}}: \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon + \gamma\}$, but ρ is symmetric. Then $\rho\left(\hat{\mathcal{B}}., \hat{\mathcal{A}}\right) \leq \varepsilon$ from condition (h4). We obtain $\rho\left(\hat{\mathcal{B}}., \hat{\mathcal{A}} \cup \hat{\mathcal{C}}\right) \leq \min\{\varepsilon, \gamma\}$, take $\{\tilde{a}\} \subseteq \hat{\mathcal{A}} \cup \hat{\mathcal{B}}., \gamma < \varepsilon$.

So that $\rho\left(\{\hat{b}\}, \{\tilde{a}\}\right) \leq \gamma\ \hat{b} \in \hat{\mathcal{B}}.$, hence $\rho\left(\{\tilde{a}\}, \{\hat{b}\}\right) \leq \gamma$. Therefore $\tilde{a} \in t_{\gamma}\left(\{\tilde{b}\}\right) = \hat{\mathcal{C}}$, implies that $\tilde{a} \in \hat{\mathcal{C}}$.

(h1) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, set $\hat{\mathcal{A}} = \hat{\mathcal{A}}^{\varepsilon}, \hat{\mathcal{B}} = \hat{\mathcal{B}}^{\varepsilon}$

$$\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}\right) = \inf\{\varepsilon \in \Re^{+}: t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \hat{\mathcal{B}}.\} = \inf\{\varepsilon \in \Re^{+}: \hat{\mathcal{A}}^{\varepsilon} = \hat{\mathcal{B}}\}$$

$$= \inf\{\varepsilon \in R^{+}: \hat{\mathcal{A}} = \hat{\mathcal{B}}^{\varepsilon}\} = \inf\{\varepsilon \in R^{+}: t_{\varepsilon}\left(\hat{\mathcal{B}}\right) = \hat{\mathcal{A}}\} = \rho(\hat{\mathcal{B}}, \hat{\mathcal{A}})$$

(h2) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in 2^{\tilde{\mathbb{X}}}$, if $\hat{\mathcal{A}} = \emptyset$, we get

$$\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}\right) = \inf\{\varepsilon \in \Re^{+}: t_{\varepsilon}(\emptyset) = \hat{\mathcal{B}}\} = \inf\left\{\varepsilon \in \Re^{+}: \emptyset = \hat{\mathcal{B}}\right\} = \infty.$$

If $\hat{\mathcal{B}} = \emptyset$, we have $\rho\left(\hat{\mathcal{A}}, \hat{\mathcal{B}}\right) = \inf\{\varepsilon \in \Re^{+}: t_{\varepsilon}\left(\hat{\mathcal{A}}\right) = \emptyset\} = \infty$

(h3) Let $\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}.) = 0$, $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}$ so $\{0\} = \inf\{\varepsilon \in R^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}.\}$. Then $\inf_{\tilde{a}\in\hat{\mathcal{A}}} \rho(\{\tilde{a}\}, \hat{\mathcal{B}}.) = 0$ and $\inf_{\tilde{b}\in\hat{\mathcal{B}}} \rho(\hat{\mathcal{A}}, \{\tilde{b}\}) = 0$, so that $\tilde{a} \in \hat{\mathcal{B}}$ and $\tilde{a} \in \hat{\mathcal{A}}$. Hence $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} \neq \emptyset$.

(h4) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}. \in 2^{\hat{\mathbb{X}}}$,

$$\begin{aligned}\rho(\hat{\mathcal{A}}, \hat{\mathcal{B}} \cup \hat{\mathcal{C}}) &= \inf\{\varepsilon \in \Re^+ : t_\varepsilon(\hat{\mathcal{A}}.) =. \hat{\mathcal{B}} \cup \hat{\mathcal{C}}\} \\ &= \inf\{\varepsilon \in .\Re^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}. \cup\ t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\} \\ &= \inf\{\varepsilon \in \Re^+ : \min\{t_\varepsilon(\mathcal{A}) = \hat{\mathcal{B}}., t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\}\} \\ &= \min[\inf\{\varepsilon \in \Re^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}\}, \inf\{\varepsilon \in R^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{C}}\}] \\ &= \min\{\rho(\hat{\mathcal{A}}., \hat{\mathcal{B}}.), \rho(\hat{\mathcal{A}}., \hat{\mathcal{C}}.)\}\end{aligned}$$

(h5) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}. \in 2^{\hat{\mathbb{X}}.}$, let $\theta, \varepsilon \in [0\infty]$

$$\begin{aligned}&\rho(\hat{\mathcal{A}}., \hat{\mathcal{B}}) = \inf\{\varepsilon \in \Re^+ : t_\varepsilon(\hat{\mathcal{A}}) = \hat{\mathcal{B}}.\} = \inf\{\varepsilon . \in \Re^+ : \hat{\mathcal{B}} = \rho(\{\tilde{a}\}, \hat{\mathcal{A}}) \leq \varepsilon\} \\ &\leq \rho(\hat{\mathcal{A}}, \hat{\mathcal{B}}^\varepsilon) + \varepsilon + \theta.\end{aligned}$$

5. Conclusion

In order to study the concept of a new kind of proximit space, the relationship between Dh-functional and the tower, as well as certain fundamental facts and instances, are used. It is shown to also include proximit space-related theorems. We also show the relation between proximit and metric space.

References

[1] R.K. Abbas, B Y Hussein, New Results of Completion Normed A roach Space, *Iraqi Journal of Science*, 63(5), 2103-2113, DOI: 10.24996/ijs.2022.63.5.25, 2022.

[2] R.K. Abbas, B.Y Hussein, A new kind of topological vector space:Topological approach vector space, *AIP Conference Proceedings* 2386, 060008, https://doi.org/10.1063/5.0066971, 2022.

[3] N. Ackerman, M. L. Karker, Potential isomorphisms of generalized a roach spaces, *Quaestiones Mathematicae*, 45, (2022) https://doi.org/10.2989/16073606.2021.1883764.

[4] F. Afzal, Shah Rukh, D. Afzal, M.R.Farahani, M.Cancan, S. Ediz. A study of non-commutative Von-Neumann regular rings. *Journal of Information and Optimization Sciences*. 42(7), 1425-1436 (2021). https://doi.org/10.1080/02522667.2021.1896650.

[5] Z. Ahmad, M.A. Chaudhary, A.Q. Baig, M.A. Zahid. On metric dimension of P(n,2) *K1 graph. *Journal of Discrete Mathematical Sciences*

and Cryptography. 24(2), 629-645 (2021). https://doi.org/10.1080/09720529.2021.1907017.

[6] Z. Ahmad, M.A. Chaudhary, A.Q. Baig, M.A. Zahid. Fault-tolerant metric dimension of P(n,2)*K1 graph. *Journal of Discrete Mathematical Sciences and Cryptography.* 24(2), 647-656 (2021). https://doi.org/10.1080/09720529.2021.1899209.

[7] A. Alsinai, H.M.U. Rehman, Y. Manzoor, M. Cancan, Z. Taş, M.R. Farahani, Sharp upper bounds on forgotten and SK indices of cactus graph, *Journal of Discrete Mathematical Sciences and Cryptography*, 1-22 (2022). https://doi.org/10.1080/09720529.2022.2027605.

[8] R. Batra, R. Gupta, P. Sahni, Some results related to fixed points of F-G-contractions on metric spaces equied with w-distance, *Journal of Interdisciplinary Mathematics*, 23, (2020) https://doi.org/10.1080/09720502.2020.1731954.

[9] M. Edelstein, Metric Spaces: Iteration and A lication. By Victor Bryant, *The American Mathematical Monthly*, 95, (1988). https://doi.org/10.1080/00029890.1988.11972069.

[10] S. Ediz, İdris Çiftçi, M.Cancan, M.R.Farahani, On k-total distance degrees and k-total Wiener polarity index. *Journal of Information and Optimization Sciences,* 42(7), 1469-1477 (2021). https://doi.org/10.1080/02522667.2021.1896652.

[11] H. Herrlich, TopologieII: Uniforme Räume, *Verlag*, Berlin, (1988). Zbl 0644.54001.

[12] E. Lowen, R. Lowen, A quasitopos containing CONV and MET as full subcategories, *Internat. J. Math. and Math. Sci.* 11, 417-433, (1988). https://doi.org/10.1155/S016117128800050X.

[13] R. Lowen, Approach spaces: a common supercategory of TOP and MET, *Math. Nachr*. 14, 183-226, (1989). https://doi.org/10.1002/mana.19891410120.

[14] R. Lowen and K. Robey, Completion of product of metric spaces, *Quart. J. Math. Oxford* 43, 319-338, (1991). https://doi.org/10.1093/qmath/43.3.319.

[15] R. Lowen, M. Sioen, The Wijsman and Attouch-Wets topologies onhyperspaces revisited, *Topology A* l.70, 179-197, (1996). https://doi.org/10.1016/0166-8641(95)00096-8.

[16] R. Lowen, Approach spaces: the missing link in the topologies-uniformities-metrics, Oxford University Press, (1997).

[17] R. Lowen, M. Sioen, Proximal hypertopologies revisited, *Set-Valued Anal.*, 1-19, (1998). https://doi.org/10.1023/A:1008646106442.

[18] R. Lowen, M. Sioen, Weak representations of quantified hyperspace structures, *Topology A* l.104, 169-179, (2000). https://doi.org/10.1016/S0166-8641(99)00024-3.

[19] R. Lowen, M. Sioen, Approximations in functional analysis, *Result. Math.* 37, 345-372, (2000). https://doi.org/10.1007/BF03322005.

[20] R. Lowen, Index Analysis: Approach Theory at Work, Springer Monographs in Mathematics, Springer Verlag, (2015).

[21] M. Nadeem, S. Ahmad, M.K. Siddiqui, M.A. Ali, M.R.Farahani, A.J.M. Khalaf. On some applications related with algebraic structures through different well known graphs. *Journal of Discrete Mathematical Sciences and Cryptography.* 24(2), 2021.451-471. https://doi.org/10.1080/09720529.2021.1885806.

[22] S.A. Naimpally, B.D. Warrack, Proximity Spaces, Cambridge University Press, Cambridge, (1970).

[23] M.S. Sardar,M. Alaeiyan,M.R.Farahani,M.Cancan,S. Ediz. Resistance distance in some classes of rooted product graphs obtained by Laplacian generalized inverse method. *Journal of Information and Optimization Sciences* 42(7), 2021, 1447-1467. https://doi.org/10.1080/02522667.2021.1899210.

[24] M. Sioen, W. Van, D. Haute, An Alternative Description of Approach Spaces Via A roach Cores, A l. *Categor. Struct.*, 545-558, (2016). https://doi.org/10.1007/s10485-016-9447-x.

[25] Y. Sun, Y. Guo, M. Fuming , The reciprocity gap functional method for the inverse scattering problem for cavities, A licable Analysis, 95, (2016). https://doi.org/10.1080/00036811.2015.1064519.

[26] J. D. Victor, K.P. Purpura, Metric-space analysis of spike trains: theory, algorithms and a lication, *Network: Computation in Neural Systems,* 8, (1997). https://doi.org/10.1088/0954-898X_8_2_003.

[27] L. Wei, D. Zhang, Sober metric approach spaces, *Topology and its Applications* 233, 67-88, (2018). https://doi.org/10.1016/j.topol.2017.10.019.