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Weighted shift operators and extended eigenvalues

Kareem M. Hussein *
Buthainah A. Ahmed †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

Let H be a Hilbert space. A complex number μ is named the extended eigenvalue for an operator $T \in B(H)$, if there is operator not equal zero $X \in B(H)$ so that: $TX = \mu XT$ and X are named as extended eigen operator for an operator T opposite to μ . The goal of this work is to find extended eigenvalues and extended eigen operators for shift operators J, J_α, K, K_α such that $J: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ and $K: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by:

$$Je_n = e_{2n} \text{ , and } Ke_n = \begin{cases} e_{\frac{n}{2}} \ if \ n \ even \\ 0 \ \ if \ n \ odd \end{cases} \text{ , for all } x \in l^2(\mathbb{N}) .$$

Furthermore, the closeness of extended eigenvalues for all of these shift operators under multiplication has been proven.

Subject Classification: *00A05, 00A06.*

Keywords: *Extended eigenvalues, Extended eigen operators, Weighted shift operators.*

1. Introduction

Let H represent the complex Hilbert space and *B(H)* the algebra of all continuous linear operators on *H*. Assume *E(T)* represents the set of all extended eigenvalues and $E_\mu(T)$ the set of all extended eigen operators, i.e,

$$E(T) = \{ \mu \in \mathbb{C} : TX = \mu XT \, ; 0 \neq X \in B(H)\}$$
$$E_\mu(T) = \{0 \neq X \in B(H) \, ; TX = \mu XT \}$$

He was the first to give the idea of this definition in research [1] to expand the Lomonosov theorem. S. Petrovic and Lampert in [1], they took the integral

* *E-mail:* kmhsn780@gmail.com (Corresponding Author)
† *E-mail:* buthaina.a@sc.uobaghdad.edu.iq

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Volterra operator on the space $L^2(0,1)$. Many matters have been addressed about having of non-unimodular extended eigenvalues leads to invariant or hyper in variant subspaces. S. Petrovic and Biswas in [3], given the kinship between *E(T)* of an operator T and its powers. Cassier. G and H. Alkango in [4], they described the sets of *E(T)* and extended eigen operators for the product of a positive and self-adjoint operator which are both one-one. L. K. Shaakir and A. A. Hijab [6], calculated *E(T)* of the Unilateral shift operator on the l^2 space. Leon and Petro Vic. [1], gave a complete solution for the extended eigenvalues calculation the composition operators c_φ resulting from the Hardy space $H^2(\mathbb{D})$.

2. The Main Results

Lets start by giving the definitions we need in the future in the calculations.

Definition 2.1: Let $J: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be shift operator defined by:
$Je_n = e_{2n}$, $i.e, J(x_1, x_2, x_3, \dots) = (0, x_1, 0, x_2, 0, x_3, \dots)\ \forall x \in l^2(\mathbb{N})$ ... (1)

Definition 2.2: Let $K: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be shift operator defined by:

$$Ke_n = \begin{cases} e_{\frac{n}{2}} \ if\ n\ even \\ 0 \ \ if\ n\ odd \end{cases}$$

Thus:

$$K(x_1, x_2, x_3, x_4, \dots) = (x_2, x_4, x_6, \dots) \text{ for all } x \in l^2(\mathbb{N}) \dots \ (2)$$

Let $\{\alpha_n\}$ be a sequence of numbers such that $\alpha_n > 0$ for all $n \in \mathbb{N}$ and let $\lim_{n\to+\infty} \alpha_n = \alpha$, for all $x \in l^2(\mathbb{N})$,

Definition 2.3: Let $J_\alpha: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be weighted shift operator defined by: $J_\alpha e_n = \alpha_{2n} e_{2n}$, that is
$J_\alpha(x_1, x_2, x_3, \dots) = (0, \alpha_2 x_1, 0, \alpha_4 x_2, 0, \alpha_6 x_3, \dots)$ for all $x \in l^2(\mathbb{N})$ ... (3)

Definition 2.4: Let $K_\alpha: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ be weighted shift operator defined by:

$$K_\alpha e_n = \begin{cases} \alpha_{\frac{n}{2}}\, e_{\frac{n}{2}} \ if\ n\ even \\ 0 \ \ if\ n\ odd \end{cases}$$

Thus:

$$K_\alpha(x_1, x_2, x_3, x_4, \dots) = (\alpha_1 x_2, \alpha_2 x_4, \alpha_3 x_6, \dots)\ \forall\ x \in l^2(\mathbb{N}) \dots \ (4)$$

The objective of the following two theorems is to find *E(T)* for shift operators J and J_α .

Theorem 2.1: Suppose that $J: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is shift operator defined in (2.1), Then $E(J) = \mathbb{C}$.

Proof: **(Case 1)** Assume $|\mu| \geq 1$ and let $D_{\mu^{-1}}$ non-zero operator defined as it comes: for $x \in l^2(\mathbb{N})$,

$$x = (x_1 , x_2 , x_3 , \dots , x_i , \dots) = (x_1^1 , x_2^2 , x_3^3 , \dots , x_i^j , \dots),$$

for all i , j 1 where i is component name and j *is* component position

let $D_{\mu^{-1}} (x) = \begin{cases} \mu^{-i} x_i & if \ \ i = j \\ \mu^{-i-1} x_i & if \ \ i < j \end{cases}$. Claim that $JD_{\mu^{-1}} = \mu \, D_{\mu^{-1}} J$, To prove this claim, $JD_{\mu^{-1}} (x_1 , x_2 , x_3 , \dots)$ $since \ (i = j)$

$$= J(\mu^{-1} x_1 , \mu^{-2} x_2 , \mu^{-3} x_3 , \dots) = (0 , \mu^{-1} x_1 , 0 , \ \mu^{-2} x_2 , 0 , \ \mu^{-3} x_3 , \dots)$$

$$\mu \, D_{\mu^{-1}} J(x_1 , x_2 , x_3 , \dots) = \mu \, D_{\mu^{-1}} (0 , x_1 , 0 , \ x_2 , 0 , \ x_3 , \dots) \ \ since \ \ (i < j)$$

$$= \ \mu \, (0 , \mu^{-2} x_1 , 0 , \ \mu^{-3} x_2 , 0 , \ \mu^{-4} x_3 , \dots)$$

$$= (0 , \mu^{-1} x_1 , 0 , \ \mu^{-2} x_2 , 0 , \ \mu^{-3} x_3 , \dots)$$

Assume it must be $|\mu| \geq 1 \quad so \ that \ x \in l^2(\mathbb{N})$ space So $JD_{\mu^{-1}} = \mu \, D_{\mu^{-1}} J$.

(case 2) Assume $|\mu| \leq 1$ and let D_μ be a nother non-zero operator defined as:

for $x \in l^2(\mathbb{N})$, $x = (x_1 , x_2 , x_3 , \dots , x_i , \dots) = (x_1^1 , x_2^2 , x_3^3 , \dots , x_i^j , \dots)$,

for all i , j 1 where i is component name and j is component position let

$$D_\mu (x) = \begin{cases} \mu^i x_i & if \ \ i = j \\ \mu^{i-1} x_i & if \ \ i < j \end{cases}$$

Claim that $JD_\mu = \mu \, D_\mu J$, To prove this claim,

$$JD_\mu \, (x_1 , x_2 , x_3 , \dots) \ \ since \ \ (i = j)$$

$$= J(\mu x_1 , \mu^2 x_2 , \mu^3 x_3 , \dots) \ = (0 , \ \mu x_1 , 0 , \ \mu^2 x_2 , 0 , \ \mu^3 x_3 , \dots)$$

$$\mu \, D_\mu \, J \, (x_1 , x_2 , x_3 , \dots) = \ \mu D_\mu (0 , x_1 , 0 , \ x_2 , 0 , \ x_3 , \dots) \ since \ (i < j)$$

$$= \mu \, (0 , x_1 , 0 , \ \mu x_2 , 0 , \ \mu^2 x_3 , \dots)$$

$$= (0 , \ \mu x_1 , 0 , \ \mu^2 x_2 , 0 , \ \mu^3 x_3 , \dots)$$

Assume it must $|\mu| \leq 1 \ \ so \ x \in l^2(\mathbb{N})$and $JD_\mu = \mu \, D_\mu J$, Therefore $E(J) = \mathbb{C}$.

Theorem 2.2: *Suppose that* $J_\alpha : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *be Weighted shift operator defined in (2.3), Then* $E(J_\alpha) = \{ \mu \in \mathbb{C} : |\mu| \geq sup\{\alpha_i\} \ \forall \ i \geq 1 \}$.

Proof: Suppose that $|\mu| \geq \sup \{\alpha_i\} \ \forall \ i \geq 1$ and let $D_{\mu^{-1}}$ be the diagonal operator defined as it comes:

For $x \in l^2(\mathbb{N})$, $x = (x_1 , x_2 , x_3 , \dots , x_i , \dots) = \left(x_1^1 , x_2^2 , x_3^3 , \dots , x_i^j , \dots \right), \forall i ,$ $j \geq 1$ where i is component name and j is component position let:

$$D_{\mu^{-1}} (x) = \begin{cases} \mu^{-i} x_i & if \ \ i = j \\ \mu^{-i-1} x_i & if \ \ i < j \end{cases}$$

Claim that $J_\alpha \, D_{\mu^{-1}} = \mu \, D_{\mu^{-1}} J_\alpha$, To prove this claim,

$$J_\alpha D_{\mu^{-1}} (x_1 , x_2 , x_3 , \dots) \ \ since \ i = j$$

$$= \ J_\alpha \, (\mu^{-1} x_1 , \mu^{-2} x_2 , \mu^{-3} x_3 , \dots) = (0 , \alpha_2 \mu^{-1} x_1 , 0 , \alpha_4 \mu^{-2} x_2 , 0 , \alpha_6 \mu^{-3} x_3 , \dots)$$

$$\mu D_{\mu^{-1}} J_\alpha(x_1, x_2, x_3, \dots) = \mu D_{\mu^{-1}}(0, \alpha_2 x_1, 0, \alpha_4 x_2, 0, \alpha_6 x_3, \dots) \text{ since } i < j$$
$$= \mu(0, \alpha_2 \mu^{-2} x_1, 0, \alpha_4 \mu^{-3} x_2, 0, \alpha_6 \mu^{-4} x_3, \dots)$$
$$= (0, \alpha_2 \mu^{-1} x_1, 0, \alpha_4 \mu^{-2} x_2, 0, \alpha_6 \mu^{-3} x_3, \dots)$$

We assume that it must be $|\mu| \geq sup\{\alpha_i\}\ \forall\, i \geq 1$ *so that* $x \in l^2(\mathbb{N})$ space,

So $J_\alpha\, D_{\mu^{-1}} = \mu D_{\mu^{-1}} J_\alpha$, Therefore

$$E(J_\alpha) = \{\mu \in \mathbb{C} : |\mu| \geq sup\{\alpha_i\}\ \forall\, i \geq 1\}.$$

The objective of the following two theorems is to find *E(T)* for shift operators K and K_α .

Theorem 2.3: *Assume that* $K: l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *is shift operator defined in (2.2), then* $E(K) = \mathbb{C}$.

Proof: **(case 1)** Assume $|\mu| \geq 1$ and let $D_{\mu^{-1}}$ be a non-zero operator defined as it comes: for $x \in l^2(\mathbb{N})$,
$x = (x_1, x_2, x_3, \dots, x_i, \dots) = \left(x_1^1, x_2^2, x_3^3, \dots, x_i^j, \dots\right)$, for all $i\,,\ j \geq 1$
where i is component name and j is component position let :

$$D_{\mu^{-1}}(x) = \begin{cases} \mu^{-i} x_i & if\ \ i = j \\ \mu^{-i-1} x_i & if\ \ i > j \end{cases}$$

Claim that $KD_{\mu^{-1}} = \mu\, D_{\mu^{-1}} K$, To prove this claim,

$$KD_{\mu^{-1}}(x_1, x_2, x_3, \dots) = K(\mu^{-1}x_1, \mu^{-2}x_2, \mu^{-3}x_3, \dots)$$
$$= (\mu^{-2}x_2, \mu^{-4}x_4, \mu^{-6}x_6, \dots).$$
$$\mu\, D_{\mu^{-1}} K(x_1, x_2, x_3, \dots) = \mu\, D_{\mu^{-1}}(x_2, x_4, x_6, \dots) \qquad since\ \ i > j$$
$$= \mu(\mu^{-3}x_2, \mu^{-5}x_4, \mu^{-7}x_6, \dots)$$
$$= (\mu^{-2}x_2, \mu^{-4}x_4, \mu^{-6}x_6, \dots)$$

let it must $|\mu| \geq 1$ *so* $x \in l^2(\mathbb{N})$ space and $KD_{\mu^{-1}} = \mu\, D_{\mu^{-1}} K$.

(case 2) Assume $|\mu| \leq 1$ and let D_μ be a nother non-zero operator defined as it comes: for
$x \in l^2(\mathbb{N})$, $x = (x_1, x_2, x_3, \dots, x_i, \dots) = (x_1^1, x_2^2, x_3^3, \dots, x_i^j, \dots)$, for all i, $j \geq 1$ where i is component name and j is component position let:

$$D_\mu(x) = \begin{cases} \mu^i x_i & if\ \ i = j \\ \mu^{i-1} x_i & if\ \ i > j \end{cases}$$ Claim that $KD_\mu = \mu\, D_\mu K$,

$KD_\mu(x_1, x_2, x_3, \dots)$ *since* $i = j$

$= K(\mu x_1, \mu^2 x_2, \mu^3 x_3, \dots) = (\mu^2 x_2, \mu^4 x_4, \mu^6 x_6 \dots)$
$\mu\, D_\mu K(x_1, x_2, x_3, \dots) = \mu\, D_\mu(x_2, x_4, x_6, \dots)$ *since* $i > j$
$= \mu(\mu x_2, \mu^3 x_4, \mu^5 x_6, \dots) = (\mu^2 x_2, \mu^4 x_4, \mu^6 x_6, \dots)$

We assume that it must be $|\mu| \leq 1$ *so that* $x \in l^2(\mathbb{N})$ space, So $KD_\mu = \mu D_\mu K$, Therefore $E(K) = \mathbb{C}$.

Theorem 2.4: *Suppose that* $K_\alpha : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ *is Weighted shift operator defined in (2.4), Then* $E(K_\alpha) = \{\mu \in \mathbb{C} : |\mu| \geq sup\{\alpha_i\} \forall i \geq 1\}$.

Proof: Suppose that $|\mu| \geq \sup\{\alpha_i\} \forall i \geq 1$ and let $D_{\mu^{-1}}$ be the diagonal operator defined as follow: for $x \in l^2(\mathbb{N})$, $x = (x_1, x_2, x_3, \dots, x_i, \dots) = \left(x_1^1, x_2^2, x_3^3, \dots, x_i^j, \dots\right)$ for all i, $j \geq 1$ where i is component name and j is component position let:

$$D_{\mu^{-1}}(x) = \begin{cases} \mu^{-i} x_i & if \ i = j \\ \mu^{-i-1} x_i & if \ i > j \end{cases}$$

Claim that $K_\alpha D_{\mu^{-1}} = \mu D_{\mu^{-1}} K_\alpha$, To prove this claim,

$$K_\alpha D_{\mu^{-1}}(x_1, x_2, x_3, \dots) \qquad since \ i = j$$
$$= K_\alpha(\mu^{-1}x_1, \mu^{-2}x_2, \mu^{-3}x_3, \mu^{-4}x_4, \dots)$$
$$= (\alpha_1\mu^{-2}x_2, \alpha_2\mu^{-4}x_4, \alpha_3\mu^{-6}x_6, \dots)$$
$$\mu D_{\mu^{-1}} K_\alpha(x_1, x_2, x_3, \dots) = \mu D_{\mu^{-1}}(\alpha_1 x_2, \alpha_2 x_4, \alpha_3 x_6, \dots) \quad since \ i > j$$
$$= \mu(\alpha_1\mu^{-3}x_2, \alpha_2\mu^{-5}x_4, \alpha_3\mu^{-7}x_6, \dots) = (\alpha_1\mu^{-2}x_2, \alpha_2\mu^{-4}x_4, \alpha_3\mu^{-6}x_6, \dots)$$

We assume that it must be $|\mu| \geq \sup\{\alpha_i\} \forall i \geq 1$ *so that* $x \in l^2(\mathbb{N})$. So, $K_\alpha D_{\mu^{-1}} = \mu D_{\mu^{-1}} K_\alpha$, $Therefore \ \ E(K_\alpha) = \{\mu \in \mathbb{C} : |\mu| \geq sup\{\alpha_i\} \forall i \geq 1\}$.

In the following theorems, we will show that the set of extended eigenvalues for shift operators (J, J_α, K, K_α) are closed under multiplication.

Theorem 2.5: *If J is the shift operator on* $l^2(\mathbb{N})$ *then,* $E(J)$ *is closed under multiplication, that is if* $\mu \in E(J)$ *and* $\lambda \in E(J)$ *then* $\mu\lambda \in E(J)$.

Proof: Suppose that $D_{\mu^{-1}}, D_{\lambda^{-1}}$ are two extended eigen operators for J corresponding to μ, λ respectively where $|\mu| \geq 1$ *and* $|\lambda| \geq 1$ defined as follow:

for $x \in l^2(\mathbb{N})$, $x = (x_1, x_2, x_3, \dots, x_i, \dots) = (x_1^1, x_2^2, x_3^3, \dots, x_i^j, \dots)$, $for \ all \ i, j \geq 1$ where i is component name, j is component position let :

$$D_{\mu^{-1}}(x) = \begin{cases} \mu^{-i} x_i & if \ i = j \\ \mu^{-i-1} x_i & if \ i < j \end{cases} \text{ and } D_{\lambda^{-1}}(x) = \begin{cases} \lambda^{-i} x_i & if \ i = j \\ \lambda^{-i-1} x_i & if \ i < j \end{cases}$$

Such that $JD_{\mu^{-1}} = \mu D_{\mu^{-1}} J$ and $JD_{\lambda^{-1}} = \lambda D_{\lambda^{-1}} J$

Now, $JD_{\mu^{-1}} D_{\lambda^{-1}} = \mu D_{\mu^{-1}} J D_{\lambda^{-1}} = \mu D_{\mu^{-1}} \lambda D_{\lambda^{-1}} J = \mu\lambda D_{\mu^{-1}} D_{\lambda^{-1}} J$,

So $\mu\lambda \in E(J)$, very clear that $D_{\mu^{-1}} D_{\lambda^{-1}}$ nonzero operator, since $D_{\mu^{-1}}, D_{\lambda^{-1}}$ non zero operators, for all $x \in l^2(\mathbb{N})$.

Proposition 2.6: Suppose that (K, K_α, J_α) are shift operators on $l^2(\mathbb{N})$ then all these operators are closed under multiplication.

Proof: *To prove* $\mu\lambda \in E(J_\alpha)$, let $D_{\mu^{-1}}, D_{\lambda^{-1}}$ in proposition (2,5) and by using the same method above, the proof is obtained.

But to prove that $\mu\lambda \in E(K)$ and $\mu\lambda \in E(K_\alpha)$ we know $D_{\mu^{-1}}, D_{\lambda^{-1}}$ in the following way:

$$D_{\mu^{-1}}(x) = \begin{cases} \mu^{-i}x_i & if \ \ i = j \\ \mu^{-i-1}x_i & if \ \ i > j \end{cases} \quad \text{and} \quad D_{\lambda^{-1}}(x) \begin{cases} \lambda^{-i}x_i & if \ \ i = j \\ \lambda^{-i-1}x_i & if \ \ i > j \end{cases}$$

and by using the same method above, the proof is obtained.

3. Conclusions

In this article, we calculate extended eigenvalues and extended eigen operators for shift operators J, J_α , K, K_α , In addition, we demonstrated the closure of eigenvalues under the multiplication currency for these operators.

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