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Weak Galerkin finite element method for the linear Schrodinger equation

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Abstract

The numerical technique for 2D time dependent linear Schrodinger equation is the subject of this work. The approximations are produced using the weak Galerkin finite element technique with continous and discrete FEM, on relay , using the backward Euler method in time. Using the elliptic projection operator, we provide L2 error speculation for continues and discretely weak Galerkin finite element.

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1. Introduction

The problem is consider in the next paper initial term value of a time dependent linear Schrodinger equation 2D : We're looking for a complex amount function $u = u(x, y, t)$ defined on $\Omega \times [0, T]$ satisfy

$$iu_t = -\Delta u + gu \, , (x, y, t) \in \Omega \times [0, T] \tag{1}$$

$$u(x, y, t) = 0 \, , (x, y, t) \in \partial\Omega \times [0, T] \tag{2}$$

$$u(x, y, 0) = u_0(x, y), u(x, y) \in \Omega \tag{3}$$

Where $i = \sqrt{-1}$ is the complex unit ,$\Omega \in R$. It is a convex polygonal field., Δ is usual Laplace operator and $u_0(x, y)$ is a given co-complex value

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initial data. Function $g(x, y)$ is given real valued and non-negative function bounded for all $(x, y) \in \Omega$.The Schrödinger equation may be found in many different fields of physics. Applications as quantum dynamic computation [1,2] and has gotten a lot of interest because of its usefulness as a model for various significant physical and chemical processes[3]. Numerical solution of Schrodinger equation has been proposed by a few authors. Dia (1999)[4] derives the three plane explicit scheme for solving Schrodinger equation with constant and with a variable parameter which are unconditionally stable. Rest of the research is. In Section 2, we define weak Galerkin space an introduce the semi-discrete WGFEM formulation, In Section 3, the error analysis of semi- discrete WGFEM. In Section 4, the error analysis of Fully –discrete WGFEM.(backward WGFEM).

2. Weak Galerkin Finite Element Space

We define the weak Galerkin space, Let

$$\emptyset_h = \{v = \{v^0, v^b\} \in W(K, j, l), \forall K \in T_h\} \tag{4}$$

and

$$\emptyset_h^0 = \left\{v = \{v^0, v^b\} \in \emptyset_h\,, v^b\big|_{\partial K \cap \partial \Omega} = 0, \forall K \in T_h\right\} \tag{5}$$

Then the Variational form $u(x, y, t)$ of equation (1.1) is defined as follows for all $t \in [0, T], find\ u(x, y, t) \in H_0^1(\Omega)$ such that

$$(iu_t, v) = (\nabla u, \nabla v) + (gu, v)\ \forall v \in H_0^1(\Omega), t > 0 \tag{6}$$
$$u(x, y, 0) = u_0(x, y)\,, (x, y) \in \Omega$$
$$u(x, y, t) = 0\ (x, y, t) \in \partial\Omega \times [0, T].$$

We define the norm as follow:

$$\|u\|_w = (\|\nabla u\|^2 + \|u\|^2)^{\frac{1}{2}} \tag{7}$$

Denoted by $P_r(K)$the collection of poly. on K with grade less than equal r. For every positive integer $r \geq 0$, let $V(K, r) \subset [P_r(K)]^2$ be a subspace of the space of vector-valued polynomials of grade r. Let $\emptyset_h$ be the appropriate piecewise finite element subspace in $H_0^1(\Omega)$, In general ,for every settled $t \in [0, T]$ and given $u(x, y, t) \in H_0^1(\Omega)$, we can offer its elliptic projection $\pi_h u(x, y, t) \in \emptyset_h$ such that

$$a_w(\pi_h u, v) = a_w(u, v), \forall v \in \emptyset^h \tag{8}$$

And the projection π_h is supposed to be present and it fulfills the following characteristics: For $q \in H(div, \Omega)$with moderate uniformity, $\pi_h q \in H(div, k)$ such that $\pi_h q \in V(K, r)$ for any $K \in T_h$ and satisfying

$$(\nabla. q, v^0)_K = (\nabla. \pi_h q, v^0)_K, \forall v^0 \in P_j(K^0) \tag{9}$$

$$\pi_h u(m) = u(m), m = \left(m_i, n_j\right)\ i = 1,2, .., k\ and\ j = 1,2, .., l \tag{10}$$

$$\|u - \pi_h u\| \leq Ch^{\mu+1}\|u\|_{\mu+1}\ 0 \leq \mu \leq r + 1 \tag{11}$$

Also there exist two projections are defined for each element $K \in T_h$,one is $Q_h u = \{Q_h^0 u, Q_h^b u\}$, the L^2 projection of $H^1(K)$on to $P_j(K^0) \times P_{j+1}(\partial K)$ and R_h the L^2 projection of $[L^2(K)]^2$on to $V(K, r)$.

It is simply to vision the following beneficial identity [5]:

$$\nabla_{d,r}(Q_h u) = R_h(\nabla u), \forall u \in H^1(K), \tag{12}$$

Such that $P_j(K^0)$ the collection of polynomials on K^0 with grade less than equal j, and $P_l(\partial K)$ the collection of polynomials on ∂K with grade less than equal l.

The WG FEM based on (1) Which is to find $u_h = \{u_h^0, u_h^b\} \in \emptyset_h(j, l)$ satisfying $u^b = Q^b \ on \ \partial\Omega$ such that

$$(iu_t, v^0) = a_w(u_h, v) \ \forall \ v = \{v^0, v^b\} \in \emptyset_h^0(j, l) \tag{13}$$

Where

$$a_w(u_h, v) = (\nabla_{d,r} u_h, \nabla_{d,r} v) + (g u_h, v^0) \tag{14}$$

Then the semi discrete WG FEM is find $u_h = \{u_h^0, u_h^b\} \in \emptyset_h$ such that

$$(iu_{h,t}, v^0) = a_w(u_h, v) , \forall v = \{v^0, v^b \} \in \emptyset_h^0 \tag{15}$$

3. The Error Analysis of semi –discrete WGFEM

In this part, we use the $L^2 - norm$ to calculate error speculation for the semi- discrete WGFEM

Lemma 3.1: *Let $u \in H^1(0, T, H^2(\Omega))$ be the solution of equation (1) such that*

$$(iu_t, v^0) = (\pi_h \nabla_{d,r} u, \nabla_{d,r} v) + (gu, v^0) \ \forall v \in \emptyset_h^0 \tag{16}$$

Lemma 3.2: *Let $u \in H^{1+r}(\Omega)$ $with \ r > 0$ and u_h be the solution of (1)and (15) respectively. $Q_h u = \{Q_h^0 u, Q_h^b u\}$, the $L^2 - projection$ from the exact solution $u(x, t)$.Then error equation for the WG FEM is:*

$$(i(u_h - Q_h u)_t, v^0) + a_w(u_h - Q_h u, v) = (\pi_h(\nabla_{d,r} u - R_h(\nabla u), \nabla_{d,r} v) + (g(u - Q_h^0 u), v^0) \tag{17}$$

Where $a_w(u_h - Q_h u, v) = (\pi_h(\nabla_{d,r} u) - R_h(\nabla u), \nabla_{d,r} v) + (g(u - Q_h^0 u), v^0)$

Theorem 3.1: *Let $u \in H^{1+r}\Omega)$$with \ r > 0$ be the exact solution and $u_h$be approximation solution of equation (15) denoted by $e = u_h - Q_h u$ then found a fixed C such that*

$$\|e\|^2 \leq \int_0^t C h^{2(r+1)} \|u\|_{1+r}^2 \tag{18}$$

4. The Error Analysis of Fully –Discrete WGFEM

In this part, we calculate error speculation to fully-discrete WGFEM in $L^2 - norm$. This section focuses on the backward Euler scheme for time discretization.

4.1 *Backward WGFEM*

Let $\tau = \frac{T}{N}$ be the time step of the interval $[0, T]$ the time nodes $s_j = j\tau$, $j = 0,1, \dots, N$ and time elements

$$M_j = (s_j, s_{j+1}), j = 0,1, \dots, N-1.$$

$u(x, y, t_n)$ is denoted by u^n to simplify the notation. The fully discrete finite element solution $u_h^n(x, y) \in \emptyset_h, 0 \leq n \leq N$ of equation (1) can be defined by

$$\left(i\underline{\partial_t}u_h^n, v^0\right) = \left(\nabla_{d,r}u_h^n, \nabla_{d,r}v\right) + (gu_h^n, v^0), \forall v \in \emptyset_h \tag{19}$$

Where

$$\underline{\partial_t}u_h^n = \frac{1}{\tau}(u_h^n - u_h^{n-1}).$$

Theorem 4.1: *Let u^n and u_h^n be the solution of (15) and (19) on relay ,then found a fixed C, separate of h such that*

$$\|e^n\| \leq \|e^0\| + C\tau h^{r+1}\|u^n\|_{1+r} + (C\tau^2 \int_0^{t_n} \|u_{tt}\|^2)^{\frac{1}{2}} \tag{20}$$

6. Conclusions

In this research, we introduced the weak Galerkin finite element method to solve the linear Schrodinger equation, the error was determined in two ways :semi –discrete and fully– discrete. In fully discrete WGFEM that has been addressed upon (backward).We found that the convergence order of the error $o(h^{r+1})$ is r+1 ,where r is the degree of the polynomial used in the research.

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