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The characteristic equation of the matrix over min-plus algebra

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Abstract

Max-plus algebra is one of many idempotent semi-rings. The max-plus algebraic structure is semi field while the conventional algebra is a field. Because of their similar structure, various properties and concepts in the conventional algebra such as characteristic equations have max plus algebraic equivalence. The characteristic equation has been proved in the max-plus algebra. The other semi-field is min-plus algebra. Because of the structure in the min-plus algebra is also similar to the conventional algebra, the characteristic equation also has a min-plus algebraic equivalent. In this paper, it is discussed how to prove the characteristic equation of the matrix over conventional algebra into the min-plus algebra. The results are almost the same. The addition and multiplication operations in the conventional algebra are replaced by min and plus operations in the min-plus algebra. In addition, because of the min-plus algebra does not define the subtraction operation, the formulation of the characteristic equation of the matrix over min-plus algebra is not equal to zero.

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1. Introductions

In the linear algebra we're concerned with system of n linear equation that are expressed in the form $Ax = \lambda x$, where λ is a scalar. The form $Ax = \lambda x$ can be written as $\lambda x - Ax = 0$, or by inserting an identity matrix and factoring we have $(\lambda I_n - A) = 0$. If A is a $n \times n$ matrix then $\det(\lambda I_n - A) = 0$ is the characteristic equation of A, where I_n is the identity matrix with degree n. The scalars that meet the characteristic equation are known as the eigenvalues of A. The determinant is always a polynomial $p(\lambda)$ which is known as the

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characteristic polynomial of A [1]. The characteristic polynomial $p(\lambda)$ of a $n \times n$ matrix has the form

$$(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^n - 1 + \ldots + c_n.$$

It follows from the Fundamental Theorem of Algebra that the characteristic equation is

$$\lambda n + c1\lambda n - 1 + \ldots + cn = 0,$$

has at most n distinct solutions, so a $n \times n$ matrix has at most n distinct eigenvalues [1]. The characteristic equation are related to Cayley-Hamilton theorem. According to the Cayley-Hamilton theorem, every square matrix A satisfies the characteristic equation for that matrix. [4]. Cayley-Hamilton theorem and the characteristic equation have been proved in the max-plus algebra. It is because the similarity structure of the max-plus algebra and the conventional algebra. Max-plus algebra is an idempotent semi-field, whereas conventional algebra is a field. The result is both of Cayley-Hamilton theorem and the characteristic equation have max-algebraic equivalent [2][4][7].

Min-plus algebra is another semi-field. It is defined as the set of $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ where $\mathbb{R}$ is the set of real number and $\varepsilon' = +\infty$, and is equipped with plus and min operation [8]. Because of the min-plus algebra and the max-plus algebra have the same structure, many concept of conventional algebra such as the characteristic equation, Cayley-Hamilton theorem, eigenvalue problem which have the max-algebraic equivalent also have min-algebraic equivalent. In the min-plus algebra, the Cayley-Hamilton theorem has been proven [6]. Then in this paper we will proved the characteristic equation in the min-plus algebra. We deserve a brief introduction to the characteristic equation before we prove it in the min-plus algebra. In section 2 we give some notation and definition that we will use to prove the characteristic equation, also we give some introduction to the min-plus algebra. The next section we show how to prove the characteristic equation in the min-plus algebra. We also give some examples of the characteristic equation in a different degree of a square matrix.

2. Preliminaries

2.1 *Min-Plus Algebra*

Min-plus algebra is the set of $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\}$ which is equipped with min ($\oplus'$) and plus ($\otimes'$) operation, where $\mathbb{R}$ is the set of real number and $\varepsilon' = +\infty$ [8]. Furthermore min-plus algebra can be stated as $\mathbb{R}_{min}$. If $a, b \in \mathbb{R}_{min}$, then the basic operation in min-plus algebra can be stated as follows

$$a \oplus' b = \min(a, b),$$
$$a \otimes' b = a + b,$$

Min ($\oplus'$) and plus ($\otimes'$) operations are associative and commutative, we have

$$a \oplus' (b \oplus' c) = (a \oplus' b) \oplus' c$$
$$a \otimes' (b \otimes' c) = (a \otimes' b) \otimes' c$$
$$a \oplus' b = b \oplus' a$$

$$a \otimes' b = b \otimes' a,$$

for all $a, b, c \in \mathbb{R}_{min}$. The zero element of min ($\otimes'$) operation is $\varepsilon' = +\infty$, we have

$$a \oplus' \varepsilon' = \varepsilon' \oplus' a = \min(a, +\infty) = a,$$

and the zero element of plus ($\oplus'$) operation is $e = 0$, we have

$$a \otimes' e = e \otimes' a = a + 0 = a.$$

The operation $\otimes'$ is distributive with respect to $\oplus'$:

$$a \otimes' (b \oplus' c) = (a \otimes' b) \oplus' (a \otimes' c).$$

The operation $\oplus'$ is idempotent, we can write $a \oplus' a = (a, a) = a$. Let $x \in \mathbb{R}_{min}$ and $k \in \mathbb{N}$, the k^{th} min-algebraic power of x is stated as follows

$$x^{\otimes' k} = \underbrace{x \otimes' x \otimes' \ldots \otimes' x}_{k \text{ times}}.$$

In $\mathbb{R}_{min}$ the k^{th}power of x reduce to the conventional multiplication $x^{\otimes' k} = kx$. Let $\mathbb{R}_{min}^{m \times n}$ is the set of all $m \times n$ matrices with all entries in $\mathbb{R}_{min}$, for m and n are positive integers [8]. The matrix E_n is the n-by-n identity matrix in the min-plus algebra: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon'$ for all i, j with $i \neq j$. We define some operations in $\mathbb{R}_{min}$ as follows [8].

1. If $A, B \in \mathbb{R}_{min}^{m \times n}$ then

$$(A \oplus' B)_{ij} = a_{ij} \oplus' b_{ij}$$

2. If $A \in \mathbb{R}_{min}^{m \times n}$ and $\alpha \in \mathbb{R}_{min}$ then we have

$$(\alpha \otimes' A)_{ij} = \alpha_{ij} \otimes' a_{ij}$$

3. If $A \in \mathbb{R}_{min}^{m \times k}$ and $B \in \mathbb{R}_{min}^{k \times n}$ then

$$[A \otimes' B]_{ij} = \oplus'^{k}_{\ell=1} \left(a_{i\ell} \otimes' b_{\ell j}\right)$$

4. If $A \in \mathbb{R}_{min}^{n \times n}$ and $k \in \mathbb{N}$, the k^{th} power of A is defined by

$$A^{\otimes' k} = \underbrace{A \otimes' A \otimes' \ldots \otimes' A}_{k \text{ times}}.$$

For $k = 0$ we set $A^{\otimes' 0} = E$ and $A^{\otimes' k} = A^{\otimes' k-1} \otimes' A$ for $k > 0$.

Notice that the definitions are equivalent to the definitions for the multiplication of two matrices in linear algebra, the addition of two matrices, and the multiplication of a matrix by a scalar, but with $+$ substituted by $\oplus'$ and $\times$ by $\otimes'$.

Definition 2.1: Given a matrix $A \in \mathbb{R}_{min}^{n \times n}$, the matrix e^{sA} has entries $e^{sa_{ij}}$ where $a_{ij} \in \mathbb{R}_{min}$ are the entries in A.

Definition 2.2: $\varepsilon^{m} + \varepsilon^{n} \asymp \varepsilon^{\min\{m,n\}}$

2.2 *Notation and Definition*

The characteristic equation in the min-plus algebra [2][3][4][5] is generalized using the following definition and theorem.

Definition 2.3: The determinant of $A = [a_{ij}]$, denoted by $\det(A)$ or $|A|$ is the sum of all permutation in S_n where is such product is multiplied by $sgn(\sigma)$, that is

$$|A| = \sum_{\sigma \in Sn} (sgn(\sigma)) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

with

$$\operatorname{dom}(A) = \begin{cases} 1, & \text{if } \sigma \text{ is even;} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

We introduce the $n \times n$ matrix z^A where $z^{a_{ij}}$ is element (i,j) in order to define the dominant (dom) of the $n \times n$ matrix A as follows.

$$dom(A) = \begin{cases} lowest\ exponent\ in \det(z^A), if \det(z^A) \neq 0 \\ \varepsilon' \end{cases}$$

Definition 2.4: If $p : (0, \infty)$ and $u \in [-\infty, \infty)$, we define $p \asymp e^{su}$ *to* mean $lim_{s\to\infty} s^{-1} \, ln(p) = u.$

Theorem 2.5: If $f \asymp e^{sm}$ and $g \asymp e^{sn}$, then

$$f + g \asymp e^{s(m \oplus' n)} \text{ and } fg \asymp e^{s(m \otimes' n)}.$$

Proof: We have that

$$\lim_{s\to\infty} s^{-1} \ln(fg) = \lim_{s\to\infty} s^{-1} \ln(g) + \lim_{s\to\infty} s^{-1} \ln(g) = m + n = m \otimes' n.$$

Thus $fg \asymp e^{s(a \otimes' b)}$. Now notice that $\min(f,g) \leq f + g \leq 2\min(f,g)$. By using Definition 2.1.1 we obtain

$$\begin{aligned} \lim_{s\to\infty} s^{-1} \ln(\min(e^{sm}, e^{sn})) &\leq \lim_{s\to\infty} s^{-1} \ln(e^{sm} + e^{sn}) \\ &\leq \lim_{s\to\infty} s^{-1} \ln[\min(e^{sm}, e^{sn}) + \ln(2)]. \end{aligned}$$

Since $s^{-1} \ln(\min(e^{sm}, e^{sn})) = \min(m,n)$, by implementing the squeeze theorem we have

$$\lim_{s\to\infty} s^{-1} \ln(f + g) = \min(m,n) = m \oplus' n.$$

3. The Characteristic Equation in the Min-Plus Algebra

In the conventional algebra, the characteristic equation of a square matrix A defined by $\det(\lambda I_n - A = 0)$, where I_n is the identity matrix with degree n. Let $A \in \mathbb{R}^{n\times n}$ and $\varphi \subset \{1,2,\ldots,n\}$ then $A_{\varphi\varphi}$ is the sub matrix of A acquired by eliminating all columns and rows of A except those denote by φ. Furthermore the characteristic equation in the conventional algebra can be write as follows:

$$\det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + \ldots\ c_{n-1}\lambda + c_n = 0 \qquad (3.1)$$

where $c_k = (-1)^k \sum_{\sigma \in C_n^k} \det(A_{\varphi\varphi})$ and C_n^k is the set of all subsets of k elements of the sets $\{1,2,\ldots,n\}$.

The characteristic equation can be transformed into the min-plus algebra, which is what we will demonstrate next. Let $A \in \mathbb{R}_{min}^{n\times n}$, then z^A is a real $n \times n$ matrix-valued function that is defined by $(z^A)_{ij} = z^{a_{ij}}$ for all i, j. Then the definition of the dominant of A is

$$\mathrm{dom}(A) = \begin{cases} \text{lowest exponent in } \det(z^A), & \text{if } \det(z^A) \neq 0 \\ \varepsilon', & \text{otherwise.} \end{cases}$$

In light of Theorem 2.5 and leading of Definition 2.1, we replace z by e^s. Then the dominant of A can be written as

$$\mathrm{dom}(A) = \begin{cases} \lim\limits_{s\to\infty} \frac{1}{s} \ln|\det(e^{sA})|, & \text{if } \det(e^{sA}) \not\equiv 0 \\ \varepsilon', & \text{if } \det(e^{sA}) \equiv 0\,. \end{cases}$$

According to Definition 2.4, this says that $|\det(e^{sA})| \asymp e^{s dom(A)}$.
The matrix-valued function e^{sA} has the following characteristic polynomial

$$\det(\lambda I_n - e^{sA}) = \lambda^n + \gamma(s)\lambda^{n-1} + \dots \gamma_{n-1}(s)\lambda + \gamma_{n(s)} = 0 \qquad (3.2)$$

with coefficient

$$\gamma_k(s) = (-1)^k \textstyle\sum_{\varphi\in C_n^k} \det(e^{sA_{\varphi\varphi}})\,. \qquad (3.3)$$

Therefore the characteristic equation can be write as

$$\lambda^n + \gamma(s)\lambda^{n-1} + \dots \gamma_{n-1}(s)\lambda + \gamma_{n(s)} = 0 \qquad (3.4)$$

The lowest degree in (3.3) is given by

$$d_k = \min\{\zeta \in \Gamma_k : I_k(\zeta) \neq 0\},$$

with $\Gamma_k = \{\zeta : \exists (i_1, i_2, \dots, i_k) \text{ such that } \zeta = \sum_{r=1}^{k} a_{i_r i_{\rho(r)}}\}$ for $k = 1, 2, \dots, n$.

The values of $a_{i_r i_{\rho(r)}}$ determine the value of ζ for which $e^{s\zeta}$ appears in $\gamma_k(s)$. Let P_k be the set of all permutations. Let $I_k(\zeta)$ be the coefficient of $e^{s\zeta}$ for every $\zeta \in \Gamma_k$ with $k \in \{1, 2, \dots, n\}$. We have $I_k(\zeta) = I_k^e(\zeta) - I_k^o(\zeta)$, where $I_k^e(\zeta)$ is the number of $\rho \in P_k^e$ and $I_k^o(\zeta)$ is the number of $\rho \in P_k^o$, which P_k^e is the number of even permutations and P_k^o is the number of odd permutations. Then we have

$$\gamma_k(s) = (-1)^k \textstyle\sum_{\zeta\in\Gamma_k} I_k(s) e^{s\zeta}. \qquad (3.5)$$

Let $\bar{\gamma}_k = (-1)^k I(d_k)$ be the leading coefficient of $\gamma_k(s)$ for $k = 1, 2, \dots, n$. The terms k with positive $\bar{\gamma}_k$ and those with negative $\bar{\gamma}_k$ should be separated. Let $\ell = \{k : \bar{\gamma}_k > 0\}$ and $q = \{k : \bar{\gamma}_k < 0\}$. Then the coefficient of characteristic equation e^{sA} satisfy $\gamma_k(s) \asymp \bar{\gamma}_k(s) e^{s d_k}$.

There is no subtraction operation defined in the min-plus algebra. Hence, we replace the negative leading coefficient $\gamma_k(s)$ in (3.4) from the left-hand side to the right-hand side, then we have

$$\lambda^n + \sum_{k\in\ell} \bar{\gamma}_k(s) e^{s d_k} \lambda^{n-k} = \sum_{k\in q} \bar{\gamma}_k(s) e^{s d_k} \lambda^{n-k}.$$

Next replace λ with $e^{s\lambda}$, then we have

$$e^{s\lambda^n} + \sum_{k\in\ell} \bar{\gamma}_k(s) e^{sd_k} e^{s\lambda^{n-k}} = \sum_{k\in q} \bar{\gamma}_k(s) e^{sd_k} e^{s\lambda^{n-k}}.$$

Use the Definition 2.4, and the formulation becomes

$$\lim_{s\to\infty} s^{-1} \ln(e^{s\lambda^n} + \sum_{k\in\ell} \bar{\gamma}_k(s) e^{sd_k} e^{s\lambda^{n-k}}) = \lim_{s\to\infty} s^{-1} \ln(\sum_{k\in q} \bar{\gamma}_k(s) e^{sd_k} e^{s\lambda^{n-k}}).$$

Then use Theorem 2.5, we have

$$\min_{k\in\ell}(n\lambda, d_k + \big((n-k)\lambda\big) + \ln(\bar{\gamma}_k(s))) = \min_{k\in q} \begin{array}{c}(d_k + \big((n-k)\lambda\big) \\ + \ln(\bar{\gamma}_k(s)))\,.\end{array}$$

We can ignore $\ln(\bar{\gamma}_k(s))$, so we obtain the following formulation

$$\lambda^{\otimes' n} \oplus' \bigoplus\nolimits'_{k\in\ell} d_k \otimes' \lambda^{\otimes' n-k} = \bigoplus\nolimits'_{k\in \mathrm{q}} d_k \otimes' \lambda^{\otimes' n-k}.$$

The equation can be viewed as a min-algebraic form of the characteristic equation.

The following examples are to illustrate the characteristic equations.

Example 1: Let

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

According to the definition given above, $\Gamma_1 = \{-2, 0\}, \Gamma_2 = \{-2, 2\}$. We see that

$$I_1(-2) = 1,\ I_1(0) = 1,$$
$$I_2(-2) = -1,\ I_2(2) = 1.$$

Hence $d_1 = -2, d_2 = -2, \overline{\gamma}_1 = -1, \overline{\gamma}_2 = 1$, and $\ell = \{2\}, q = \{1\}$.

Therefore, the characteristic equation of A can be stated as

$$\lambda^{\otimes' 2} \oplus' (-2) = (-2) \otimes' \lambda.$$

Example 2: Let

$$A = \begin{pmatrix} -2 & 1 & \varepsilon' \\ 1 & 0 & 0 \\ \varepsilon' & -2 & 1 \end{pmatrix}.$$

We found that

$\Gamma_1 = \{-2, 0, 1\}, \Gamma_2 = \{-2, -1, 1, 2, \varepsilon'\}, \Gamma_3 = \{-4, -1, 0, 3\}$, with

$$I_1(-2) = 1, I_1(0) = 1, I_1(1) = 1$$
$$I_2(-2) = 0, I_2(-1) = 1, I_2(1) = 1, I_2(2) = -1, I_2(\varepsilon') = -1$$
$$I_3(-4) = -1, I_3(-1) = 1, I_3(0) = 1, I_3(3) = -1.$$

Hence $d_1 = -2, d_2 = -1, d_3 = -4, = -1, \bar{\gamma}_2 = 1, \bar{\gamma}_3 = 1,$

and $\ell = \{2, 3\}, q = \{1\}$.

Therefore, the characteristic equation of A can be stated as

$$\lambda^{\otimes'3} \oplus' (-1) \otimes' \lambda \oplus' (-4) = (-2) \otimes' \lambda \otimes' 2.$$

Example 3: Let

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 3 & 2 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 1 & -4 \end{pmatrix}$$

We found that $\Gamma_1 = \{-4, -1, 1\}, \Gamma_2 = \{-5, -3, 0, 2, 3, 4, 5\}$,

$\Gamma_3 = \{-4, -2, 1, 3, 4, 5, 7\}, \Gamma_4 = \{-3, 0, 3, 4, 5, 7\}$, with

$I_1(-4) = 1, I_1(-1) = 1, I_1(1) = 1$

$I_2(-5) = 0, I_2(-3) = 1, I_2(0) = 1, I_2(2) = -1, I_2(3) = -1, I_2(4) = -1$

$I_2(5) = -2$

$I_3(-4) = -1, I_3(-2) = -1, I_3(1) = 2, I_3(3) = 2, I_3(4) = 0, I_3(5) = 0$

$I_3(7) = -2$

$I_4(-3) = 1, I_4(0) = -1, I_4(3) = -1, I_4(4) = 1, I_4(5) = -1, I_4(7) = 1.$

Hence $d_1 = -4, d_2 = -5, d_3 = -4,\ d_4 = -3,\ \bar{\gamma}_1 = -1,\ \bar{\gamma}_2 = 1$,

$\bar{\gamma}_3 = 1, \bar{\gamma}_4 = -1$ and $\ell = \{2, 3\}, q = \{1, 4\}$.

Therefore, the characteristic equation of A can be stated as

$\lambda^{\otimes'4} \oplus' (-5) \otimes' \lambda^{\otimes'2} \oplus' (-4) \otimes' \lambda = (-4) \otimes' \lambda^{\otimes'3} \oplus' (-3) \otimes' E.$

4. Conclusion

We have shown the characteristic equation related to the min-plus algebra. We can say that the characteristic equation is also true in the min-plus algebra. The results of the formulation are almost the same. The addition and multiplication operations in conventional algebra are replaced by the min and plus operations in the min-plus algebra. In addition, because of the min-plus algebra does not define the subtraction operation, The characteristic equation of the matrix over min-plus algebra is not equal to zero according to its specification.

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