Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 5, pp. 829–834
DOI : 10.47974/JIM-1503

Studying the result involution graphs for HS and McL Leech lattice groups

Ali Talib Jawad *
Ali Abd Aubad †
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

Abstract

Let G be a finite group, the result is the involution graph of G, which is an undirected simple graph denoted by the group G as the vertex set and x, y ∈ G adjacent if xy and $(xy)^2 = 1$. In this article, we investigate certain properties of G, the Leech lattice groups HS and McL. The study involves calculating the diameter, the radius, and the girth of Γ_G^{RI}.

Subject Classification: *Primary 05C25, Secondary 20D08.*

Keywords: *Leech lattice Group, Result Involution Graph, Connectedness, Girth and Market share.*

1. Introduction

One of the most essential approaches to analyzing group structure is to use the action of a group on a graph. The wide references in [1,4 and 5] highlight the several distinct graphs that have been investigated for various groups. If an element of a group has order 2, it is considered an involution. The structure of a group is significantly influenced by its involutions. For example, the classification of finite simple groups depends heavily on the study of involutions. In [2] Devillers and Giudici introduced an S3-involution graph for a group G as a graph with a vertex set, all G-class involutions, and two vertices adjacent if they produce an S3-subgroup in a given G-class. In that

* *E-mail:* ali.taleb1203a@sc.uobaghdad.edu.iq (Corresponding Author)
† *E-mail:* ali.abd@sc.uobaghdad.edu.iq

paper, the structure of the S_3-involution graph for a group PSL (2,q), for q>3 has been analyzed along with the general properties of the graph.

Consider a finite group G and let I(G) be the set of all involution elements in G. The result involution graph, Γ_G^{RI} is an undirected simple graph that has G, as a vertex set, and two vertices are linked if they are distinct and their product in I(G). The result involution graph and its properties were originally presented by Asaad and Salih [3]. In their work, they showed that Γ_{Sn}^{RI} and Γ_{An}^{RI} are connected with diameter and radius at most 3 for n>4, as well as, girth equal to 3. This paper aims to study the result involution graphs for Higman–Sims Leech lattice group HS and McLaughlin Leech lattice group McL. In particular, we determine some features of the graph, such as the radius, the diameter, and the girth.

The following is how the paper is organized: In Section 2, for result involution graphs, we present some notations and basic results. In section 3, we establish certain results for a set of finite groups that include Leech lattice groups HS and McL. A variety of results graph attributes are offered. Finally, in section 4, we establish conclusions and make recommendations for further studies.

2. General Results

We start by defining certain terms and mentioning a few findings that will be useful in this study. From now on, we may let G be isomorphic to one of the Leech lattice groups HS or McL. Furthermore, we set $\mathbf{sI}_G = |I(G)|$ to be the size of the set I(G). We begin by giving the formula to get the number of edges in the result involution graph:

Proposition 1: [3] Let G be a finite group, and the number of edges in the result involution graph is given by the formula ½ $(\boldsymbol{sI_G}\,|G| - F)$ such that is F the order of 4 elements of G.

In [3] a resized graph has a vertex set that G-classes, two G-classes X, Y ⊆ G, are linked by an edge if X≠Y and their exit $x_0 \in X$ and $y_0 \in Y$ such that x_0,y_0 are adjacent in Γ_G^{RI}. We may denote the resized graph of the group G by Γ_G^{RS}. Next the relation between Γ_G^{RS} and Γ_G^{RI} are given:

Proposition 2: [3] in the case of a finite group G. The result involution graph Γ_G^{RI} is connected if and only if the resize graph Γ_G^{RS} is connected.

The following example will help to demonstrate these findings.

Example 1: Let $G \cong D_{10}$ be the dihedral group of order 10. Then the vertex set of Γ_{D10}^{RI} describe in the following set {e, (2,5)(3,4), (1,2)(3,5), (1,2,3,4,5), (1,3)(4,5),(1,3,5,2,4), (1,4)(2,3), (1,4,2,5,3), (1,5,4,3,2), (1,5)(2,4)}. Thus, by using computational techniques mainly Gap [7]. The result involution graph Γ_{D10}^{RI} is 5-regular with 25 edges. Moreover, Γ_{D10}^{RI} is connected with clique number, radius, and diameter 2, also girth 4.

In Figure 1, the result involution graphΓ_{D10}^{RI}, is presented. Note that the vertices appear in the below figure as numbers and according to their order in the above set. For example, the first element of the vertex set that is *e*, is labeled 1, and the second element (2, 5) (3, 4) is labeled 2, and so on.

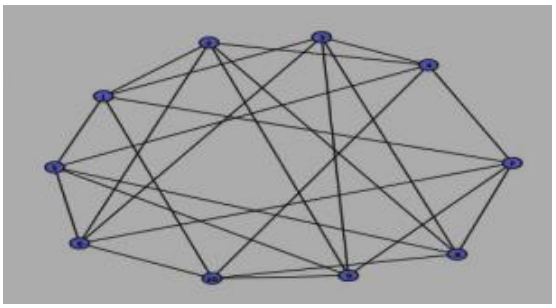

Figure 1
The Result Involution Graph Γ_G^{RI}, G ≅ D10

The resize graph Γ_{D10}^{RS} with vertex {1A, 5A, 5B, 2A} are given in below:

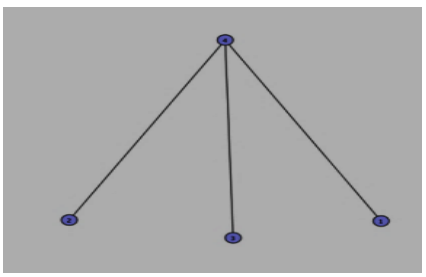

Figure 2
The Resize Graph Γ_G^{RS}, G ≅ D10

From Figure 2 we see that Γ_{D10}^{RS} is (1, 3) - biregular with 4 edges. Also, the graph is a connected tree with diameter 2, clique number 2, and radius 1.

3. Main Results

In this section, we study the Γ_G^{RI} when G≅ HS or G≅McL. In case McL we have, $\boldsymbol{sI}_{\mathrm{McL}}$= 22275, while $\boldsymbol{sI}_{\mathrm{HS}}$= 21175. Thus, we will study the resized graph Γ_G^{RS} to achieve our goal. This work will do computationally by using Gap and the Online Atlas [6].

3.1 *The Resize Graph* Γ_{HS}^{RS}

The vertex set of Γ_{HS}^{RS} are {1A, 2A, 2B, 3A, 4A, 4B, 4C, 5A, 5B, 5C, 6A, 6B, 7A, 8A, 8B, 8C, 10A, 10B, 11A, 11B, 12A, 15A, 20A, 20B}. The graph describes as:

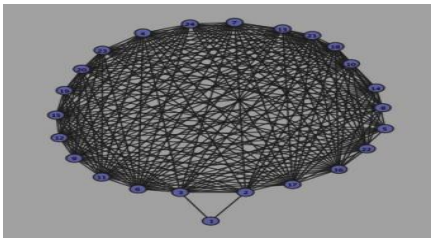

Figure 3
The Resize Graph Γ_{HS}^{RS}

From Figure 3 we conclude that Γ_{HS}^{RS} is connected with 233 edges. Furthermore, diameter 3, clique number 19, radius 2, and girth 3.

3.2 *The Resize Graph* Γ_{McL}^{RS}

Γ_{McL}^{RS} describe in below with vertex set are {1A,2A,3A,3B,4A,5A,5B, 6A, 6B,7A,7B,8A, 9A, 9B, 10A, 11A, 11B, 12A, 14A, 14B, 15A, 15B, 30A, 30B}.

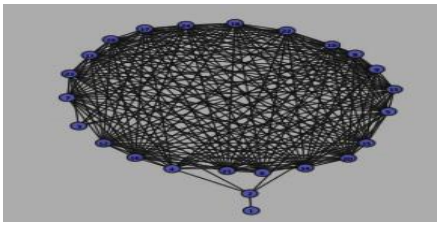

Figure 4
The Resize Graph Γ_{McL}^{RS}

From Figure 4 we have Γ_{HS}^{RS} is connected with 206 edges. Moreover, diameter 3, clique number 18, radius 2, and girth 3.

We are now in a position to give the main theorem for this work:

Theorem 1: *Consider the result involution graph of the Leech lattice groups HS and McL. Then we have the following results:*

1. Γ_{HS}^{RI} is connected with girth 3, diameter 3, and radius 2.

2. Γ_{McL}^{RI} is connected with girth 3, diameter 3, and radius 2.

Proof:

1) From the information in Section 3.1 we have Γ_{HS}^{RS} is connected. Thus, by applying Proposition 2 we get the connectivity of Γ_{HS}^{RI}. Moreover, all the vertices of classes 2A and 2B are adjacent to the identity element. Also, there are 11,55,000 edges between these classes. Thus, the graph Γ_{HS}^{RI} has girth 3. Furthermore, the number of the edges in 6A with the HS-classes

as organized in the Online Atlas {3696000, 5913600, 110880000, 110880000, 103488000, 170016000, 177408000, 177408000, 155232000, 22176000, 199584000, 199584000, 199584000, 399168000, 136752000, 152768000, 133056000, 7392000, 4435200, 77616000, 27720000, 1848000, 16077600}. The exception is the identity element, which is connected with the vertices in class 6A during classes 2A or 2B. Therefore, the radius, in this case, is 2. The maximum distance for the other vertices is 3. Thus, the diameter is 3.

2) In Section 3.2, the resize graph$\Gamma_{McL}^{RS}$is connected. Therefore, Proposition **2** yields that Γ_{HS}^{RI} is also connected. Also, there are 3,17,625 edges between the elements of class 2A in the graph Γ_{HS}^{RI} and each vertex is connected to the identity element of McL. Thus, the graph Γ_{McL}^{RI} has girth 3. In addition, the edges generated by class 6B are linked with other McL-classes as organised in the Online Atlas {149688000, 99792000, 573804000, 8981280000, 898128000, 26045712000, 299376000, 18860688000, 18860688000, 15268176000, 15268176000, 39966696000, 39966696000, 50295168000, 46702656000, 46702656000, 11675664000, 18860688000, 18860688000, 71401176000, 42661080000, 42661080000, 10328472000}.The identity element is the exception that is linked with class 2A and this class is connected with 6B. Thus, the radius of Γ_{McL}^{RI} is 2. The other vertices have a distance of at most 3. Then the diameter of Γ_{McL}^{RI} is 3.

4. Discussion and Conclusion

The result involution graphs for the Leech lattice groups HS and McL are investigated. In particular, the radius, the diameter, and the girth, as well as full information on resized graphs is provided. This work can be used to study more complicated simple groups, such as monster groups and Pariahs groups.

References

[1] Ali Aubad, Peter Rowley, Commuting Involution Graphs for Certain Exceptional Groups of Lie Type, *Graphs and Combinatorics* 37.4, pp. 1345–1355 (2021).

[2] Alice Devillers, Michael Giudici, Involution graphs where the product of two adjacent vertices has order three, *Journal of the Australian Mathematical Society* 85.3, pp. 305-322 (2008).

[3] Asaad Jund, Haval Salih, Result involution graphs of finite groups, *Journal of Zankoy Sulaimani* 23.1, pp. 113–118 (2021).

[4] Asmaa Oudah and Ali Aubad, Analysing the structure of A4-graphs for Mathieu groups. *Journal of Discrete Mathematical Sciences and Cryptography*, Taylor & Francis 24.7, pp.1907-1913 (2021).

[5] Raju Muneshwar and Kirankumar Bondar, Automorphism group of the open subset inclusion graph of a topological space, *Journal of Interdisciplinary Mathematics*, Taylor & Francis 24.3, pp.657-665 (2021).

[6] Robert Wilson, Peter Walsh, Jonathan Tripp, Ibrahim Suleiman, Richard Parker, Simon Norton, Simon Nickerson, Steve Linton, John Bray, and Rachel Abbott, A world wide web atlas of group representations. http://brauer.maths.qmul.ac.uk/Atlas/v3/(2021).

[7] The GAP Group. GAP Groups, Algorithms, and Programming. Version 4.11.1, http://www.gap-system.org (2021).

Received March, 2022

Revised April, 2022