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Ss-supplemented semimodules

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Abstract

A semimodule T is named ss-supplemented if all subsemimodule N of T have supplement L in T such that $N \cap L$ is semisimple. T is named strongly local if $Rad(T)$ is semisimple and T is local. It is shown that every strongly local semimodule is amply ss-supplemented. If T is an ss-supplemented π-projective subtractive semimodule, at that time T is amply ss-supplemented.

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1. Introduction

Throughout the paper, R will indicate a commutative semiring through identity besides T be a left R-semimodule. A semiring R is definite as an algebraic structure $(R, +, \cdot)$ where $(R, +)$, $(R, \cdot)$ are common semigroups, joined thru $a(b+c) = ab + ac$ for $a, b, c \in R$, besides there exists $0,1 \in R$ such that $r + 0 = r$ besides $r0 = 0r = 0$ for all $r \in R$ [3]. If N is a subsemimodule N of T we indicate $X \leq T$. $X \leq T$ is named small in T (we inscribe $N \ll T$), if for all $X \leq T$, with $N + X = T$ implies that $X = T$ [8]. The radical of T, indicated by $Rad(T)$, is the sum of all small subsemimodules of T. T is named hollow if every proper subsemimodule of T is small in T. T is named local, if T has a single maximal subsemimodule. T is said to be simple if T has no non-trivial subsemimodule. $Soc(T)$ the socle of T, is the summation of all simple subsemimodules of T [7]. Let $L \leq T$ besides $K \leq T$. K is named a supplement of L in T if K is minimal with respect for $T = L + K$. $K \leq T$ is a supplement of L

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in T iff $T = L + K$ besides $L \cap K \ll K$ [3]. Similarly to [9] T is supplemented if each subsemimodule L of T has a supplements in T. $L \leq T$ has ample supplements in T if each subsemimodule K of T with $T = L + K$ contains a supplement of L in T. T is named amply supplemented if any subsemimodule of T has ample supplements in T. $M \leq T$ is a subtractive subsemimodule of T if $a, a + b \in M$ then $b \in M$ [7]. If every $M \leq T$ is subtractive, then T is named subtractive. Local semimodules are hollow, besides hollow semimodules are amply supplemented. Similary to [10], we define the notion of socle to T to that of $Soc_s(T)$ To T by, $Soc_s(T) = \sum\{L \ll T | L \text{ is simple}\}$. Evidently $Soc_s(T) \subseteq Rad(T)$, $Soc_s(T) \subseteq Soc(T)$. T is named strongly local if $Rad(T)$ is semisimple besides T is local. In Sec. 2, we learning ss-Supplement subsemimodules. In Sec.3, we provide properties of ss-supplemented semimodules.

2. ss-Supplement subsemimodules

Here, we learning on ss-supplement subsemimodules besides strongly local semimodule.

Definition 2.1: [7] T is a direct sum of subsemimodules $T_1, \ldots, T_n$ to T, if all $t \in T$ can be written only as $t = t_1 + \cdots + t_n$ where $t_i \in T_i$, $1 \leq i \leq n$. It is denoted by $T = T_1 \oplus \ldots \oplus T_n$. All T_i is a summand of T.

Remark 2.2:[7] If T is a subtractive semisimple semimodule then $T = T^1 \oplus T^2$ iff $T = T^1 + T^2$ besides $T^1 \cap T^1 = \{0\}$.

Similar to [4, 2.8(9)] we ascertain the next lemma for semimodules.

Lemma 2.3: *If T is a subtractive semimodule. At that time $Soc(Rad(T)) \ll T$.*

Proof: Put $N = Soc\big(Rad(T)\big)$ and some $T = N + K$ for certain $K \leq T$. Setting $V = N \cap K$, we get $N = V \oplus V'$ for certain $V' \leq N$ besides $T = N + K = (V \oplus V') + K = V' \oplus K$, by Remark 2.2. At this time any simple subsemimodule of V' is a summand of (V' as well as) T besides is small in T besides thus is zero. Hence $V' = 0$ besides so $K = T$, $N \ll T$.

Lemma 2.4: *Let N, L, and $\mathcal{F}$ subsemimodules of T. At that moment*

(1) *If $\mathcal{F} \ll N$ and $N \leq L$, then $\mathcal{F} \ll L$.*
(2) *If $\mathcal{F} \ll N$ and $\leq \mathcal{F}$, then $L \ll N$.*
(3) *If $\mathcal{F} \ll T$, $f: T \to Y$ a homo., at that moment $f(\mathcal{F}) \ll Y$.*

Proof: Similar to the proof in the case of modules [5, 5.1.3].

Lemma 2.5:

(1) *Every subsemimodule of a semisimple semimodule is semisimple.*
(2) *Every epimorphism image of a semisimple semimodule is semisimple.*
(3) *Every sum of semisimple semimodules is semisimple.*

Proof: Similar to the proof in the case of modules [5, 8.1.5].

Lemma 2.6: *Lease T be a subtractive semimodule. If $K \leq T$ is semisimple contained in $Rad(T)$, then $K \ll T$.*

Proof: Presume that $K \leq T$ which is a semisimple. Then $Soc(K) = K$ and since $K \leq Rad(T)$, $Soc(K) \leq Soc(Rad(T))$. Also by Lemma 2.3, $Soc(Rad(T)) \ll T$. By Lemma 2.4, $K \ll T$.

From now on, we take T to be a cancellative, yoked, and subtractive.

Lemma 2.7: *If T is a subtractive semimodule. Then $Soc_s(T) = Rad(T) \cap Soc(T)$.*

Proof: Show $Rad(T) \cap Soc(T) \leq Soc_s(T)$. Let $x \in Rad(T) \cap Soc(T)$. Then $Rx \leq Rad(T)$ and Rx is semisimple. So there exist $F_i \leq T$ $(1 \leq i \leq n)$ such that $Rx = F_1 \oplus F_2 \dots . \oplus F_n$ besides F_i is simple. By Lemma 2.6, $Rx \ll T$. Since for each, $F_i \leq Rx$, we get $F_i \ll T$ according to [8]. Thus $x \in Rx \leq Soc_s(T)$. Hence, $Rad(T) \cap Soc \leq Soc_s(T)$. The reverse inclusion is strong.

The next lemma is a generalization of [6, Lemma 3].

Lemma 2.8: *Let $\mathcal{F}, L \leq T$. Now the next is equivalent.*
(a) $T = \mathcal{F} + L$ and $\mathcal{F} \cap L \leq Soc_s(L)$.
(b) $T = \mathcal{F} + L, \mathcal{F} \cap L \leq Rad(L)$ *and* $\mathcal{F} \cap L$ *is semisimple.*
(c) $T = \mathcal{F} + L, \mathcal{F} \cap L \ll L$ *and* $\mathcal{F} \cap L$ *is semisimple.*

Proof: (a) $\Longrightarrow$ (b) It follows that $\mathcal{F} \cap L \leq Soc_s(L) \leq Rad(L) \cap Soc(L)$. So $\mathcal{F} \cap L \leq Rad(L)$ besides $\mathcal{F} \cap L$ is semisimple. (b) $\Longrightarrow$ (c) Using Lemma 2.6. (c) $\Longrightarrow$ (a) Using Lemma 2.7.

Definition 2.9: L is an ss-supplement of $\mathcal{F}$ in T if $T = \mathcal{F} + L, \mathcal{F} \cap L \ll L$ and $\mathcal{F} \cap L$ is semisimple. If T is subtractive then L an ss-supplement of $\mathcal{F}$ in T if any one of the equal conditions in the above lemma are pleased.

Definition 2.10: We call a semimodule T ss-supplemented if for all $\mathcal{F} \leq T$, there exists a subsemimodule $\mathcal{H}$ in T with $T = \mathcal{F} + \mathcal{H}$, $\mathcal{F} \cap \mathcal{H} \ll \mathcal{H}$ besides $\mathcal{F} \cap \mathcal{H}$ is semisimple. T is ss-supplemented if every subsemimodule of T has an ss-supplement in T. $\mathcal{F} \leq T$ has ample ss-supplements in T if every $\mathcal{H} \leq T$ with $T = \mathcal{F} + \mathcal{H}$ contains an ss-supplement of $\mathcal{F}$ in T. T is amply ss-supplem. if any subsemimodule of T has ample ss-supplements in T. It is perfect that every ss-supplem. semimodule is supplem.

Definition 2.11: Let $\mathcal{F}, \mathcal{S} \leq T$. L is named a Rad-supplement of $\mathcal{F}$ if $T = \mathcal{F} + \mathcal{S}$ besides $\mathcal{F} \cap \mathcal{S} \leq Rad(\mathcal{S})$.

Plainly, Direct summand $\Longrightarrow$ ss-supplement $\Longrightarrow$ supplement $\Longrightarrow$ Rad-supplement.

Note the local $\mathbb{Z}$-semimodule $T = \mathbb{Z}_8$ is T is a not strongly local semimodule. If $\mathbb{N}$ is the semiring of non-negative integers. $\mathbb{N}$ is local but not strongly local $\mathbb{N}$-semimodule.

Theorem 2.12*: If T is strongly local then any factor semimodule of T is strongly local.*

Proof: Lease $L \leq T$. If $A \leq T$ with $L \leq A$, then A is a maximal in T iff $\frac{A}{L}$ is a maximal of $\frac{T}{L}$; so $\frac{T}{L}$ is local. Supposition, $Rad\left(\frac{T}{L}\right) = \frac{Rad(T)}{L} \leq \frac{Soc(T)}{L} = f\big(Soc(T)\big) \leq Soc\left(\frac{T}{L}\right)$; where f from T into $\frac{T}{L}$, as a result $Rad(\frac{T}{L})$ is semisimple. So $\frac{T}{L}$ is strongly local.

3. Some properties of ss-supplemented semimodules

Here, we find descriptions of ss-supplemented semimodules. Similarly to [6] take the next prop.

Proposition 3.1: Assume T is a semimodule besides U be a maximal subsemimodule of T. A subsemimodule V of T is an ss-supplement of U in T iff $T = U + V$ and V is strongly local.

Proof: Lease V be an ss-supplement of U in T. U be local and $U \cap V = Rad(V)$ is a single maximal subsemimodule of V. As $U \cap V$ be semisimple, we consume $Rad(V) \subseteq Soc(V)$. Thus V is strongly local. Conversely, since V is local besides $T = U + V$, one can write $U \cap V \subseteq Rad(V)$. It follows from the assumption that $L \cap V$ is semisimple. So, V is ss-supplement of U.

Lemma 3.2: *Assume T is an ss-supplement semimodule besides $N \ll T$. Then $N \leq Soc_s(T)$.*

Proof: By the assumption, T is the unique ss-supplement of N in T besides as a result $N \cap T = N$ is semisimple. Using Lem. 2.7, $N \leq Soc_s(T)$.

Corollary 3.3: *Lease T be a ss-supplem. semimodule and $Rad(T) \ll T$. Then $Rad(T) \leq Soc(T)$.*

Theorem 3.4: *Every one strongly local semimodule is amply ss-supplemented*

Proof: T is local besides thus it is amply supplemented. As $Rad(T) \leq Soc(T)$, so T is amply ss-supplemented.

Proposition 3.5: T is (ample) ss-supplemented iff it is strongly local whenever T be a hollow R-semimodule.

Proof: Take up that T is ss-supplem. Lease $x \in Rad(T)$. Then, as in [5, Lemma 5.1.4], $Rx \ll T$. As T is ss-supplemented, by Lemma 3.2, $Rx \leq Soc_s(T)$. So $x \in Soc(T)$, and so $Rad(T) \leq Soc(T)$. Suppose $T = Rad(T)$. Since $T = Rad(T) = Soc(T)$, and the radical of a semisimple semimodule is zero, we get $T = 0$, a contradiction since T is hollow. So $T \neq Rad(T)$, that is, T is local. So T stays strongly local. The reverse is clear from Theorem 3.4.

Example 3.6: For q is prime integer, $r \geq 3$. Let $T = \mathbb{Z}_{q^r}$ be a $\mathbb{Z}$-semimodule. $Soc_s(\mathbb{Z}_{q^r}) = Soc(\mathbb{Z}_{q^r}) \cong \mathbb{Z}_q$ and $Rad(T) = q\mathbb{Z}q^r$, T is not strongly local and so it is not ss-supplemented.

Lemma 3.7: *If T is a supplemented semimodule besides $Rad(T) \leq Soc(T)$. Then T is ss-supplemented.*

Proof: Let $L \leq T$. Since T are supplemented, there is $M \leq T$ with $T = L + M$ besides $L \cap M \ll M$. Then $L \cap M \leq Rad(M) \leq Rad(T) \leq Soc(T)$. So $L \cap M$ is semisimple. As a result M is ss-supplement of L.

Theorem 3.8: *If $Rad(T) \ll T$, then the next is equivalent*

(a) *T is ss-supplemented,*

(b) *$Rad(T) \subseteq Soc(T)$ and T is supplemented,*

(c) *$Rad(T)$ has an ss-supplement in T and T is supplemented.*

Proof: (a)⟹(c) Palpable. (c)⟹(b) Via Lem. 3.2. (b)⟹(a) By Lem. 3.7.

Theorem 3.9: *Let $T = \oplus_{i \in I} T_i$, wherever every single T_i is a strong local semimodule. At that time, T are ss-supplemented.*

Proof: $Rad(T) = \oplus_{i \in I} Rad(T_i) \leq \oplus_{i \in I} Soc(T_i) = Soc(T)$ by [9, 21.6 (5) and 21.2 (5)]. By Lemma 2.6, $Rad(T) \ll T$. Since strongly local semimodules are local, T is supplemented. By Theorem 3.8, T is ss-supplemented.

Lemma 3.10: *Let T_1, L be subsemimodules of T with T_1 ss-supplemented. If $T_1 + L$ has an ss-supplement in T, L too has an ss-supplement*

Proof: Take up A is an ss-supplement of $T_1 + L$ in $T$and B is an ss-supplement of $(A + L) \cap T_1$. Then $T = A + B + L$ and $(A + B) \cap L \ll A + B$. In addition, $A \cap (B + L)$ is semisimple as a subsemimodule of the semisimple semimodule $A \cap (T_1 + L)$. But, $B \cap [(A + L) \cap T_1] = B \cap (A + L)$ is semisimple. Via Lemma 2.5, we take $(A + B) \cap L$ is semisimple. $A + B$ is an ss-supplement of L .

Proposition 3.11: Let T_1, T_2 be any subsemimodules of T such that $T = T_1 + T_2$. Then if T_1 and T_2 are ss-supplemented, T is ss-supplemented.

Proof: Let $L \leq T$. The subsemimodule 0 is ss-supplement of $T = T_1 + T_2 + L$. As T_1 is ss-supplemented, $T_2 + L$ has an ss-supplement in T by Lemma 3.10. By Lemma 3.10, L has ss-supplement in T.

Proposition 3.12: Every factor semimodule of ss-supplemented is also ss-supplemented.

Proof: Let $L \leq T$, to show that $\frac{T}{L}$ is ss-supplemented semimodule. Let $\frac{K}{L}$ be a subsemimodule of $\frac{T}{L}$, now K is a subsemimodule of T containing L besides as T is ss-supplemented, exists $A \leq T$ besides $K + A = T$ and $K \cap A \ll A$ and is semisimple. $\frac{T}{L} = \frac{K}{L} + \frac{A+L}{L}$ and $\frac{K}{L} \cap \frac{A+L}{L} = \frac{K \cap (A+L)}{L} = \frac{(K \cap A)+L}{L}$ which is small in $\frac{A+L}{L}$ since it is the image of $K \cap A$ under the natural epimorphism from T onto $\frac{T}{L}$ by Lem 2.5. As $U \cap V$ is semisimple, again by Lemma 2.5 the spitting image of $K \cap A$ under the natural epimorphism from T onto $\frac{T}{L}$ is equivalent to $\frac{(K \cap A)+L}{L} = \frac{K}{L} \cap \frac{A+L}{L}$ is semisimple. $\frac{A+L}{L}$ is an ss-supplement to $\frac{K}{L}$ of $\frac{T}{L}$, as required.

Corollary 3.13: *Any summand of an ss-supplemented semimodule is ss-supplemented.*

Remark 3.14: T is amply ss-supplemented if every subsemimodule of T is ss-supplemented.

Definition 3.15: T is named π-projective if for any A, $L \leq T$ with $A + L = T$, there exists $f, g \in \mathrm{End}_R(T)$ with $f + g = 1_T$, $f(T) \leq A$ besides $g(T) \leq L$.

Theorem 3.16: *Let T be an ss-supplemented π-projective subtractive R-semimodule. Then T amply ss-supplemented.*

Proof: Let $T = C + D$ besides N a ss-supplement of C of T. Since T is π-projective, so there exist homos h and q with $h(T) \leq D, q(T) \leq C$ and $h + q = 1_T$. we claim $h(C) \leq C,\ T = C + h(N),\ h(C \cap N) = C \cap h(N)$. To verify this claim, let $c \in C$, then $q(c) \in C$, $c = h(c) + q(c) \in C\ (h + q = 1_T)$ since T is subtractive then $h(c) \in C$, so $h(C) \leq C$. Since N is ss-supplement ot C, now $T = C + N$ implies $h(T) = h(C) + h(N) \leq C + h(N)$ and since T is π-projective, then $T = h(T) + q(T) \leq C + C + h(N) = C + h(N)$, hence $T = C + h(N)$. To prove that $h(C \cap N) = C \cap h(N)$, let $t \in C \cap h(N)$, indicates that $t \in C$ and $t \in h(N)$, that is $t = h(n)$, $n \in N$. But $n = h(n) + q(n)$, $h(n) \in C, q(n) \in C$, so n belong to C then $t \in h(C \cap N)$, then $C \cap h(N) \leq h(C \cap N)$. Clearly, $h(C \cap N) \leq h(C) \cap h(N) \leq C \cap h(N)$. $h(N) = C \cap h(N)$.

As N is ss-supplement of C, so $C \cap N \ll N$ besides $C \cap N$ is semisimple. By Lemma 2.4, $h(C \cap N) \ll h(N)$, so $C \cap h(N)$. As $C \cap N$ is semisimple. then $h(C \cap N)$ is semisimple via Lem 2.5, $C \cap h(N)$ is semisimple. As a result, $h(N)$ is an ss-supplement of C, $h(N) \leq D$. T is amply ss-supplemented.

Lemma 3.17: *Let T be an amply ss-supplemented semimodule and L be an ss-supplement in T. At that time L is amply ss-supplemented.*

Proof: Take up L be an ss-supplement of $N \leq T$. Let $X, Y \leq L$ with $L = X + Y$. So $T = (N + X) + Y$. T is ss-supplemented, $N + X$ has an ss-supplement $Y' \leq Y$ in T. So $X + Y' \leq L$. Via the minimality of L, $L = X + Y'$. Also, $X \cap Y' \leq (N + X) \cap Y' \ll Y'$. So $X \cap Y' \ll Y'$. As $(N + X) \cap Y'$ is semisimple, $X \cap Y'$ is semisimple by Lemma 2.5. As a result Y' is ss-supplement of X.

Theorem 3.18: *T is amply ss-supplemented iff all $N \leq T$, $N = \mathfrak{D} + Y$ wherever $\mathfrak{D}$ is ss-supplemented and $Y \leq Soc_s(T)$*

Proof: Let T be ss-supplemented besides $N \leq T$. N has a ss-supplement $\mathbb{E}$. $T = \mathbb{E} + N$. There is a $\mathfrak{D} \leq N$ with $\mathfrak{D}$ is an ss-supplement of $\mathbb{E}$ in T. Put $Y = N \cap \mathbb{E}$. $\mathbb{E}$ is an ss-supplement of N in T, so $Y \leq Soc_s(\mathbb{E}) \leq Soc_s(T)$. As T is subtractive, by [3, Proposition 14.22] $N = N \cap T = N \cap (\mathfrak{D} + \mathbb{E}) = \mathfrak{D} + (N \cap \mathbb{E}) = \mathfrak{D} + Y$, where $Y \leq Soc_s(T)$ and $\mathfrak{D}$ is ss-supplemented by Lemma 3.17. Conversely, let $N \leq T$, then $N = \mathfrak{D} + Y$ with $\mathfrak{D}$ is ss-supplemented, $Y \leq Soc_s(T)$. By Prop. 3.11, N is ss-supplemented. By Rem. 3.14, T is amply ss-supplemented.

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