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Some properties of an v-open set

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Abstract

This article aims to introduce some concepts related to the v-open set such as v-closure, v-interior, and v-exterior. Basic properties, examples, and characterizations of these concepts have been investigated. Finally, this work aims to study these ideas, and some propositions, theorems, and relations between purposed concepts have been given.

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1. Introduction

Topology is an offshoot of mathematics that deals with the study of those properties of certain stuff that remain invariant under a certain type of transformation as bending or stretching. In simple terms, topology is the discussion of continuity, compactness, and connectivity [1].

Njastad in 1965 [2] first introduced α $-$open set concept as a generalization of open set, where $D \subseteq \mathcal{X}$ is a α $-$open iff $D \subseteq (int(cl(int(D))))$. Many authors are interested studies $\alpha -$ open , see [3] and [4]. The notion of $\beta - the$ open set was studied in [5], where $D \subseteq \mathcal{X}$ is a β $-$ open iff $D \subseteq (cl(int(cl(D))))$.

A subset $D \subseteq \mathcal{X}$ is a θ - open if $\forall d \epsilon D$ there is open E such that $d \epsilon E \subseteq cl(E) \subseteq D$ [6].

In this paper, the idea of $\mathcal{v} - open$ is studied as a generalization of $open$ sets that exists in the intersection of a family of topologies on $\mathcal{X}$.

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2. $\mathcal{v}$ −Open Set

Definition 2.1: Let $\{\mathfrak{F}_\kappa\}_{\kappa\in J}$ be a topology on $\mathcal{X}$ and $\mathcal{N} \subseteq \mathcal{X}$. Then $\mathcal{N}$ is called $\mathcal{v}-open$ set in $\mathcal{X}$ if there is $T\epsilon \bigcap_{\kappa\in J}\mathfrak{F}_\kappa$ such that $\phi \neq T \subseteq \mathcal{N}$ where $\mathcal{N} \neq \phi$ and $T = \phi$ where $\mathcal{N} = \phi$.

The set of all $\mathcal{v}-open$ in $\mathcal{X}$ is denoted by $\mathcal{v}O_{\mathcal{X}}$ and defined as:

$\mathcal{v}O_{\mathcal{X}} = \{\mathcal{N}: \mathcal{N}$ is $\mathcal{v}-open\ in\ \mathcal{X}\}$.

The complement of a $\mathcal{v}-open$ set is called $\mathcal{v}-closed$ set and the set of all $\mathcal{v}-closed$ is denoted by $\mathcal{CS}_{\mathcal{v}}$.

An ordered pair $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is called $\mathcal{v}-space$ where $\mathcal{X}$ is a universal set and $\mathcal{v}O_{\mathcal{X}}$ is a set of all $\mathcal{v}-open$ in $\mathcal{X}$.

Proposition 2.2: Let $\{\mathfrak{F}_\kappa\}_{\kappa\in J}$ be a topology on $\mathcal{X}$. Then a $\mathcal{v}-space$ $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is satisfies the conditions of topology on $\mathcal{X}$.

Theorem 2.3: *If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$. Then:*

1. $\phi, \mathcal{X}\epsilon\mathcal{CS}_{\mathcal{v}}$.
2. If $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n\epsilon\mathcal{CS}_{\mathcal{v}}$, then $\bigcup_{i=1}^{n}\mathcal{N}_i\ \epsilon\mathcal{CS}_{\mathcal{v}}$.
3. If $\{\mathcal{N}_{\beta_i}\}_{\beta_i\epsilon I}\epsilon\mathcal{CS}_{\mathcal{v}}$, then $\bigcap_{\beta_i\epsilon I}\mathcal{N}_{\beta_i} \epsilon\ \mathcal{CS}_{\mathcal{v}}$.

Definition 2.4: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$ and $\mathfrak{b}\epsilon\mathcal{X}$. A subset $\mathcal{N}$ of $\mathcal{X}$ is called $\mathcal{v}-neighbourhood$ of $\mathfrak{b}$ (in short $\mathcal{N}$ is $\mathcal{v}-nhd$ of $\mathfrak{b}$) if there is $\mathcal{D}\epsilon\mathcal{v}O_{\mathcal{X}}$ containing $\mathfrak{b}$ such that $\mathcal{D} \subseteq \mathcal{N}$.

Remark 2.5: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$ and $\mathfrak{b}\epsilon\mathcal{N} \subseteq \mathcal{X}$. If $\mathcal{N}\epsilon\mathcal{v}O_{\mathcal{X}}$, then $\mathcal{N}$ is $\mathcal{v}-nhd$ of $\mathfrak{b}$.

Theorem 2.6: *Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$ and $\mathcal{N} \subseteq \mathcal{X}$. Then $\mathcal{N}\epsilon\mathcal{v}O_{\mathcal{X}}$ iff $\mathcal{N}$ includes $\mathcal{v}-nhd$ of each of its points.*

Proposition 2.7: Let $\{\mathfrak{F}_\kappa\}_{\kappa\in J}$ be a family of topology on $\mathcal{X}$. If $T\epsilon\mathfrak{F}_\kappa$ for all $\kappa\epsilon J$, then $T\epsilon\ \mathcal{v}O_{\mathcal{X}}$.

3. The Main Results

Definition 3.1: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$. A $\mathcal{v}$ −closure of $\mathcal{N} \subseteq \mathcal{X}$ is denoted by $cl_{\mathcal{v}}(\mathcal{N})$ and it is intersection of all $\mathcal{v}$ −closed sets that contain $\mathcal{N}$.

Theorem 3.2: *Assume $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{N} \subseteq \mathcal{X}$, then $cl_{\mathcal{v}}(\mathcal{N})$ is the smallest $\mathcal{v}$ −closed set that contains $\mathcal{N}$.*

Proof From [Theorem (2.3): (3)], we note that an arbitrary intersection of $\mathcal{v}$ −closed sets is a $\mathcal{v}$ −closed set, so we have $cl_{\mathcal{v}}(\mathcal{N})$ is a $\mathcal{v}$ −closed set.
$\mathcal{N} \subseteq cl_{\mathcal{v}}(\mathcal{N})$ by the definition of $cl_{\mathcal{v}}(\mathcal{N})$. Let $\mathcal{N}^*$ be a $\mathcal{v}$ −closed set that contains $\mathcal{N}$. Then $\mathcal{N}^*$ includes the intersection of all $\mathcal{v}$ −closed sets that contain $\mathcal{N}$. Hence, $cl_{\mathcal{v}}(\mathcal{N}) \subseteq \mathcal{N}^*$. This completes the proof.

Corollary 3.3: *Assume* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N} \subseteq \mathcal{X}$, *then* $\mathcal{N}$ *is a* $\mathcal{v}$ *−closed set iff* $\mathcal{N} = cl_{\mathcal{v}}(\mathcal{N})$.

Theorem 3.4: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{X}$. *Then*
1. $\mathcal{N}_1 \subseteq \mathcal{N}_2$, then $cl_{\mathcal{v}}(\mathcal{N}_1) \subseteq cl_{\mathcal{v}}(\mathcal{N}_2)$.
2. $cl_{\mathcal{v}}(\mathcal{N}_1 \cup \mathcal{N}_2) = cl_{\mathcal{v}}(\mathcal{N}_1) \cup cl_{\mathcal{v}}(\mathcal{N}_2)$.
3. $cl_{\mathcal{v}}(\mathcal{N}_1 \cap \mathcal{N}_2) \subseteq cl_{\mathcal{v}}(\mathcal{N}_1) \cap cl_{\mathcal{v}}(\mathcal{N}_2)$.
4. $cl_{\mathcal{v}}\big(cl_{\mathcal{v}}(\mathcal{N})\big) = cl_{\mathcal{v}}(\mathcal{N})$, for any $\mathcal{N} \subseteq \mathcal{X}$.
5. $cl_{\mathcal{v}}(\phi) = \phi$ and $cl_{\mathcal{v}}(\mathcal{X}) = \mathcal{X}$.

Definition 3.5: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v} - space$ and $\mathcal{N} \subseteq \mathcal{X}$. A point $\mathfrak{b} \in \mathcal{X}$ is a $\mathcal{v}$ −limit point of $\mathcal{N}$ if $\mathcal{D}$-$\{\mathfrak{b}\} \cap \mathcal{N} \neq \phi$ $\forall \mathcal{D} \in \mathcal{v}O_{\mathcal{X}}$ containing $\mathfrak{b}$. The set of all $\mathcal{v} - L.P.$ of $\mathcal{N}$ is denoted by $D_{\mathcal{v}}(\mathcal{N})$.

Theorem 3.6: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N} \subseteq \mathcal{X}$. *Then* $\mathcal{N}$ *is* $\mathcal{v}$ *−closed iff* $D_{\mathcal{v}}(\mathcal{N}) \subseteq \mathcal{N}$.

Theorem 3.7: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N} \subseteq \mathcal{X}$. *Then* $D_{\mathcal{v}}(\mathcal{N})$ *is a* $\mathcal{v}$ *−closed.*

Theorem 3.8: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N} \subseteq \mathcal{X}$. *Then* $\mathcal{N} \cup D_{\mathcal{v}}(\mathcal{N})$ *is a* $\mathcal{v}$ *−closed.*

Theorem 3.9: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v} - space$ *and* $\mathcal{N} \subseteq \mathcal{X}$. *Then* $cl_{\mathcal{v}}(\mathcal{N}) = \mathcal{N} \cup D_{\mathcal{v}}(\mathcal{N})$.

Proposition 3.10: Let $\{\mathfrak{F}_\kappa\}_{\kappa \in J}$ be a topology on $\mathcal{X}$ and $T \in \mathfrak{F}_\kappa$ for all $\kappa \in J$, then $cl(T^c)$in $(\mathcal{X}, \mathfrak{F}_\kappa)$ for all $\kappa \in J$ is equal to $cl_{\mathcal{v}}(T^c)$ in $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$.

Corollary 3.11: *Let* $\{\mathfrak{F}_\kappa\}_{\kappa \in J}$ *be a topology on* $\mathcal{X}$ *and* $T \in \bigcap_{\kappa \in J} \mathfrak{F}_\kappa$. *Then* $cl(T^c)$ *in* $(\mathcal{X}, \bigcap_{\kappa \in J} \mathfrak{F}_\kappa)$ = $cl_{\mathcal{v}}(T^c)$ *in* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ =T^c.

Definition 3.12: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. A point $\mathfrak{b} \in \mathcal{A}$ is called an $\mathcal{v}$ −interior point of $\mathcal{A}$ if there is $\mathcal{N} \in \mathcal{v}O_{\mathcal{X}}$ such that $\in \mathcal{N} \subseteq \mathcal{A}$.

Definition 3.13: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. The set of all $\mathcal{v}$ −interior points of $\mathcal{A}$ is known as the $\mathcal{v}$ −interior of $\mathcal{A}$ and is denoted by $lnt_{\mathcal{v}}(\mathcal{A})$.

Theorem 3.14: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v}-space$ *and* $\mathcal{A} \subseteq \mathcal{X}$. *Then*

$$lnt_{\mathcal{v}}(\mathcal{A}) = \bigcup_{i\in I} \{\mathcal{N}_i : \mathcal{N}_i \text{ is } \mathcal{v}-open\ ,\ \mathcal{N}_i \subseteq \mathcal{A}\ \forall i\in I\}.$$

Theorem 3.15: If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $lnt_{\mathcal{v}}(\mathcal{A})$ is the largest $\mathcal{v}-open$ set contained in $\mathcal{A}$.

Theorem 3.16: *If* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *is a* $\mathcal{v}-space$ *and* $\mathcal{A} \subseteq \mathcal{X}$. *Then* $\mathcal{A}$ *is a* $\mathcal{v}-open$ *iff* $lnt_{\mathcal{v}}(\mathcal{A}) = \mathcal{A}$.

Proof: Assume $\mathcal{A}$ is a $\mathcal{v}-open$. From Theorem (3.16), we have

$lnt_{\mathcal{v}}(\mathcal{A}) \subseteq \mathcal{A}$. But $\mathcal{A}$ is a $\mathcal{v}-open$ & $\mathcal{A} \subseteq \mathcal{A}$ and $lnt_{\mathcal{v}}(\mathcal{A})$ is the largest $\mathcal{v}-open$ set contained in $\mathcal{A}$. Then $\mathcal{A} \subseteq lnt_{\mathcal{v}}(\mathcal{A})$.

Hence $lnt_{\mathcal{v}}(\mathcal{A}) = \mathcal{A}$.

Conversely): suppose that $lnt_{\mathcal{v}}(\mathcal{A}) = \mathcal{A}$. By Theorem (3.16), we have $lnt_{\mathcal{v}}(\mathcal{A})$ is a $\mathcal{v}-open$ set. This completes the proof.

Theorem 3.17: *Let* $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ *be a* $\mathcal{v}-space$ *and* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$. *Then*

1. If $\mathcal{A} \subseteq \mathcal{B}$, then $lnt_{\mathcal{v}}(\mathcal{A}) \subseteq lnt_{\mathcal{v}}(\mathcal{B})$.
2. $lnt_{\mathcal{v}}(\mathcal{A}\cap\mathcal{B}) = lnt_{\mathcal{v}}(\mathcal{A})\cap lnt_{\mathcal{v}}(\mathcal{B})$.
3. $lnt_{\mathcal{v}}(lnt_{\mathcal{v}}(\mathcal{A})) = lnt_{\mathcal{v}}(\mathcal{A})$.
4. $lnt_{\mathcal{v}}(\phi) = \phi$ and $lnt_{\mathcal{v}}(\mathcal{X}) = \mathcal{X}$.

Proposition 3.18: Let $\{\mathfrak{F}_\kappa\}_{\kappa\in J}$ be a topology on $\mathcal{X}$ and $\mathcal{A}\in\mathfrak{F}_\kappa$ for all $\kappa\in J$, then $lnt(\mathcal{A})$ in $(\mathcal{X}, \mathfrak{F}_\kappa)$ for all $\kappa\in J$ is equal to $lnt_{\mathcal{v}}(\mathcal{A})$ in $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$.

Proof: Assume that $\mathcal{A}\in\mathfrak{F}_\kappa$ for all $\kappa\in J$, then by Proposition (2.15), we have $\mathcal{A}\in \mathcal{v}O_{\mathcal{X}}$, thus $\mathcal{A}$ is a $\mathcal{v}-open$, therefore by Theorem (3.17), $lnt_{\mathcal{v}}(\mathcal{A}) = \mathcal{A}$.
Now, $\mathcal{A}\in\mathfrak{F}_\kappa$ for all $\kappa\in J$, then $\mathcal{A}$ is a $open$ set in $(\mathcal{X}, \mathfrak{F}_\kappa)$ for all $\kappa\in J$, thus $lnt(\mathcal{A}) = \mathcal{A}$. Consequentially, $lnt(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A}) = \mathcal{A}$.

Proposition 3.19: Let $\{\mathfrak{F}_\kappa\}_{\kappa\in J}$ be a topology on $\mathcal{X}$ and $\mathcal{A}\in \bigcap_{\kappa\in J}\mathfrak{F}_\kappa$. Then $lnt(\mathcal{A})$ in $(\mathcal{X}, \bigcap_{\kappa\in J}\mathfrak{F}_\kappa) = lnt_{\mathcal{v}}(\mathcal{A})$ in $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$.

Definition 3.20: Let $\mathcal{X}$ be a nonempty set and $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$. Then a $\mathcal{v}$ −interior operator is a map $\mathcal{I} : (\mathcal{X}, \mathcal{v}O_{\mathcal{X}}) \to (\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ which satisfies the following axioms:

1. $\mathcal{I}(\mathcal{X}) = \mathcal{X}$.
2. $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{A}\ \forall\mathcal{A}\in\mathcal{v}O_{\mathcal{X}}$.
3. If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\in\mathcal{v}O_{\mathcal{X}}$, then $\mathcal{I}(\bigcap_{i=1}^{n}\mathcal{A}_i) = \bigcap_{i=1}^{n}\mathcal{I}(\mathcal{A}_i)$.
4. $\mathcal{I}(\mathcal{I}(\mathcal{A})) = \mathcal{I}(\mathcal{A})\ \forall\mathcal{A}\in\mathcal{v}O_{\mathcal{X}}$.

Theorem 3.21: *Let $\mathcal{X}$ be a nonempty set and $\mathcal{I} : (\mathcal{X}, \mathcal{v}O_{\mathcal{X}}) \to (\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}$ −interior operator. Define a class $\mathfrak{F}$ as follows:*

$\mathfrak{F} = \{\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}} : \mathcal{A} = \mathcal{I}(\mathcal{A})\}$. Then $\mathfrak{F}$ is a topology on $\mathcal{X}$ and

$\mathcal{I}(\mathcal{A}) = lnt_{\mathfrak{F}}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A})$ for all $\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$.

Proof: Since $\phi, \mathcal{X} \epsilon \mathcal{v}O_{\mathcal{X}}$ and $\mathcal{I}$ is a $\mathcal{v}$ −interior operator, $\mathcal{X} \epsilon \mathfrak{F}$. Now, $\phi, \phi \epsilon \mathcal{v}O_{\mathcal{X}}$. Assume that $\{\mathcal{A}_i\}_{i \epsilon \mathrm{I}} \epsilon \mathfrak{F}$. Then $\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i \epsilon \mathcal{v}O_{\mathcal{X}}$. Now, $\mathcal{A}_i = \mathcal{I}(\mathcal{A}_i) \subseteq \mathcal{I}(\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i)$, then $\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i \subseteq \mathcal{I}(\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i)$. $\mathcal{I}(\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i) \subseteq \bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i$. then $\bigcup_{i \epsilon \mathrm{I}} \mathcal{A}_i \epsilon \mathfrak{F}$.

Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \epsilon \mathfrak{F}$. Then as above $\bigcap_{i=1}^{n} \mathcal{A}_i \epsilon \mathfrak{F}$.Therefore, $\mathfrak{F}$ is a topology on $\mathcal{X}$. To prove $\mathcal{I}(\mathcal{A}) = lnt_{\mathfrak{F}}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A})$ for all $\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$. Let $\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$. Then $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{A}$ and $\mathcal{I}(\mathcal{I}(\mathcal{A})) = \mathcal{I}(\mathcal{A}) \epsilon \mathfrak{F}$, But $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{A}$, then $\mathcal{I}(\mathcal{A})$ is an open set contained in $\mathcal{A}$. Now, let $\mathcal{B}$ is an open set in $\mathfrak{F}$ such that $\mathcal{B} \subseteq \mathcal{A}$. Then $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. That is, $\mathcal{I}(\mathcal{A}) = lnt_{\mathfrak{F}}(\mathcal{A})$.

Now, $\mathcal{I}(\mathcal{A}) \epsilon \mathfrak{F}$, then by the definition of $\mathfrak{F}$, we get $\mathcal{I}(\mathcal{A}) \epsilon \mathcal{v}O_{\mathcal{X}}$. But $\mathcal{I}(\mathcal{A}) \subseteq \mathcal{A}$, then $\mathcal{I}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A})$. Therefore, $\mathcal{I}(\mathcal{A}) = lnt_{\mathfrak{F}}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A})$.

Definition 3.22: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. A point $\mathfrak{b} \epsilon\ \mathcal{X}$ is called an $\mathcal{v}$ − exterior point of $\mathcal{A}$ if there is $\mathcal{N} \epsilon\ \mathcal{v}O_{\mathcal{X}}$ such that $\mathfrak{b} \epsilon \mathcal{N} \subseteq \mathcal{A}^c$.

Definition 3.23: Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. The set of all $\mathcal{v}$ − exterior points of $\mathcal{A}$ is known as the $\mathcal{v}$ − exterior of $\mathcal{A}$ and is denoted by $ext_{\mathcal{v}}(\mathcal{A})$.

Theorem 3.24: *If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then*

$$ext_{\mathcal{v}}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A}^c).$$

Proof: Suppose that $\mathfrak{b} \epsilon ext_{\mathcal{v}}(\mathcal{A})$, then $\mathfrak{b}$ is an $\mathcal{v}$ − exterior point of $\mathcal{A}$, so there is $\mathcal{N} \epsilon\ \mathcal{v}O_{\mathcal{X}}$ such that $\mathfrak{b} \epsilon \mathcal{N} \subseteq \mathcal{A}^c$, that is $\mathfrak{b}$ is an $\mathcal{v}$ −interior point of $\mathcal{A}^c$, hence $\mathfrak{b} \epsilon lnt_{\mathcal{v}}(\mathcal{A}^c)$, thus $ext_{\mathcal{v}}(\mathcal{A}) \subseteq lnt_{\mathcal{v}}(\mathcal{A}^c)$. Assume $\mathfrak{b} \epsilon lnt_{\mathcal{v}}(\mathcal{A}^c)$, then $\mathfrak{b}$ is an $\mathcal{v}$ −interior point of $\mathcal{A}^c$, hence there is $\mathcal{N} \epsilon\ \mathcal{v}O_{\mathcal{X}}$ such that $\mathfrak{b} \epsilon \mathcal{N} \subseteq \mathcal{A}^c$, thus $\mathfrak{b} \epsilon ext_{\mathcal{v}}(\mathcal{A})$. Therefore, $ext_{\mathcal{v}}(\mathcal{A}) \supseteq lnt_{\mathcal{v}}(\mathcal{A}^c)$. Hence, $ext_{\mathcal{v}}(\mathcal{A}) = lnt_{\mathcal{v}}(\mathcal{A}^c)$.

Theorem 3.25: *If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then*

$$ext_{\mathcal{v}}(\mathcal{A}^c) = lnt_{\mathcal{v}}(\mathcal{A}).$$

Theorem 3.26: Assume that $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $ext_{\mathcal{v}}(\mathcal{A}) = (cl_{\mathcal{v}}(\mathcal{A}))^c$.

Corollary 3.27: *Suppose that $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v} - space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $cl_{\mathcal{v}}(\mathcal{A}) = (ext_{\mathcal{v}}(\mathcal{A}))^c$.*

Proposition 3.28: Assume that $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $(cl_{\mathcal{v}}(\mathcal{A}))^c = lnt_{\mathcal{v}}(\mathcal{A}^c)$.

Proof: The result is direct by Theorem (3.24) and Theorem (3.26).

Proposition 3.29: If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $(lnt_{\mathcal{v}}(\mathcal{A}))^c = cl_{\mathcal{v}}(\mathcal{A}^c)$.

Proposition 3.30: Assume that $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $(ext_{\mathcal{v}}(\mathcal{A}^c))^c = cl_{\mathcal{v}}(\mathcal{A}^c)$.

Proof: The result is direct by Theorem (3.25) and Proposition (3.29).

Proposition 3.31: If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $\mathcal{A}$ is an $\mathcal{v}-open$ iff $ext_{\mathcal{v}}(\mathcal{A}^c) = \mathcal{A}$

Proof: The result is direct by Theorem (3.16) and Theorem (3.25).

Proposition 3.32: If $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a $\mathcal{v}-space$ and $\mathcal{A} \subseteq \mathcal{X}$. Then $\mathcal{A}$ is an $\mathcal{v}-closed$ iff $(ext_{\mathcal{v}}(\mathcal{A}))^c = \mathcal{A}$

Proof: The proof follows by Corollary (3.3) and Corollary (3.27).

Theorem 3.33: *Let $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$. Then*

1. $ext_{\mathcal{v}}(\phi) = \mathcal{X}$ and $ext_{\mathcal{v}}(\mathcal{X}) = \phi$.
2. $ext_{\mathcal{v}}(\mathcal{A}) \subseteq \mathcal{A}^c$ and $ext_{\mathcal{v}}(\mathcal{A}) = ext_{\mathcal{v}}((ext_{\mathcal{v}}(\mathcal{A}))^c)$.
3. If $\mathcal{A} \subseteq \mathcal{B}$, then $ext_{\mathcal{v}}(\mathcal{B}) \subseteq ext_{\mathcal{v}}(\mathcal{A})$.
4. $lnt_{\mathcal{v}}(\mathcal{A}) \subseteq ext_{\mathcal{v}}\big(ext_{\mathcal{v}}(\mathcal{A})\big)$ and $ext_{\mathcal{v}}(\mathcal{A} \cup \mathcal{B}) = ext_{\mathcal{v}}(\mathcal{A}) \cap ext_{\mathcal{v}}(\mathcal{B})$.

Definition 3.34: Let $\mathcal{X}$ be a nonempty set and $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ be a $\mathcal{v}-space$. Then a $\mathcal{v}-$ exterior operator on a $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ is a map
$\mathcal{E} : (\mathcal{X}, \mathcal{v}O_{\mathcal{X}}) \rightarrow (\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ which satisfies the following axioms:

1. $\mathcal{E}(\mathcal{X}) = \phi$, $\mathcal{E}(\phi) = \mathcal{X}$.
2. $\mathcal{E}(\mathcal{A}) \subseteq \mathcal{A}^c \ \forall \mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$.
3. If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \epsilon \mathcal{v}O_{\mathcal{X}}$, then $\mathcal{E}(\bigcup_{i=1}^{n} \mathcal{A}_i) = \bigcap_{i=1}^{n} \mathcal{E}(\mathcal{A}_i)$.
4. $\mathcal{E}((\mathcal{E}(\mathcal{A}))^c) = \mathcal{E}(\mathcal{A}) \ \forall \mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$.

Theorem 3.36: *Let $\mathcal{E}$ be a $\mathcal{v}-exterior\ operator$ on $(\mathcal{X}, \mathcal{v}O_{\mathcal{X}})$ such that $\mathcal{v}O_{\mathcal{X}}$ is closed under complements. Define a class $\mathfrak{F}$ as follows:*
$\mathfrak{F} = \{\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}} : \mathcal{A} = \mathcal{E}(\mathcal{A}^c)\}$. Then $\mathfrak{F}$ is a topology on $\mathcal{X}$ and
$\mathcal{E}(\mathcal{A}) = ext(\mathcal{A}) \ \forall \mathcal{A} \epsilon \mathfrak{F}$.

Proof: Since $\phi, \mathcal{X} \epsilon \mathcal{v}O_{\mathcal{X}}$, then $\mathcal{E}(\mathcal{X}) = \phi$ and $\mathcal{E}(\phi) = \mathcal{X}$, hence $\phi, \mathcal{X} \epsilon \mathfrak{F}$. Let $\{\mathcal{A}_i\}_{i\epsilon \mathrm{I}} \epsilon \mathfrak{F}$. Then $\mathcal{A}_i \epsilon \mathcal{v}O_{\mathcal{X}}$ and $\mathcal{A}_i = \mathcal{E}(\mathcal{A}_i^c)$ $\forall i \epsilon \mathrm{I}$. Now, $\mathcal{E}((\bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i)^c) \subseteq ((\bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i)^c)^c = \bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i \Longrightarrow \bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i = \mathcal{E}((\bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i)^c) \Longrightarrow \bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i \epsilon \mathfrak{F}$.

Let $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \epsilon \mathfrak{F}$. Then by the definition of $\mathfrak{F}$ and from De-Morgan Laws, we get $(\bigcap_{i=1}^{n} \mathcal{A}_i)^c = \bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i^c$. So, $\mathcal{E}((\bigcap_{i=1}^{n} \mathcal{A}_i)^c) = \mathcal{E}(\bigcup_{i\epsilon \mathrm{I}} \mathcal{A}_i^c)$ $= \bigcap_{i=1}^{n} \mathcal{E}(\mathcal{A}_i^c) = \bigcap_{i=1}^{n} \mathcal{A}_i$ Consequentially, $\bigcap_{i=1}^{n} \mathcal{A}_i \epsilon \mathfrak{F}$. Therefore, $\mathfrak{F}$ is a topology on $\mathcal{X}$. Let $\mathcal{A} \epsilon \mathfrak{F}$. Then $\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$ and $\mathcal{A} = \mathcal{E}(\mathcal{A}^c)$. Now, $\mathcal{A} \epsilon \mathcal{v}O_{\mathcal{X}}$ and $\mathcal{E}$ is $\mathcal{v} -$ exterior operator, then $\mathcal{E}((\mathcal{E}(\mathcal{A}))^c) = \mathcal{E}(\mathcal{A})$ and $\mathcal{E}(\mathcal{A}) \subseteq \mathcal{A}^c$. So, we have $\mathcal{E}(\mathcal{A}) \epsilon \mathfrak{F}$ and $\mathcal{E}(\mathcal{A}) \subseteq \mathcal{A}^c$. Hence $\mathcal{E}(\mathcal{A})$ is an open set contained in $\mathcal{A}^c$. We claim $\mathcal{E}(\mathcal{A}) = int(\mathcal{A}^c)$, let $\mathcal{B} \epsilon \mathfrak{F}$ and $\mathcal{B} \subseteq \mathcal{A}^c$. Then $\mathcal{B} \epsilon \mathcal{v}O_{\mathcal{X}}$ and $\mathcal{B} = \mathcal{E}(\mathcal{B}^c)$. Since $\mathcal{B} \subseteq \mathcal{A}^c$, then $\mathcal{A} \subseteq \mathcal{B}^c$. Hence $\mathcal{E}(\mathcal{B}^c) \subseteq \mathcal{E}(\mathcal{A})$.

But $\mathcal{B} = \mathcal{E}(\mathcal{B}^c)$, then $\mathcal{B} \subseteq \mathcal{E}(\mathcal{A})$. Thus $\mathcal{E}(\mathcal{A}) = int(\mathcal{A}^c)$.

But $int(\mathcal{A}^c) = ext(\mathcal{A})$. So, we have $\mathcal{E}(\mathcal{A}) = ext(\mathcal{A})$.

4. Conclusions

This work studies the concept of $\mathcal{v} -$ open set and its basic properties which are investigated. Furthermore, some ideas that are defined in $\mathcal{v} -$ space were introduced. Characterization and theorems of the proposed ideas are presented, as well as several different properties of these concepts are studied.

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