Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 5, pp. 863–872
DOI : 10.47974/JIM-1518

Some games via semi-generalized regular spaces

Afraa. R. Sadek *
Department of Mathematics
College of Science
University of Baghdad
Baghdad
Iraq

M. A. Abdel Karim †
Department of Mathematics
College of Education for Pure Sciences (Ibn Al-Haitham)
University of Baghdad
Baghdad
Iraq

Abstract

In this research, a new application has been developed for games by using the generalization of the separation axioms in topology, in particular regular, Sg-regular and SSg- regular spaces. The games under study consist of two players and the victory of the second player depends on the strategy and choice of the first player. Many regularity, Sg, SSg regularity theorems have been proven using this type of game, and many results and illustrative examples have been presented.

Subject Classification: *14A20, 14M17.*

Keywords: *Games, Regular space, Sg open set, Semi-closed, SgR space, Sg continuous function.*

1. Introduction

The concept of generalized separation axioms was studied by many authors. N.Levine [1], P. Bhattacharyya, and B.K. Lahiri [2], they are introduced and investigated semi-$open$ sets and $semi$-genaralized $open$ of sets. The complements of these sets are caused by the same types of $closed$ sets.

A.R.Sadek and J.H.Hussein [3] paired the previous concepts and introduced a new type of $regular$ity of a given $topological$ spaces. In this work, a new

* *E-mail:* **sadek.r.afraa@gmail.com (Corresponding Author)**
† *E-mail:* **maherjaleel89@yahoo.com**

branch of games was created by using this kind of $regular$ity. In what follows $(\mathcal{H}, \mho)$ (or $\mathcal{H}$) indicate a $topological$ space and if $\mathcal{C} \subset \mathcal{H}$, the closure of $\mathcal{C}$ and the interior of $\mathcal{C}$ are encoded by $cl(\mathcal{C})$, $int(\mathcal{C})$ respectively. A subset $\mathcal{C}$ of a space $\mathcal{H}$ is said to be a $semi\text{-}open$ (resp. $semi\text{-}closed$) [1] if $\mathcal{C} \subset cl(int(\mathcal{C}))$ (resp. $int(cl(\mathcal{C})) \subset \mathcal{C}$) and $\mathcal{C}$ is called a $semi\text{-}generalized\ closed$ set [2] (simply $Sg\text{-}closed$) if $scl(\mathcal{C}) \subset \mathcal{U}$, whenever $\mathcal{C} \subset \mathcal{U}$ and $\mathcal{U}$ is a $semi\text{-}open$ in $\mathcal{H}$. The complement of $semi\text{-}generalized\ closed$ set is $semi\text{-}\ generalized\ open$ (simply $Sg\text{-}open$). We mean by $scl(\mathcal{C})$ (see also [1]) the intersection of all semi-closed subsets in $\mathcal{H}$ containing $\mathcal{C}$. The collection of all $semi\text{-}open$, $semi\text{-}closed$, $Sg\text{-}open$ and $Sg\text{-}closed$ in $\mathcal{H}$ epitomize them by $SO(\mathcal{H})$, $SC(\mathcal{H})$,$Sg\text{-}O(\mathcal{H})$ and $Sg\text{-}C(\mathcal{H})$ respectively. The set of all $Sg\text{-}closed$ sets in $(\mathcal{H}, \mho)$ simply $Sg\text{-}C(\mathcal{H})$ and the set of all $Sg\text{-}open$ sets in in $(\mathcal{H}, \mho)$ simply $Sg\text{-}O(\mathcal{H})$. A space $\mathcal{H}$ is called $regular$ [4](respectively, $semi\text{-}generalized\ regular$[3], strongly $semi\text{-}generalized\ regular$ [3]) space and shortly $\mathcal{R}$-space (respectively, $Sg\text{-}\mathcal{R}$-space, $SSg\text{-}\mathcal{R}$-space) if for each $c \in \mathcal{H}$ and for every $closed$ (respectively, $semi\text{-}closed$, $semi\text{-}closed$) set $\digamma$ in $\mathcal{H}$ such that $c \notin \digamma$, there are two disjoint $open$ (respectively, $Sg\text{-}open$, $open$) sets $\mathcal{U}$ and $\mathcal{V}$ where $c \in \mathcal{U}$ and $\digamma \subset \mathcal{V}$.

The games is defined between two players, Player I and Player II and there are many options L_1, L_2, L_3, …, L_n. There are two types of games, finished games and unfinished games, and in each game there are two players with n options. The type of game that serves our work is the "Two.Zero.Sum" Games , which means the loss of the first player meets the profit of the second player [5,6,7,8]. Each type of finished and unfinished games includes two forms of alternating and concurrent games. The first form comes with the choice of the second player after the first player has completed his choice, as in the game of chess. As for the concurrent form of games, unlike the first form, that is, the movement of the two players in the selection takes place at the same time as in competing companies [5].

We linked games to the concept of three types of regularities in topological spaces, namely, regular, Sg – regular and SSg – regular, respectively. This types of spaces was studied through the concept of games. Finally, we will denote the winning strategy with the symbol ↗ and the losing strategy with the symbol ↘. In the event that the player does not have a winning strategy, it will be Code ⤯ .

2. The game via regular and semi-generalized regular space

In this section, a new game by linking them with regular and semi-generalized regular spaces via open (respectively, Sg-open) sets for Player I and Player II were inserted.

Definition 1: For a space $(\mathcal{H}, \mho)$, determine a game via $\mathcal{R}$-space (respectively, Sg-$\mathcal{R}$-space) denoted by $\acute{G}(\mathcal{R}, \mathcal{H})$ (respectively, Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$) as follows:

Player I and Player II are playing an inning for each positive integer in this game in the r-th inning:

In the first step, Player I : Choose $c_r \in \mathcal{H}$ and a closed (respectively, semi-closed) set $\mathfrak{F}$ in $\mathcal{H}$ such that $c \notin \mathfrak{F}$.

In the second step, Player II : Choose $\mathcal{U}_r, \mathcal{V}_r$ are two disjoint open (respectively, Sg-open) sets in $\mathcal{H}$ such that $c \in \mathcal{U}$ and $\mathfrak{F} \subset \mathcal{V}$.

Then Player II wins in the game $\acute{G}(\mathcal{R},\mathcal{H})$ (respectively, Sg-$\acute{G}(\mathcal{R},\mathcal{H})$) if

$\mathcal{D} = \{\{\mathcal{U}_1,\mathcal{V}_1\}, \{\mathcal{U}_2,\mathcal{V}_2\},\{\mathcal{U}_3,\mathcal{V}_3\},\dots,\{\mathcal{U}_r,\mathcal{V}_r\},\dots\}$ be a selection of open (resp., Sg-open) sets in $\mathcal{H}$ such that for each $c \in \mathcal{H}$ and for every closed (respectively, semi-closed) set $\mathfrak{F}$ in $\mathcal{H}$ such that $c \notin \mathfrak{F}$, there are two disjoint open (respectively, Sg-open) sets $\mathcal{U}_r$ and $\mathcal{V}_r$ in $\mathcal{H}$, where $c \in \mathcal{U}_r$ and $\mathfrak{F} \subset \mathcal{V}_r$.

Otherwise, Player I wins the game $\acute{G}(\mathcal{R},\mathcal{H})$ (respectively, -$\acute{G}(\mathcal{R},\mathcal{H})$).

Example 1: Consider the game $\acute{G}(\mathcal{R},\mathcal{H})$ (respectively, Sg-$\acute{G}(\mathcal{R},\mathcal{H})$), where $\mathcal{H} = \mathrm{N}$ is the set of all natural numbers and $\mho$ is cofinite topology as follows:

Severeal innings will be played by player I and player II. Hence the first inning will be as follows: Player I : Choose an element say $1 \in \mathrm{N}$ and a closed (respectively, semi-closed) set $\{2\}$ in $\mathcal{H}$ such that $1 \notin \{2\}$.

Player II cannot find two disjoint open (respectively, Sg-open) sets $\mathcal{U}_1$ and $\mathcal{V}_1$ in $\mathcal{H}$, such that $1 \in \mathcal{U}_1$ and $\{2\} \subset \mathcal{V}_1$.

Thus there is no winning strategy for Player II in $\acute{G}(\mathcal{R},\mathcal{H})$ (respectively, -$\acute{G}(\mathcal{R},\mathcal{H})$).

Hence, Player I $\nearrow$ $\acute{G}(\mathcal{R},\mathcal{H})$ (respectively, Player I $\nearrow$ Sg-$\acute{G}(\mathcal{R},\mathcal{H})$).

Example 2: Consider the game -$\acute{G}(\mathcal{R},\mathcal{H})$, where $\mathcal{H} = \{c_1, c_2, c_3\}$,

$\mho = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}\}$ then

$SO(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}, \{c_1, c_3\}, \{c_2, c_3\}\} =$ Sg-$O(\mathcal{H})$, so

$SC(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_1, c_3\}, \{c_2, c_3\}\} =$ Sg-$C(\mathcal{H})$ as follows:

In this game player I and player II will playing three innings, so the first one as follows:

Player I: Choose $c_1 \in \mathcal{H}$ and a semi-closed set say $\{c_2\}$ in $\mathcal{H}$ such that $c_1 \notin \{c_2\}$.

Player II: Choose $\mathcal{U}_1 = \{c_1\}$, $\mathcal{V}_1 = \{c_2\}$ which are disjoint Sg-open sets, such that $c_1 \in \mathcal{U}_1$ and $\{c_2\} \subset \mathcal{V}_1$.

Now the second step in choosing first player is as follows:

Player I: Choose $c_2 \in \mathcal{H}$ and a semi-closed set say $\{c_1, c_3\}$ in $\mathcal{H}$ such that $c_2 \notin \{c_1, c_3\}$

Player II: Choose $\mathcal{U}_2 = \{c_2\}$, $\mathcal{V}_2 = \{c_1, c_3\}$ which are disjoint Sg-open sets, such that $c_2 \in \mathcal{U}_2$ and $\{c_1, c_3\} \subset \mathcal{V}_2$.

Then the next inning (the third inning) is as follows:

Player I: Choose $c_3 \in \mathcal{H}$ and a semi-closed set say $\{c_1\}$ in $\mathcal{H}$ such that $c_3 \notin \{c_1\}$

Player II: Choose $\mathcal{U}_3 = \{c_2, c_3\}$, $\mathcal{V}_3 = \{c_1\}$ which are disjoint open Sg-open sets, such that $c_3 \in \mathcal{U}_3$ and $\{c_1\} \subset \mathcal{V}_3$.

Then $\mathcal{D} = \{\{\mathcal{U}_1, \mathcal{V}_1\}, \{\mathcal{U}_2, \mathcal{V}_2\}, \{\mathcal{U}_3, \mathcal{V}_3\}, \ldots\}$ is the winning strategy for Player II in Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Hence, Player II $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Now, consider the game $\acute{G}(\mathcal{R}, \mathcal{H})$ instead of Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$ in the same previous example as follows:

Several innings will be played by player I and player II. Hence the first inning will be as follows:

Player I: Choose $c_1 \in \mathcal{H}$ and a closed set say $\{c_3\}$ in $\mathcal{H}$ such that $c_1 \notin \{c_3\}$.

Player II: Cannot find two disjoint open sets $\mathcal{U}_1$ and $\mathcal{V}_1$ in $\mathcal{H}$, such that $c_1 \in \mathcal{U}_1$ and $\{c_3\} \subset \mathcal{V}_1$.

Thus there is no winning strategy for Player II in $\acute{G}(\mathcal{R}, \mathcal{H})$.

Hence, Player I $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$.

From (example 2) not each Sg-$\mathcal{R}$-space is $\mathcal{R}$-space, but it is easy to show that every $\mathcal{R}$-space is Sg-$\mathcal{R}$-space.

The following remark describes what has been obtained until now.

Remark 1: For a space $(\mathcal{H}, \mathfrak{T})$:

If Player II $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$, then Player II $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

If Player I $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$, then Player I $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$.

Theorem 1: *A space $(\mathcal{H}, \mathfrak{T})$ is $\mathcal{R}$-space(resp. Sg-$\mathcal{R}$-space) if and only if, Player II $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$ (resp. $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$).*

Proof: Necessity; Suppose that $(\mathcal{H}, \mathfrak{T})$ is $\mathcal{R}$-space (resp. Sg-$\mathcal{R}$-space). For any choice $c_r \in \mathcal{H}$ and a closed (resp. semi-closed) set $\mathfrak{F}$ in $\mathcal{H}$ such that $c \notin \mathfrak{F}$ of Player I, then Player II will find two disjoint open (resp. Sg-open) sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{H}$ such that $c \in \mathcal{U}$ and $\mathfrak{F} \subset \mathcal{V}$. Hence, Player II $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$ (resp. $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$).

Sufficient; If Player II $\nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$ (resp. $\nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$), this means that for any choice $c_r \in \mathcal{H}$ and a closed (resp. semi-closed) set $\mathfrak{F}$ in $\mathcal{H}$ such that $c \notin \mathfrak{F}$ of Player I, implies that Player II can choose two disjoint open (resp. Sg-open) sets $\mathcal{U}$ and $\mathcal{V}$ in $\mathcal{H}$ such that $c \in \mathcal{U}$ and $\mathfrak{F} \subset \mathcal{V}$. So, $(\mathcal{H}, \mathfrak{T})$ is $\mathcal{R}$-space (resp. Sg-$\mathcal{R}$-space).

Corollary 1: *A space* $(\mathcal{H}, \mho)$ *is* $\mathcal{R}$-*space* $(resp. Sg\text{-}\mathcal{R}\text{-}space)$ *if and only if, Player* $I \nearrow\!\!\!\!\times$ $\acute{G}(\mathcal{R}, \mathcal{H})$ *(resp.* $\nearrow\!\!\!\!\times$ $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$*).*

3. Games of $\mathcal{R}$-space and Sg-$\mathcal{R}$-space via some types of functions

First recall that [1] a map f, from a space $\mathcal{H}$ to a space $\mathcal{K}$ is called $Semi\ closed$ (resp. $irresolute\ contiuous$) map if the $image$ (resp. the $inverse\ image$) of each $semi\text{-}closed$ set in $\mathcal{H}$ is $semi\text{-}closed$ set in $\mathcal{K}$ (resp. each $semi\text{-}open$ set in $\mathcal{K}$ is $semi\text{-}open$ set in $\mathcal{H}$). Further f is called $Sg\text{-}irresolute$ (resp. $Sg\text{-}open$) map if the $image$ (resp. the $inverse\ image$) of each $Sg\text{-}open$ set in $\mathcal{H}$ is $Sg\text{-}open$ set in $\mathcal{K}$ (resp. each $Sg\text{-}open$ set in $\mathcal{K}$ is $Sg\text{-}open$ set in $\mathcal{H}$) [3]. In this section, some results from the above functions with $regular$ and $semi\text{-}generalized\ regular$ spaces via $open$ (respectively, $Sg\text{-}open$) sets in some games will be discussed.

Proposition 1: Let $f : (\mathcal{H}, \mho) \to (\mathcal{K}, \mathcal{Y})$ be a semi-closed, Sg-irresolute continuous and injective function, and Player $II \nearrow Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$, then Player $II \nearrow Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$.

Proof: **In the first inning** Player I in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_1 \in \mathcal{H}$ and a $semi\text{-}closed$ set $\digamma_1$ in $\mathcal{H}$ such that $c_1 \notin \digamma_1$, Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$ will hold account $f(c_1) \in \mathcal{K}$ and the $semi\text{-}closed$ set $f(\digamma_1)$ in $\mathcal{K}$ (since f is $semi\ closed$ function) such that $f(c_1) \notin f(\digamma_1)$, but Player $II \nearrow Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$, $\exists\, u_1, w_1$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{K}$ such that $f(c_1) \in u_1$ and $f(\digamma_1) \subset w_1$ and since f is $Sg\text{-}irresolute\ contiuous$ function, then $f(u_1)$ and $f^{-1}(w_1)$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{H}$. For Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_1 \in f^{-1}(u_1)$ and $\digamma_1 \subset f^{-1}(w_1)$.

In the next inning Player I in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_2 \in \mathcal{H}$ and a $semi\text{-}closed$ set $\digamma_2$ in $\mathcal{H}$ such that $c_2 \notin \digamma_2$, Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$ will hold account $f(c_2) \in \mathcal{K}$ and the $semi\text{-}closed$ set $f(\digamma_2)$ in $\mathcal{K}$ such that $f(c_2) \notin f(\digamma_2)$, but Player $II \nearrow Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$, $\exists\, u_2, w_2$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{K}$ such that $f(c_2) \in u_2$ and $f(\digamma_2) \subset w_2$ and since f is $Sg\text{-}irresolute\ contiuous$ function, then $f^{-1}(u_2)$ and $f^{-1}(w_2)$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{H}$. For Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_2 \in f^{-1}(u_2)$ and $\digamma_2 \subset f^{-1}(w_2)$.

In the r-th inning of the game $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ Player I in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_r \in \mathcal{H}$ and a $semi\text{-}closed$ set $\digamma$ in $\mathcal{H}$ such that $c_r \notin \digamma_r$, $r = 1,2, \dots$. Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$ will hold account $f(c_r) \in \mathcal{K}$ and the $semi\text{-}closed$ set $f(\digamma_r)$ in $\mathcal{K}$ such that $f(c_r) \notin f(\digamma_r)$, but Player $II \nearrow Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{K})$, $\exists\, u_r, w_r$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{K}$ such that $f(c_r) \in u_r$ and $f(\digamma_r) \subset w_r$ and since f is $Sg\text{-}irresolute\ contiuous$ function, then $f^{-1}(u_r)$ and $f^{-1}(w_r)$ are two disjoint $Sg\text{-}open$ sets in $\mathcal{H}$. For Player II in $Sg\text{-}\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_r \in f^{-1}(u_r)$ and $\digamma_r \subset f^{-1}(w_r)$.

Then $\mathcal{D} = \{ \{f^{-1}(u_1), f^{-1}(w_1)\}, \{f^{-1}(u_2), f^{-1}(w_2)\}, \{f^{-1}(u_3), f^{-1}(w_3)\}, \dots, \{f^{-1}(u_r), f^{-1}(w_r)\}, \dots \}$ is the winning strategy for Player II in Sg-$\mathrm{G}(\mathcal{R}, \mathcal{H})$. Hence Player $II \nearrow Sg$-$\mathrm{G}(\mathcal{R}, \mathcal{H})$.

Proposition 2: Let $f : (\mathcal{H}, \mathcal{C}) \to (\mathcal{K}, \mathcal{Y})$ be a $closed$, $continuous$, and injective function and Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{K})$, then Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{H})$.

Proof: Similar proof of Proposition 1 (see [9]).

Proposition 3: Let $f : (\mathcal{H}, \mathcal{C}) \to (\mathcal{K}, \mathcal{Y})$ be $contiuous$, $open$ and bijective function, Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{H})$. Then Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{K})$.

Proof: **In the first inning** Player I in $\mathrm{G}(\mathcal{R}, \mathcal{K})$ choose $c_1 \in \mathcal{K}$ and a $closed$ set F_1 in $\mathcal{K}$ such that $c_1 \notin \mathrm{F}_1$, Player II in $\mathrm{G}(\mathcal{R}, \mathcal{H})$ will hold an account $f^{-1}(c_1) \in \mathcal{H}$ and the $closed$ set $f^{-1}(\mathrm{F}_1)$ in $\mathcal{H}$ (since f is $contiuous$ function) such that $f^{-1}(c_1) \notin f^{-1}(\mathrm{F}_1)$, but Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{H})$, $\exists\, u_1, w_1$ are two disjoint $open$ sets in $\mathcal{H}$ such that $f^{-1}(c_1) \in u_1$ and $f^{-1}(\mathrm{F}_1) \subset w_1$ and since f is $open$ function, then $f(u_1)$ and $f(w_1)$ are two disjoint $open$ sets in $\mathcal{K}$. For Player II in $\mathrm{G}(\mathcal{R}, \mathcal{K})$ choose $c_1 \in f(u_1)$ and $\mathrm{F}_1 \subset f(w_1)$.

In the r-th inning of the game Sg-$\mathrm{G}(\mathcal{R}, \mathcal{K})$ Player I in $\mathrm{G}(\mathcal{R}, \mathcal{K})$ choose $c_r \in \mathcal{K}$ and a $closed$ set F_r in $\mathcal{K}$ such that $c_r \notin \mathrm{F}_r$, r = 1,2, … . Player II in $\mathrm{G}(\mathcal{R}, \mathcal{H})$ will hold an account $f^{-1}(c_r) \in \mathcal{H}$ and the $closed$ set $f^{-1}(\mathrm{F}_r)$ in $\mathcal{H}$ (since f is $contiuous$ function) such that $f^{-1}(c_r) \notin f^{-1}(\mathrm{F}_r)$, but Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{H})$, $\exists\, u_r, w_r$ are two disjoint $open$ sets in $\mathcal{H}$ such that $f^{-1}(c_r) \in u_r$ and $f^{-1}(\mathrm{F}_r) \subset w_r$ and since f is $open$ function, then $f(u_r)$ and $f(w_r)$ are two disjoint $open$ sets in $\mathcal{K}$. For Player II in $\mathrm{G}(\mathcal{R}, \mathcal{K})$ choose $c_r \in f(u_r)$ and $\mathrm{F}_r \subset f(w_r)$.

Then $\mathcal{D} = \{ \{f(u_1), f(w_1)\}, \{f(u_2), f(w_2)\}, \{f(u_3), f(w_3)\}, \dots, \{f(u_r), f(w_r)\}, \dots \}$ is the winning strategy for Player II in $\mathrm{G}(\mathcal{R}, \mathcal{K})$. Hence Player $II \nearrow \mathrm{G}(\mathcal{R}, \mathcal{K})$.

Proposition 4: Let $f : (\mathcal{H}, \mathcal{C}) \to (\mathcal{K}, \mathcal{Y})$ be $irresolute\ contiuous$, Sg-$open$ and bijective function. If Player $II \nearrow Sg$-$\mathrm{G}(\mathcal{R}, \mathcal{H})$, then Player $II \nearrow Sg$-$\mathrm{G}(\mathcal{R}, \mathcal{K})$.

Proof: Similar proof of Proposition 3.

4. The game via strongly *semi-generalized regular* spaces

In this section, some games via SSg-$\mathcal{R}$-space are investigated, so at the beginning, the following definition was provided.

Definition 2: For a space $(\mathcal{H}, \mathcal{C})$, determine a game via S$Sg$-$\mathcal{R}$-space denoted by S$Sg$-$\mathrm{G}(\mathcal{R}, \mathcal{H})$ as follows:

In this game, player I and player II will play an endless number of innings as follows:

First, Player I : Choose $c_r \in \mathcal{H}$ and a *semi-closed* set $\digamma$ in $\mathcal{H}$ such that $c \notin \digamma$.

Second, Player II: Choose $\mathcal{C}_r, \mathcal{D}_r$ are two disjoint *open* sets in $\mathcal{H}$ such that $c \in \mathcal{C}$ and $\digamma \subset \mathcal{D}$.

Then Player II wins in the game SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ if $\mathcal{D} = \{\{\mathcal{C}_1, \mathcal{D}_1\}, \{\mathcal{C}_2, \mathcal{D}_2\}, \{\mathcal{C}_3, \mathcal{D}_3\}, \ \ldots\ , \{\mathcal{C}_r, \mathcal{D}_r\}, \ldots\}$ be a collection of an *open* sets in $\mathcal{H}$ such that for each $c \in \mathcal{H}$ and for every *semi-closed* set $\digamma$ in $\mathcal{H}$ such that $c \notin \digamma$, there are two disjoint *open* sets $\mathcal{C}_r$ and $\mathcal{D}_r$ in $\mathcal{H}$, where $c \in \mathcal{C}_r$ and $\digamma \subset \mathcal{D}_r$.

Otherwise, Player I wins in the game SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Example 3: Let $\mathcal{H} = \{c_1, c_2, c_3, c_4\}$, $\mathcal{T} = \{\mathcal{H}, \emptyset, \{c_3, c_4\}\}$ then $SO(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_3, c_4\}, \{c_1, c_3\ c_4\}, \{c_2, c_3, c_4\}\}$, so $SC(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}\}$ and it is not difficult to check that Sg-$C(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}, \{c_1, c_2\ c_3\}, \{c_1, c_2, c_4\}\}$, so Sg-$O(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_3\}, \{c_4\}, \{c_3, c_4\}, \{c_1, c_3\ c_4\}, \{c_2, c_3, c_4\}\}$. We can show easily that Player $II \nearrow Sg$-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Now Consider the game SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$, where $\mathcal{H} = \{c_1, c_2, c_3, c_4\}$, $\mathcal{T} = \{\mathcal{H}, \emptyset, \{c_3, c_4\}\}$ then $SO(\mathcal{H}) = \big\{\mathcal{H}, \emptyset, \{c_3, c_4\}, \{c_1, c_3\ c_4\}, \{c_2, c_3, c_4\}\big\}$, so $SC(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}\}$ and it is not difficult to check that Sg-$C(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_1\}, \{c_2\}, \{c_1, c_2\}, \{c_1, c_2\ c_3\}, \{c_1, c_2, c_4\}\}$, so Sg-$O(\mathcal{H}) = \{\mathcal{H}, \emptyset, \{c_3\}, \{c_4\}, \{c_3, c_4\}, \{c_1, c_3\ c_4\}\ , \{c_2, c_3, c_4\}\}$ as follows:

Several innings will be played by player I and player II. Hence the first inning will be as follows: Player I : Choose $c_1 \in \mathcal{H}$ and a *semi closed* set say $\{c_2\}$ in $\mathcal{H}$ such that $c_1 \notin \{c_2\}$.

Player II cannot find two disjoint *open* sets $\mathcal{C}_1$ and $\mathcal{D}_1$ in $\mathcal{H}$, such that $c_1 \in \mathcal{C}_1$ and $\{c_2\} \subset \mathcal{D}_1$.

Thus there is no winning strategy for Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Hence, Player $I \nearrow SSg$-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Remark 2: For a space $(\mathcal{H}, \mathcal{T})$:
- If Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$, then Player $II \nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$.
- If Player $I \nearrow$ $\acute{G}(\mathcal{R}, \mathcal{H})$ then Player $I \nearrow SSg$-$\acute{G}(\mathcal{R}, \mathcal{H})$.
- If Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$, then Player $II \nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$.
- If Player $I \nearrow$ Sg-$\acute{G}(\mathcal{R}, \mathcal{H})$, then Player $I \nearrow$ SSg-$\acute{G}\acute{G}(\mathcal{R}, \mathcal{H})$.

Theorem 2: *A space* $(\mathcal{H}, \mathcal{T})$ *is* SSg-$\mathcal{R}$-*space if and only if, Player* $II \nearrow SSg$-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Proof: Necessity; Since $(\mathcal{H}, \mathcal{C})$ is SSg-$\mathcal{R}$-$space$. So for any choice $c_r \in \mathcal{H}$ and a $semi$-$closed$ set $\digamma$ in $\mathcal{H}$ such that $c \notin \digamma$ of Player I, then Player II will find two disjoint $open$ sets $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{H}$ such that $c \in \mathcal{C}$ and $\digamma \subset \mathcal{D}$. That's mean Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Sufficient; If Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$, this means that for any choose $c_r \in \mathcal{H}$ and a $semi$-$closed$ set $\digamma$ in $\mathcal{H}$ such that $c \notin \digamma$ of Player I, implies that Player II can choose two disjoint $open$ sets $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{H}$ such that $c \in \mathcal{C}$ and $\digamma \subset \mathcal{D}$. So $(\mathcal{H}, \mathcal{C})$ is SSg-$\mathcal{R}$-$space$.

Corollary 2: *A space* $(\mathcal{H}, \mathcal{C})$ *is* SSg-$\mathcal{R}$-$space$ *if and only if, Player* $I \not\nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Finally, we have the following result.

Proposition 5: Let $f : (\mathcal{H}, \mathcal{C}) \to (\mathcal{K}, \mathfrak{I})$ be a $semi$-$closed$, $continuous$ and injective function and Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{K})$, then Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

Proof: **In the first inning** Player I in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_1 \in \mathcal{H}$ and a $semi$-$closed$ set $\digamma_1$ in $\mathcal{H}$ such that $c_1 \notin \digamma_1$, Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{K})$ will hold account $f(c_1) \in \mathcal{K}$ and the $semi$-$closed$ set $f(\digamma_1)$ in $\mathcal{K}$ such that $f(c_1) \notin f(\digamma_1)$, but Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{K})$, $\exists\, \mathcal{C}_1, \mathcal{D}_1$ are two disjoint $open$ sets in $\mathcal{K}$ such that $f(c_1) \in \mathcal{C}_1$ and $f(\digamma_1) \subset \mathcal{D}_1$ and since f is $continuous$ function $f^{-1}(\mathcal{C}_1)$ and $f^{-1}(\mathcal{D}_1)$ are two $open$ indeed disjoint (see [10]) sets in $\mathcal{H}$. For Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_1 \in f^{-1}(\mathcal{C}_1)$ and $\digamma_1 \subset f^{-1}(\mathcal{D}_1)$.

In the r-th inning of the game SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ Player I in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_r \in \mathcal{H}$ and the $semi$-$closed$ set $\digamma_r$ in $\mathcal{H}$ such that $c_r \notin \digamma_r$, $r = 1,2, \dots$. Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{K})$ will hold account $f(c_r) \in \mathcal{K}$ and a $semi$-$closed$ set $f(\digamma_r)$ in $\mathcal{K}$ such that $f(c_r) \notin f(\digamma_r)$, but Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{K})$, $\exists\, \mathcal{C}_r, \mathcal{D}_r$ are two disjoint $open$ sets in $\mathcal{K}$ such that $f(c_r) \in \mathcal{C}_r$ and $f(\digamma_r) \subset \mathcal{D}_r$ and since f is $continuous$ function then $f^{-1}(\mathcal{C}_r)$ and $f^{-1}(\mathcal{D}_r)$ are two disjoint $open$ sets in $\mathcal{H}$. For Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$ choose $c_r \in f^{-1}(\mathcal{C}_r)$ and $\digamma_r \subset f^{-1}(\mathcal{D}_r)$.

Then
$\mathcal{D} = \{ \{f^{-1}(\mathcal{C}_1), f^{-1}(\mathcal{D}_1)\}, \{f^{-1}(\mathcal{C}_2), f^{-1}(\mathcal{D}_2)\}, \{f^{-1}(\mathcal{C}_3), f^{-1}(\mathcal{D}_3)\}, \dots,$

$\{f^{-1}(\mathcal{C}_r), f^{-1}(\mathcal{D}_r)\}, \dots \}$ is the winning strategy for Player II in SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$. Hence Player $II \nearrow$ SSg-$\acute{G}(\mathcal{R}, \mathcal{H})$.

From the previous information, the following Figure 1 was created.

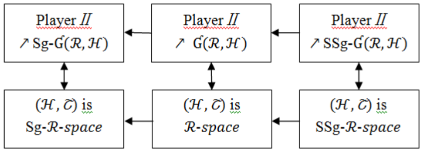

Figure 1

Explanation of linking winning strategy for Player II and strongly semi-generalized regular spaces

5. Conclusion

Some games appeared in new branches and generalizations by using *semi-g-openness*, in addition, the winning and the losing strategies for any player in these games have been linked with some branches of regular spaces with properties and examples.

References

[1] N. Levine, Semi-open sets and semi-continuity in topological spaces. Am Math Mon., 70, pp. 36-41 (1963).

[2] P. Bhattacharyya and B. K. Lahiri, “Semi Generalized Closed Sets in Topology”, *J. Indian Math.*, 29.3, pp. 373-382 (1987).

[3] A. R. Sadek, J.H. Hussein, Regularity via semi- generalized open set, JAT, 12.4, pp. 4631-4635 (2020).

[4] B. J. Conway, Course in point set topology, Springer Cham Heidelberg, New York Dordrecht London, (2014).

[5] R. J. Aumann, Survey of repeated games. Essays in game theory and mathematical economics in honor of Oskar Morgenstern., 4, pp. 11-42 (1981).

[6] F. Jordan, Consonant spaces and topological games, Topology Appl., 274, pp. 107-121, (2020).

[7] L. F. Aurichi and R. R. Dias, A minicourse on topological games, Topology Appl., 258, pp. 305-335 (2019).

[8] R. J. Mohammad and R. B. Esmaeel, *Ibn Al-Haitham Jour. for Pure & Appl. Sci.*, 33.4., pp. 122-136, (2020).

[9] A. Schwarz, Mathematical Foundations of Quantum Field Theory, UC Davis, pp. 460 (2020).

[10] A. R. Sadek, P, PL. Compact Topological Ring, *IJS*, 57.4B, pp. 27542759 (2016).

[11] Shubham Sharma & Ahmed J. Obaid, Mathematical modeling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox, *Journal of Interdisciplinary Mathematics*, 23:4, 843-849 (2020). DOI: 10.1080/09720502.2020.1727611.

[12] A. Kalavathi & G. Sai Sundara Krishnan, Soft g* closed and soft g*open sets in soft topological spaces, *Journal of Interdisciplinary Mathematics*, 19:1, 65-82 (2016), DOI: 10.1080/09720502.2015.1103110.

Received May, 2022

Revised June, 2022