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Some chaoticity properties of the Henon and the Lozi maps and comparison between them

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Abstract

In this paper, we note there are some chaotic and general properties of Henon and Lozi maps, In this work, we introduce some general properties of the Henon and Lozi maps, and we study the type of fixed points concerning the parameter spaces. has been studied chaotic properties of Henon and Lozi maps, we show that the Henon and Lozi maps have sensitive depends on initial condition by using the software (Matlab). we investigate its transitivity of it by varying the parameter of maps.

Subject Classification: *34C28, 34K23.*

Keywords: *General properties of the Henon and Lozi Map, Sensitive depends on initial condition, Transitive map.*

1. Introduction

About 40 years ago, French mathematician Michel Henon was searching for a map in the second dimension. The Henon map is a chaotic dynamic iterative system[1]. Probably, discrete mathematical models are obtained directly by using the Poincare, or department to study a continuous model scientific expertise [2]. Wadih Fayed Hassan AL-Shammari presents the

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algorithm to obtain the Henon attraction and analyzes the fixed points of the Henon map. By using Matlab software this dynamic system will be implemented used to plot the Henon diagram bifurcation in phase space and the attractor [3]. Zeraoulia Elhadj and J.C Sprott present a new seamless multi-part plane map as a separate unified chaotic system of time containing the original Henon and Lozi systems as extreme systems and other systems as a transition between them which contains strong homogeneous chaos [4]. The Lozi map is a simplification of the Henon map, the quadratic term in (x^2) replaced by $|x|$. In this work, we proved the dynamical properties of Henon and Lozi maps that show the transition to chaos. Next, we find the fixed points of the Henon and Lozi maps, see figure 1(a,b). The Henon map is given by the following difference equations [5]:

$x_{n+1} = y_n, y_{n+1} = a - bx_n - y_n^2$, where a, b are real parameters. The Lozi map is a 2D noninvertible iterated map denoted by $L_{a,b}$ proposed by Lozi as follow : $x_{n+1} = y_n, y_{n+1} = a - bx_n - |y_n|$ In this work, we will study a Henon map of the form

$$H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a - bx - y^2 \end{pmatrix} \quad \dots\dots\dots 1$$

And we study the Lozi map with the form

$$L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a - bx - |y| \end{pmatrix} \quad \dots\dots\dots 2$$

We will prove the chaotic properties of the two functions by using Wiggin's definition: Let $F: X \to X$ is a continuous map of metric space X, it is called chaotic(**Wiggins**)if satisfies the following conditions [6]:

- Topological transitive.
- SDIC

2. General Properties of the Henon and Lozi Map

Theorem 2.1: *$H_{a,b}$ has two fixed points when $a \neq 0$, and $a \geq -{}^{1}\!/_{4}\,(b+1)^2$*

Proof: The point $\begin{pmatrix} x \\ y \end{pmatrix} = H_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a - bx - y^2 \end{pmatrix}$ is a fixed point of $H_{a,b}$ provided that

Article I. The left-hand and right-hand vectors are equal if x=y and y=$a - bx - y^2$, which implies that

Article II. y= $a - by - y^2$This is equivalent to y^2 +y(b+1)-a=0, which by the quadratic formula yields

Article III. y= $\frac{-(b+1)\mp\sqrt{(b+1)^2+4a}}{2}$, such an x exists if $(b+1)^2 + 4a \geq 0$, that is, if a≥ $-\frac{1}{4}(b+1)^2$

The Lozi map for two cases to find the fixed point is shown in the next table:

Table 1

The cases for fixed points of Lozi

	y>0	**y<0**
X	$\frac{a}{b+2}$	$\frac{-a}{b}$
Y	$\frac{a}{b+2}$	$\frac{-a}{b}$

If y = 0 then $L_{a,b}$ has a unique fixed point is $\binom{0}{0}$

Proposition 2.2: Let $H_{a,b}$: $R^2 \to R^2$ be the Henon map and a,b be any real constants Then

1- J $H_{a,b}\binom{x}{y} = b$.

2- If y^2 -b ≥ 0 then the eigenvalues of D $H_{a,b}\binom{x}{y}$ are the real numbers

3- If b≠0 $H_{a,b}\binom{x}{y}$ is a one-to-one map.

4- $H_{a,b}\binom{x}{y}$ is C∞

5- $H_{a,b}$ is diffeomorphism.

Proof:

1: To find the Jacobian of $H_{a,b}$, let $F\binom{x}{y} = y$, $G\binom{x}{y}$ =a-bx-y^2, so $\frac{\partial f(x,y)}{\partial x} = 0, \frac{\partial f(x,y)}{\partial y} = 1, \frac{\partial G(x,y)}{\partial x} = -b, \frac{\partial G(x,y)}{\partial y} = -2$, $DH_{a,b}\binom{x}{y} = \begin{bmatrix} 0 & 1 \\ -b & -2y \end{bmatrix}$, J=D $H_{a,b}\binom{x}{y} = b$ …… 3

2: To find the eigenvalues of $DH_{a,b}\binom{x}{y}$ then must be satisfied the characteristic equation $\lambda^2+2y\lambda+b=0$ and the solutions of this equation are $\lambda_{1,2}$ where $\lambda_{1,2} = -y \mp \sqrt{y^2 - b}$ …….. 4 which are real since $y^2 - b \geq 0$.

3: If b=0 then it is easy to prove that $H_{a,b}$ is one-to-one.

If $b \neq 0$, Let $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in R^2$ such that $H_{a,b}\binom{x_1}{y_1} = H_{a,b}\binom{x_2}{y_2}$ then $\binom{y_1}{a - bx_1 - y_1^2} = \binom{y_2}{a - bx_2 - y_2^2}$ then $y_1 = y_2$ and $a - bx_1 - y_1^2 = a - bx_2 - y_2^2$, hence $b \neq 0$ so $x_1 = x_2$, that is $H_{a,b}$ one-to-one

4: Since F and G are polynomial of degree two, so all first partial derivatives exist and are continuous, we get that all it's mixed k^{th} partial derivatives exist, so $H_{a,b}$ is C^{∞}

5 : By part 3 and part 4 of this proposition so $H_{a,b}$ is a one-to-one map and C^{∞} To show it is onto let $\binom{z}{w}$ be any element in R^2 , then there exist y = z, x = $\frac{a-w-z^2}{b}$ in R such that $\binom{x}{y}$ in R^2 and $H_{a,b}\binom{x}{y}=\binom{z}{w}$ hence $H_{a,b}\binom{x}{y}$ is onto, and $H_{a,b}\binom{x}{y}$ is C^{∞}. Hence $H_{a,b}$ is diffeomorphism

Proposition 2.3: Let $H_{a,b}: R^2 \to R^2$ be the Henon map and $\binom{x}{y}$ is a fixed point then:

1. If |b|<0 then$H_{a,b}\binom{x}{y}$ is an area-contracting map.
2. If |b|>0 then $H_{a,b}\binom{x}{y}$ is an area-expanding map.

Proof: From Definition 1 and Proposition (3. 2) the proposition above is held

Proposition 2.4: Let $L_{a,b}$: $R^2 \to R^2$ be the Lozi map and a, b be any real constants and y>0 Then

1. J $L_{a,b}\binom{x}{y} = b$.
2. If 1-4b ≥ 0 then the eigenvalues of det $L_{a,b}\binom{x}{y}$ are the real numbers $\lambda_{1,2} = \frac{-1\mp\sqrt{1-4b}}{2}$
3. If b≠0 $L_{a,b}\binom{x}{y}$is a one-to-one map.
4. $L_{a,b}\binom{x}{y}$ is C^{∞}
5. If b ≠ 0 then $L_{a,b}$ is diffeomorphism.

Proof:

1. To find the Jacobian of $L_{a,b}$, let F $\binom{x}{y}$ = y, G$\binom{x}{y}$ =a-bx-y ,so $\frac{\partial f(x,y)}{\partial x} = 0, \frac{\partial f(x,y)}{\partial y} = 1, \frac{\partial G(x,y)}{\partial X} = -b,$

$$\frac{\partial G(x,y)}{\partial y} = -1,\ DL_{a,b}\binom{x}{y}=\begin{bmatrix}\frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y}\\ \frac{\partial G(x,y)}{\partial x} & \frac{\partial G(x,y)}{\partial y}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -b & -1\end{bmatrix},$$

$$J = D\, L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = b \quad \ldots\ldots..5$$

2: To find the eigenvalues of $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ then must be satisfied the characteristic equation $\lambda^2+\lambda+b=0$ and the solutions of this equation are $\lambda_{1,2}$ where $\lambda_{1,2} = \frac{-1\mp\sqrt{1-4b}}{2}$ ………… 6

which are real since $1-4b \geq 0$.

3: Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in R^2$ such that $L_{a,b}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = L_{a,b}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ then $\begin{pmatrix} y_1 \\ a - bx_1 - y_1 \end{pmatrix} = \begin{pmatrix} y_2 \\ a - bx_2 - y_2 \end{pmatrix}$ then $y_1 = y_2$ and $a - bx_1 - y_1 = a - bx_2 - y_2$, hence $b \neq 0$ so $x_1 = x_2$, that is $L_{a,b}$ one-to-one.

4: Since F and G are polynomial of degree two, so all first partial derivatives exist & continuous, we get that all it's mixedk^{th} partial derivatives exist from section two $L_{a,b}$ is C^∞.

5: By part 3 and part 4 of this proposition so $L_{a,b}$ is a one-to-one map and C^∞ To show it is onto let $\begin{pmatrix} z \\ w \end{pmatrix}$ be any element in R^2 then there exist $y=z, x=\frac{a-z-w}{b}$ in R such that $\begin{pmatrix} x \\ y \end{pmatrix}$ in R^2 and $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$ hence $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is onto, and $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is C^∞ Hence $L_{a,b}$ is diffeomorphism.

Proposition 2.5: Let $L_{a,b}: R^2 \to R^2$ be the Lozi map, y> 0 and $\begin{pmatrix} x \\ y \end{pmatrix}$ is a fixed point then:

1. If $|b|<0$ then $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area-contracting map.
2. If $|b|>0$ then $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area-expanding map.

Proof: From definition (2.1) and proposition 2 the proposition above are held

Proposition 2.6: Let $L_{a,b}: R^2 \to R^2$ be the Lozi map and a,b be any real constants and y<0 Then

1. $J\, L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = -b$ ………………… 7
2. If $1+4b \geq 0$ then the eigenvalues of det $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are the real numbers $\lambda_{1,2} = \frac{1\mp\sqrt{1+4b}}{2}$ ……… 8
3. If $b \neq 0$ $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is the one-to-one map.

4. $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is C^{∞}
5. If $b \neq 0$ then $L_{a,b}$ is diffeomorphism.

Proof: The proof of 4 is similar to the proof of 2.

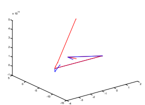
Proposition 2.7: Let $L_{a,b}: R^2 \rightarrow R^2$ be the Lozi map, y < 0 and $\begin{pmatrix} x \\ y \end{pmatrix}$ is the fixed point then:

1. If |-b|<0 then $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area-contracting map.
2. If |-b|>0 then $L_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is an area-expanding map.

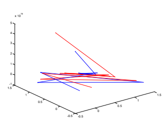
Proof: From Definition (2.1) and Proposition (3.4) the Proposition above is held.

3. Sensitive Dependent on Initial Condition of The Henon and Lozi Maps:

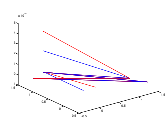
To study the chaotic behavior of the Henon and Lozi maps, we found the Henon and Lozi maps have sensitive dependence on initial conditions by varying the control parameters (a,b)by using Matlab, this is shown in figures (1,2) on the initial points(x_i, y_i) as follows ($i = 1,2$) .



$a = 2, b = 0.001$

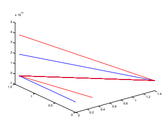


$a = 1.3, b = 0.2$

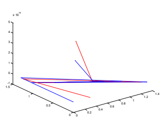


$a = 1.3, b = 0$

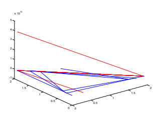
Figure 1
Sensitive dependent on the initial condition of the Henon map in different parameters with initial conditions (0,0) and (0.3,0).



$a = 1, b = 0.001$



$a = 1.3, b = 0.2$



$a = 1.9, b = 0.9$

Figure 2
Sensitive dependent on the initial condition of the Lozi map in different parameters with initial conditions $(\mathbf{0}, \mathbf{0})$ and $(\mathbf{0.3}, \mathbf{0})$

4. Topological Transitive of The Henon and Lozi Map

To study the chaotic behavior of the Henon and Lozi maps, we found the Henon and Lozi maps have topological conjugate by varying the control parameters (a,b)by using Matlab , this show as in figure (3,4) on the initial points(x_i, y_i) as follows ($i = 1,2$) .

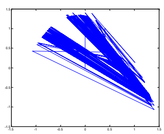
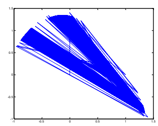
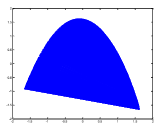
$a = 1, b = -0.4$	$a = 1, b = -0.399$	$a = 1.04, b = -0.5$
		

Figure 3
Topological Transitive of the Henon Map in different parameters with initial condition.

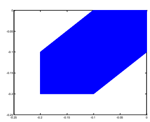
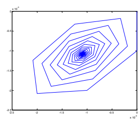
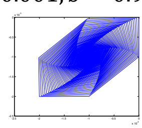
$a = 0.1, b = 0.999$	$a = 0.001, b = 0.92$	$a = 0.001, b = 0.99$a=-
		

Figure 4
Topological Transitive of the Lozi Map in different parameters with initial condition

5. Conclusion

The results presented in this paper indicate that the Henon and Lozi maps have the characteristic of sensitivity depending on the initial conditions and as well as topological transitive of it by varying the parameter by using the software (Matlab).

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