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Recent modification of decomposition method for solving wave-like equation

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Abstract

In this article exact solution for nonlinear wave-like equations with variable coefficients will be obtain by using reliable manner depend on combined Laplace transform with decomposition technique and the results has shown a high-precision, smooth and the series solution is converge rapidly to exact analytic solution compared with other classic approaches. Suggested approach not needs any discretization by data of domain or presents assumption or neglect for a perturbation parameter in problems and not need to use any assumption to convert the non-linear terms into linear. Two examples of strongly nonlinear 2-dimensional space high order have been presented to show the convergence of solution obtained by suggested method to the exact.

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Keywords: *Wave-like equations, Nonlinear PDEs, ADM, Modified decomposition method.*

1. Introduction

Differential equations overall nonlinear and 2-dimensional arise in all branches of science and engineering, hence the fundamental importance of having appropriate methods to find their solution [1, 2]. Since exact solution is available only in limited cases, the construction of efficient numerical [3, 4] or approximate approaches is essential [5 -8].

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Many researchers have been paying attention in recent years to studying how to solve non-linear system of PDEs and so has been used various approaches [9-11]. Among these are the ADM [12] its modifications [13], HAM, HPM [14], DTM [15], Laplace Decomposition Method [16], Quadratic Trigonometric-Spline Method [17], Collocation Method [18, 19], Coupled Method [20, 21], Parallel Processing Technique [22-25], Simulink Technique [26] and Semi Analytic Technique [27, 29]. In this article we suggest efficient reliable modification to solve strongly nonlinear 2-dimentionals based on combine two high performance methods such Laplace transform ($£\mathbb{T}$) and ADM has been used to solve wave-like equations with variable coefficients.

2. Suggested Method for Solving Nonlinear Wave-Like Equation

This section presents the procedure of suggested method to solve non-linear wave-like equation with initial condition (ICs). The modification depends on combine ADM with Laplace transforms and denoted by LATDM. Consider the wave-like equation (WLE):

$$u_{tt} = \sum_{i,j=1}^{n} F_{1ij}(\chi,t,u)\frac{\partial^{k+m}F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k \partial x_j^m} + \sum_{i=1}^{n} G_{1i}(\chi,t,u)\frac{\partial^p G_{2i}(u_{xi})}{\partial x_i^p} + H(\chi,t,u) + K(\chi,t) \tag{1}$$

$$\text{with ICs}: u(\chi,0) = f_1(\chi),\ u_t(\chi,0) = f_2(\chi)$$

Here $\chi = (x_1, x_2, \ldots, x_n)$ and H, K are nonlinear functions. F_{1ij}, G_{1i} are nonlinear function for χ, t and u. While F_{2ij}, G_{2i} are nonlinear function for derivatives of x_i, x_j and $k, m, p \in$ R.

The procedures of modification is started by taking Laplace transformation (denoted by £) on two sides of the eq. (1), to obtain:

$$£u_{tt} = £\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k \partial x_j^m} + \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p G_{2i}(u_{xi})}{\partial x_i^p} + H(X,t,u) + K(X,t)\right] \tag{2}$$

Using Laplace property for derivative, i.e., $£u_{tt}(X,t) = S^2£(\mathrm{u}(X,t)) - Su(X,0)$, so equation (2) can be written as:

$$S^2£(\mathrm{u}(X,t)) - Su(X,0) =$$

$$= £\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k \partial x_j^m} + \sum_{i=1}^{n} G_{1i}(\chi,t,u)\frac{\partial^p G_{2i}(u_{xi})}{\partial x_i^p} + H(\chi,t,u) + K(\chi,t)\right] \tag{3}$$

Then use initial conditions in equation (3) to get

$$S^2 £(\mathrm{u}(X,t)) - Sf_1(X) =$$
$$= £\left[\sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k \partial x_j^m} + \sum_{i=1}^{n} G_{1i}(\chi,t,u)\frac{\partial^p G_{2i}(u_{xi})}{\partial x_i^p} + H(\chi,t,u) + K(\chi,t)\right] \tag{4}$$

So, we have:

$$£(\mathbf{u}(\boldsymbol{X},\boldsymbol{t})) = \frac{1}{s} f_1(\boldsymbol{X}) + \frac{1}{s^2} £\left[\sum_{i,j=1}^{n} \boldsymbol{F}_{1ij}(\boldsymbol{X},\boldsymbol{t},\boldsymbol{u})\frac{\partial^{k+m}\boldsymbol{F}_{2ij}(\boldsymbol{u}_{xi},\boldsymbol{u}_{xj})}{\partial \boldsymbol{x}_i^k \partial \boldsymbol{x}_j^m} + \sum_{i=1}^{n} \boldsymbol{G}_{1i}(\boldsymbol{X},\boldsymbol{t},\boldsymbol{u})\frac{\partial^p \boldsymbol{G}_{2i}(\boldsymbol{u}_{xi})}{\partial \boldsymbol{x}_i^p} + \boldsymbol{H}(\boldsymbol{X},\boldsymbol{t},\boldsymbol{u}) + \boldsymbol{K}(\boldsymbol{X},\boldsymbol{t})\right] \tag{5}$$

Then we will use the series of decomposition for the linear term (equation 6) and for the non- linear term we will use the infinite series of Adomian polynomials (equation 7). That is

$$u(X,t) = \sum_{n=0}^{\infty} u_n(X,t) = u_0 + u_1 + u_2 + \dots \tag{6}$$

For the nonlinear term

$$N_1 u = \sum_{i,j=1}^{n} F_{1ij}(X,t,u)\frac{\partial^{k+m}F_{2ij}(u_{xi},u_{xj})}{\partial x_i^k \partial x_j^m} \text{ and } N_2 u = \sum_{i=1}^{n} G_{1i}(X,t,u)\frac{\partial^p G_{2i}(u_{xi})}{\partial x_i^p}$$

According to the ADM, we have:

$$N_1 u = \sum\nolimits_{n=0}^{\infty} A_{1n} \text{ and } N_2 u = \sum\nolimits_{n=0}^{\infty} A_{2n} \text{ and} \tag{7}$$

Where A_n is Adomian polynomials can be calculated by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\sum_{i=0}^{\infty}(\lambda^i u_i)\right]_{\lambda=0}, \; n = 0,1,2,\dots \tag{8}$$

Then, using the property of linearity for the Laplace transformation. So, the solution u $(\chi,$ t) are getting simply by taking the inverse of Laplace transform. Putting these components into expansion form given by equation (6). Hence, completes the solution

3. Applications

This section present some examples to illustrated suggested modification that is LATDM, for the nonlinear wave-like equations with variable coefficients.

Example 1: Consider the high order nonlinear wave-like equation with variable coefficients

$$u_{tt} = \frac{\partial^2}{\partial x \partial y}(u_{xx}u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyu_x u_y) - u$$

Subject to ICs : $u(x,y,0) = e^{xy}$ and $u_t(x,y,0) = e^{xy}$

We solve example 1, by LATDM and we start by taking Laplace transform (£) on two sides of the equation then using derivative property for Laplace to get:

$$£\{u_{tt}\} = £\left\{\frac{\partial^2}{\partial x \partial y}(u_{xx}u_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyu_x u_y) - u\right\}$$

$$s^2 £\{u\} - su(x,y,0) - u_t(x,y,t) = \frac{\partial^2}{\partial x \partial y}£\{u_{xx}u_{yy}\} - \frac{\partial^2}{\partial x \partial y}£\{xyu_x u_y\} - £\{u\}$$

$$£\{u\} = \frac{1}{s}e^{xy} + \frac{1}{s^2}e^{xy} + \frac{1}{s^2}\frac{\partial^2}{\partial x \partial y}£\{u_{xx}u_{yy}\} - \frac{1}{s^2}\frac{\partial^2}{\partial x \partial y}£\{xyu_x u_y\} - \frac{1}{s^2}£\{u\}$$

Now, using the decomposition procedures as follow: $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$

$$£\{u_0\} = \frac{1}{s}e^{xy} + \frac{1}{s^2}e^{xy}$$

Then, we take invers Laplace transformation for two sides of equation to get u_0 as:

$$u_0 = e^{xy} + te^{xy}$$

Now, calculate the nonlinear form as: $F(u(x,y,t)) = \sum_{n=0}^{\infty} A_n = u_{xx}u_{yy}$

Where A_n are Adomian polynomials computed by

$A_0 = u_{0_{xx}} u_{0_{yy}}$, $A_1 = u_{1_{xx}} u_{0_{yy}} + u_{0_{xx}} u_{1_{yy}}$, …and so on. Also,

$$N(u(x,y,t)) = \sum_{n=0}^{\infty} B_n = xyu_x u_y$$

Where B_n are Adomion polynomials computed by

$B_0 = xyu_{0_x} u_{0_y}$, $B_1 = xyu_{1_x} u_{0_y} + xyu_{0_x} u_{1_y}$, and so on

$$£\{u_1\} = \frac{1}{s^2}\frac{\partial^2}{\partial x \partial y}£\{A_0\} - \frac{1}{s^2}\frac{\partial^2}{\partial x \partial y}£\{B_0\} - \frac{1}{s^2}£\{u_0\}$$

$$A_0 = u_{0_{xx}} u_{0_{yy}} = x^2 y^2 e^{2xy}(1+t)^2$$

$$B_0 = xyu_{0_x} u_{0_y} = xy(ye^{xy}(1+t))(xe^{xy}(1+t)) = x^2y^2e^{2xy}(1+t)^2$$

$$£\{u_1\} = \frac{1}{s^2}\frac{\partial^2}{\partial x\partial y}£\{x^2y^2e^{2xy}(1+t)^2\} - \frac{1}{s^2}\frac{\partial^2}{\partial x\partial y}£\{x^2y^2e^{2xy}(1+t)^2\} - \frac{1}{s^2}£\{e^{xy} + te^{xy}\}$$

$$£\{u_1\} = -\frac{1}{s^2}\left\{\frac{1}{s}e^{xy} + \frac{1}{s^2}e^{xy}\right\} = -\frac{1}{s^3}e^{xy} - \frac{1}{s^4}e^{xy}; u_1 = e^{xy}\left(-\frac{t^2}{2!} - \frac{t^3}{3!}\right)$$

$$£\{u_2\} = \frac{1}{s^2}\frac{\partial^2}{\partial x\partial y}£\{A_1\} - \frac{1}{s^2}\frac{\partial^2}{\partial x\partial y}£\{B_1\} - \frac{1}{s^2}£\{u_1\}$$

$$A_1 = u_{1_{xx}}u_{0_{yy}} + u_{0_{xx}}u_{1_{yy}} = 2x^2y^2e^{2xy}(1+t)\left(-\frac{t^2}{2!} - \frac{t^2}{3!}\right)$$

$$B_1 = xyu_{1_x}u_{0_y} + xyu_{o_x}u_{1_y} = 2x^2y^2e^{2xy}(1+t)\left(-\frac{t^2}{2!} - \frac{t^3}{3!}\right)$$

$$£\{u_2\} = -\frac{1}{s^2}£\left\{-\frac{t^2}{2!}e^{xy} - \frac{t^3}{3!}e^{xy}\right\} = -\frac{1}{s^2}\left\{-\frac{1}{s^3}e^{xy} - \frac{1}{s^4}e^{xy}\right\} = \frac{1}{s^5}e^{xy} + \frac{1}{s^6}e^{xy}$$

$u_2 = e^{xy}\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right)$ and soon.

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t) = u_0 + u_1 + u_2 + \ldots = e^{xy}\left[1 + t - \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \ldots\right]$$

$$u(x,y,t) = e^{xy}\left[\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots\right) + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\right)\right]$$

That is closed to exact solution $u(x,y,t) = e^{xy}(\cos t + \sin t)$

Example 2: Consider the nonlinear high order wave-like equation with variable coefficients

$$u_{tt} = u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^{\ 2}\frac{\partial^2}{\partial x^2}u_x^{\ 3} - 18u^5 + u$$

With ICs : $u(x,0) = e^x$; $u_\tau(x,0) = e^x$

We start by taking Laplace transform (£) on two sides of the equation then using derivative property of Laplace to get:

$$£\{u_{tt}\} = £\left\{\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) + u_x^{\ 2}\frac{\partial^2}{\partial x^2}(u_x^{\ 3}) - 18u^5 + u\right\}$$

$$s^2 £\{u\} - su(x,0) - u_t(x,0) = £\left\{u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})\right\} + £\left\{u_x^{\ 2}\frac{\partial^2}{\partial x^2}(u_x^{\ 3})\right\} - 18£\{u^5\} + £\{u\}$$

$$£\{u\} = \tfrac{1}{s}u(x,0) + \tfrac{1}{s^2}u_t(x,0) + \tfrac{1}{s^2}£\left\{u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})\right\} + \tfrac{1}{s^2}£\left\{u_x^{\ 2}\frac{\partial^2}{\partial x^2}(u_x^{\ 3})\right\} - \tfrac{18}{s^2}£\{u^5\} + \tfrac{1}{s^2}£\{u\}$$

$$£\{u\} = \tfrac{1}{s}e^x + \tfrac{1}{s^2}e^x + \tfrac{1}{s^2}£\left\{u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx})\right\} + \tfrac{1}{s^2}£\left\{u_x^{\ 2}\frac{\partial^2}{\partial x^2}(u_x^{\ 3})\right\} - \tfrac{18}{s^2}£\{u^5\} + \tfrac{1}{s^2}£\{u\}$$

Now, using the decomposition procedures as follow:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t);\ \ £\{u_0\} = \tfrac{1}{s}e^x + \tfrac{1}{s^2}e^x$$

Then, we applied inverse Laplace transformation to get u_0

$$u_0 = e^x + te^x = e^x(1+t)$$

Now, calculate the nonlinear form as:

$$N_1(u(x,t)) = u^2\frac{\partial^2}{\partial x^2}(u_x u_{xx} u_{xxx}) = \sum_{n=0}^{\infty} A_n;\ N_2(u(x,t)) = u_x^{\ 2}\frac{\partial^2}{\partial x^2}u_x^{\ 3} = \sum_{n=0}^{\infty} B_n$$

$$D(u(x,t)) = u^5 = \sum_{n=0}^{\infty} C_n$$

$$A_0 = u_0^{\ 2}\frac{\partial^2}{\partial x^2}(u_0)_x(u_0)_{xx}(u_0)_{xxx}$$

$$A_1 = 2u_0u_1\frac{\partial^2}{\partial x^2}(u_0)_x(u_0)_{xx}(u_0)_{xxx} + u_0^{\ 2}\frac{\partial^2}{\partial x^2}[(u_1)_x(u_0)_{xx}(u_0)_{xxx} + (u_0)_x(u_1)_{xx}(u_0)_{xxx} + (u_0)_x(u_0)_{xx}(u_1)_{xxx}]$$

$$B_0 = (u_{0_x})^2\frac{\partial^2}{\partial x^2}(u_{0_x})^3;\ B_1 = 2u_{0x}u_{1x}\frac{\partial^2}{\partial x^2}(u_{0x})^3 + 3(u_{0x})^2\frac{\partial^2}{\partial x^2}[(u_{0x})^2u_{1x}]$$

$$C_0 = (u_0)^5;\ C_1 = 5(u_0)^4u_1$$

$$£\{u_1\} = \tfrac{1}{s^2}£\{A_0\} + \tfrac{1}{s^2}£\{B_0\} - \tfrac{18}{s^2}£\{C_0\} + \tfrac{1}{s^2}£\{u_0\}$$

$$A_0 = 9e^{5x}(1+t)^5;\ B_0 = 9e^{5x}(1+t)^5;\ C_0 = e^{5x}(1+t)^5$$

$$£\{u_1\} = \frac{9}{s^2}£\{e^{5x}(1+t)^5\} + \frac{9}{s^2}£\{e^{5x}(1+t)^5\} - \frac{18}{s^2}£\{e^{5x}(1+t)^5\} + \frac{1}{s^2}£\{e^x(1+t)\}$$

$$£\{u_1\} = \frac{1}{s^2}£\{e^x + e^x t)\} = \frac{1}{s^2}\left\{\frac{1}{s}e^x + \frac{1}{s^2}e^x)\right\} = \frac{1}{s^3}e^x + \frac{1}{s^4}e^x$$

$$u_1 = \frac{e^x t^2}{2!} + \frac{e^x t^3}{3!} = e^x\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

$$£\{u_2\} = \frac{1}{s^2}£\{A_1\} + \frac{1}{s^2}£\{B_1\} - \frac{18}{s^2}£\{C_1\} + \frac{1}{s^2}£\{u_1\}$$

$$A_1 = 45e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right);\ B_1 = 45e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

$$C_1 = 5e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)$$

$$\begin{aligned} £\{u_2\} = {} & \frac{45}{s^2}£\left\{e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)\right\} + \frac{45}{s^2}£\left\{e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)\right\} \\ & - \frac{90}{s^2}£\left\{e^{5x}(1+t)^4\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)\right\} + \frac{1}{s^2}£\left\{e^x\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)\right\} \end{aligned}$$

$$£\{u_2\} = \frac{1}{s^2}£\left\{e^x\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right)\right\} = \frac{1}{s^2}\left\{e^x\left(\frac{1}{s^3} + \frac{1}{s^4}\right)\right\} = e^x\left(\frac{1}{s^5} + \frac{1}{s^6}\right)$$

$$u_2 = e^x\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right) \text{ and so on}$$

$$u(x,t) = u_0 + u_1 + u_2 + \ldots = e^x + te^x + e^x\left(\frac{t^2}{2!} + \frac{t^3}{3!}\right) + e^x\left(\frac{t^4}{4!} + \frac{t^5}{5!}\right) + \ldots$$

$$u(x,t) = e^x\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right]$$

That is $u(x,t) = e^x.e^t = e^{x+t}$ closed to the exact solution.

4. The Convergence Analysis of LATDM

In this section the convergence of suggested modification will be discuss. Now, consider the general form of nonlinear equation as:

$$\text{ʉ} - \mathcal{N}\,\text{ʉ} = f;\ u \in \breve{\mathrm{H}} \qquad (8) \tag{8}$$

where $\breve{\mathrm{H}}$ is the Hilbert space, $\mathcal{N}$ is the nonlinear operator $\mathcal{N} : \breve{\mathrm{H}} \to \breve{\mathrm{H}}$ and f is also in $\breve{\mathrm{H}}$. Substituting the series of decomposition of equations (6) and (7) in equation (8) to get:

$$\sum_{n=0}^{\infty} \text{ʉ}_n(x,t) - \sum_{n=0}^{\infty} A_n(x,t) = f$$

So the recursive terms are got by: $\text{ʉ}_0 = \text{ʉ}$ and $\text{ʉ}_{n+1} = A_n(\text{ʉ}_0, \text{ʉ}_1, \ldots, \text{ʉ}_n)$

The sufficient requirement for convergence of suggested modification is addressed. That is the obtained series solution will be show closed to exact solution.

Theorem 1 : *Let A_n presented as $\text{ʉ}_0 + \ldots + \text{ʉ}_n$ be an operator from a Hilbert space $\breve{\mathrm{H}}$ to $\breve{\mathrm{H}}$. The series solution*

$$u = \sum_{k=0}^{\infty} \text{ʉ}_k(x,y)t^k$$

is convergent if $\exists 0 < \lambda < 1$ *when* $\| A_{n+1} \| \le \lambda \| A_n \|$ *(such that* $\| \text{ʉ}_{n+1} \| \le \lambda \| \text{ʉ}_n \|$*)* $\forall\, \mathrm{n} = 0, 1, \cdots$

Theorem 1, is a specific case from the *Banach's fixed point theorem* which is a sufficient condition to study the convergence of the proposed method.

Theorem 2 : *If the series solution* $u = \sum_{k=0}^{\infty} \text{ʉ}_k(x,y)t^k$ *convergent, then this series will consider the exact solution of the present non-linear problem.*

Now the following theorem show the series solution $u = \sum_{k=0}^{\infty} \text{ʉ}_k(x,y)t^k$ is convergent

Theorem 3 (Sufficient Condition for Convergence)

"If χ and Υ are Banach spaces and $\aleph : \chi \to \Upsilon$ is a contractive nonlinear mapping, that is

$$\forall\ \omega, \omega^* \in \chi; \| \aleph(\omega) - \aleph(\omega^*) \| \le \gamma \| \omega - \omega^* \|, 0 < \gamma < 1$$

Then according to Banach's fixed point theorem, $\aleph$ has a unique fixed point $\omega,$ consider the exact solution of the present non-linear problem.

Proof: Assume that the sequence generated by suggested method can be written as "(1)

$$\omega_n = \aleph(\omega_{n-1}), \omega_{n-1} = \sum_{i=0}^{n-1} \omega_i, n = 1, 2, 3, \ldots$$

Suppose that $\omega_0 \in B_r(\omega)$ where $B_r(\omega) = \{\omega^* \in \chi; | \| \omega^* - \omega \| < r\}$. Then must show

i. $\omega_n \in B_r(\omega)$

ii. $\lim\limits_{n\to\infty} \omega_n = \omega$

(i) From inductive approach, for $n = 1$,It can be got:

$$\| \omega_1 - \omega \| = \| \aleph(\omega_0) - \aleph(\omega) \| \le \gamma \| \omega_0 - \omega \|$$

Assume that

$$\begin{aligned} \| \omega_{n-1} - \omega \| &\le \gamma \| \omega_{n-2} - \omega \| \\ &\le \gamma^2 \| \omega_{n-3} - \omega \| \\ &\le \gamma^3 \| \omega_{n-4} - \omega \| \\ &\le \gamma^{n-1} \| \omega_0 - \omega \| \end{aligned}$$

As induction hypothesis, then

$$\| \omega_n - \omega \| = \| \aleph(\omega_{n-1}) - \aleph(\omega) \| \le \gamma \| \omega_{n-1} - \omega \| \le \gamma^n \| \omega_0 - \omega \|$$

Using (i), to get

$$\| \omega_n - \omega \| \le \gamma^n \| \omega_0 - \omega \| \le \gamma^n r < r \Rightarrow \omega_n \in B_r(\omega)$$

Because of $0 < \gamma < 1$, so

$$\lim_{n\to\infty} \gamma^n = 0, \lim_{n\to\infty} \| \omega_n - \omega \| \le \lim_{n\to\infty} \gamma^n r = 0$$

that is : $\lim\limits_{n\to\infty} \omega_n = \omega$

Theorem's 1, 2 and 3 show that the achieved solution from the suggested method is convergent to the exact solution under the given condition, $\exists 0 < \lambda < 1$, such that $\| \cancel{u}_{n+1} \| \le \lambda \| \cancel{u}_n \|$, $\forall n = 0, 1, \ldots$

5. Conclusions

In this article, the effective method for treating non-linear, high order PDEs is implemented. A new decomposition technique has been introduced to compute exact analytic solutions for the non-linear (2+1) D- such as: wave-like equations with variable coefficients. Series formulation is used throughout the entire procedure, which leads to series solution is made use within the new procedure. The method is generally based on the well selected base functions and produces exact solution. Two example shown that the proposed modification has better accuracy with easy implementation. Furthermore, the results are observed that when the number of iterations increases, the series solution closed to exact as well.

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