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On fixed points of the transcendental meromorphic functions

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Abstract

In this research, the fixed points of the functions in $F = \{f_c(x) = c\csc(x) : c, x \in \mathbb{R}\}$ are investigated. And also, the types of fixed points of the functions in F are investigated. In addition, the some important conditions for convergence of $\{f_c^n(x)\}$ as $n \to \infty$ are obtained.

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1. Introduction

The real dynamics of functions have been considered by Devaney [1], Fadil [2], and Sajid [3], but, Akbari and Rabii [4], Magrenan and Gutierrez [5], and Radwan [6] have considered the real dynamics of one-parameter family of transcendental functions. Faris [7] has studied the dynamics of one parameter complex functions family $H = \{h_k(z) = ke^z/(z-1) : k \in \mathbb{R}\}$ and $H = \left\{g_k(z) = \frac{k\cosh(z)}{z^2} : k > 0\right\}$ of critically and finite non-critically finite transcendental meromorphic complex functions respectively. This research

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obtains fixed points of the functions in $F = \{f_c(x) = c\csc(x): c, x \in \mathbb{R}\}$. A divide is studied points such that the sequence $\{f_c^n(x)\}$ bounded as $n \to \infty$ and points such that the sequence $\{f_c^n(x)\} \to \infty$. So, theorem 1describes this results.

Theorem 1: *Let $f_c \in F$; hence there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ where $c_1 < c_2 > 0 > c_3 > c_4$ such that:*

(1) At c_1, there are two fixed points x_1 and r_1 and $b_1 \in \mathbb{R}$ such that $f_{c_1}(b_1) = x_1$ satisfies: as $n \to \infty$, $f_{c_1}^n(x) \to x_1$for all $x \in (b_1, r_1)$, $f_{c_1}^n(x) \to \infty$ for all $x \in (2\pi, b_1) \cup (r_1, 3\pi)$, $f_{c_1}^n(x) \to -\infty$for all $x \in (3\pi, 4\pi)$. (2) At c_2, there are two fixed points $x_2$and r_2, and $b_2 \in \mathbb{R}$ such that $f_{c_2}(b_2) = x_2$ as $n \to \infty$, $f_{c_2}^n(x) \to x_2$for all $x \in (b_2, r_2)$, $f_{c_2}^n(x) \to \infty$ whenx $\in (2\pi, b_2) \cup (r_2, 4\pi)$, $f_{c_2}^n(x) \to -\infty$ whenx $\in (3\pi, 4\pi)$. (3)At c_3, there are two fixed points x_3 and r_3, and $b_3 \in \mathbb{R}$ where $f_{c_3}(b_3) = x_3$: as $n \to \infty$, $f_{c_3}^n(x) \to x_3$whenx $\in (b_3, r_3)$, $f_{c_3}^n(x) \to \infty$whenx $\in (3\pi, b_3) \cup (x_3, 4\pi)$, $f_{c_3}^n(x) \to -\infty$when $x \in (2\pi, 3\pi)$. (4)At c_4, there are fixed point x_4 and $b_4 \in \mathbb{R}$ where $f_{c_4}(b_4) = x_4$: as $n \to \infty$, $f_{c_4}^n(x) \to x_4$whenx $\in (b_4, x_4)$, $f_{c_4}^n(x) \to \infty$whenx $\in (3\pi, b_4) \cup (x_4, 4\pi)$, $f_{c_3}^n(x) \to -\infty$when $x \in (2\pi, 3\pi)$. (5)At $c \in (c_1, c_2)$, there are two fixed points a_1, r_4 and $b_5 \in \mathbb{R}$ where $f_c(b_5) = r_4$: as $n \to \infty$, $f_c^n(x) \to a_1$whenx $\in (b_5, r_4)$,$f_c^n(x) \to \infty$whenx $\in (2\pi, b_4) \cup (r_4, 2\pi)$,$f_c^n(x) - \infty$whenx $\in (3\pi, 4\pi)$. (6)At $c \in (c_1, 0)$, there exist two fixed points r_5, r_6 and $b_6 \in \mathbb{R}$ where $f_c(b_6) = r_6$ as $n \to \infty$, $f_c^n(x) \to \infty$ when $x \in (2\pi, b_6) \cup (r_6, 3\pi)$, $f_c^n(x) \to -\infty$ when $x \in (3\pi, 4\pi)$, The orbit $\{f_c^n(x)\}$ is periodic or chaotic for $x \in (r_5, r_6)$. (8)At $c \in (0, c_3)$, there are two fixed points r_7, r_8 and $b_7 \in \mathbb{R}$ where $f_c(b_8) = r_8$ as $n \to \infty$, $f_c^n(x) \to \infty$ when $x \in (3\pi, b_8) \cup (r_8, 4\pi)$, $cf_c^n(x) \to -\infty$ when $x \in (2\pi, 3\pi)$, The orbit $\{f_c^n(x)\}$ is periodic or chaotic for $x \in (r_7, r_8)$ (9) At $c \in (c_4, c_3)$, there are two fixed points a_2, r_9 and $b_8 \in \mathbb{R}$ where $f_c(b_9) = r_9$ asn $\to \infty$, $f_c^n(x) \to a_2$whenx $\in (b_9, r_9)$, $f_c^n(x) \to \infty$whenx $\in (3\pi, b_8) \cup (r_9, 4\pi)$, $f_c^n(x) \to -\infty$whenx $\in (2\pi, 4\pi)$. (9)At $c \in (-\infty, c_4) \cup (c_2, \infty)$, $f_c^n \to \infty$ as $n \to \infty$, for all $x \in (2\pi, 4\pi)\backslash\{3\pi\}$

2. Basic Main Results

In this section, we explain the dynamic of the functions in one parameter family F. Let $\mu(x): \mathbb{R} \to \mathbb{R}$ be a mapping which is defined by $\mu(x) = x\sin(x)$, for all function $f_c \in F$, a fixed point of f_c satisfy the equation $\mu(x) = c$, hence f_c has two fixed points $x_2 \simeq 7.979$ and $x_4 \simeq 11.086$. Then μ has two critical values $c_2 \simeq 7.916$ and $c_4 \simeq -11.086$. because μ is even and continuous function so, the domain of μ is $(2\pi, 4\pi)$.

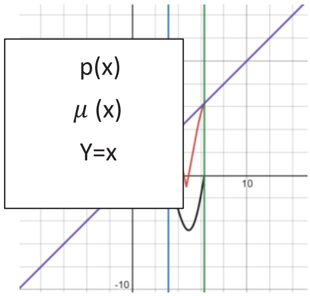
Proposition 2: Let $f_c \in F$, hence there exists three fixed points for f_c for all c: so there is no fixed point if $c < c_4$ or $c > c_2$. f_c has one fixed point when $c = c_2$ and at $c = c_4$. f_c has two fixed points for all points in $(0, c_2)$ and in $(c_4, 0)$.

Proof: Since $\mu'(x) = sin(x) + x\,cos(x)$, then: $\mu''(x) = 2\cos(x) - x\sin(x)$, $\mu''(x_3) < 0$, $\mu''(x_1) > 0$ where $x_1 \simeq 4.913$ and $x_3 \simeq 2.029$. Therefore x_3 is a maxima point and x_1 is a minima point for $\mu(x)$ in the interval $(2\pi, 4\pi)$. Hence $\mu(x) = c$ has no solutions for $c < c_1 (c > c_3)$. Then that $f_c$have no fixed points in this step. When $c = c_1 (c = c_3)$, because of $c_1 = \mu(x_1)(c_3 = \mu(x_3))$ is the minima(maxima) value of $\mu(x)$ in $(2\pi, 4\pi)$. So f_c has one fixed point at $x = x_1 (x = x_3)$. When $c \in (c_1, 0)$, the point $x = x_1$ is a minima value in $(2\pi, 4\pi)$. because μ is decreasing in (x_3, x_1) and μ is increasing function in $(x_1, 2\pi)$, so the line $c = k$ intersects the graph of μ at only one point for each $x \in (x_3, x_1)$ and $x \in (x_1, 2\pi)$. In similar way, for all $c \in (0, c_3)$ the point $x = x_3$ is a maxima value in $(2\pi, 4\pi)$. Then μ is increasing for all $x \in (0, x_3)$ and it is decreasing for all $x \in (x_3, 2\pi)$,then the line $c = k$ crosses the graph of μ at only one point for all $x \in (0, x_3)$ and for all $x \in (x_3, 2\pi)$. Hence f_c has two fixed points in the interval $(c_1, 0)$ and $(0, c_3)$ The following proposition studies the types of fixed points of the function f_c on $\mathbb{R}$. Since $f'_c(x_1) = 1 = f'_c(x_3)$, so$x_1, x_3$ are nature fixed points of f_c. as well as the positive solution of the equation$f'_c(x) = \tan(x) - (x) = -1$, is $x_2 \simeq 4.493$, so $c_2 = \phi(x_2) \simeq -4.385$

Proposition 3: Let $f_c \in F$, so when $c = c_2$, there are two fixed points for f_c:x_2 is indifferent point, and $r_1 \in (x_2, 2\pi)$ is unstable point, if $c \in (c_1, c_2)$, the two fixed points of f_c are:$a_1 \in (x_1, x_2)$ is stable point, and $r_2 \in (x_1, r_1)$ is unstable point, if $c \in (c_2, 0)$, the two fixed points of f_c are: $r_3 \in (\pi, x_2)$ is unstable point and $r_4 \in (r_1, 2\pi)$ is unstable point, if $c \in (0, c_3)$, the two fixed points of f_c are:$a_2 \in (0, x_3)$ is stable point and $r_5 \in (x_3, \pi)$ is unstable point.

Proof: Since $f'_c(x) = -c\csc(x)\cot(x)$ and the solutions of equation $c = \frac{x}{\csc(x)} = \mu(x)$ is the function of fixed points of f_c.Thus the types of fixed point x is obtained by $|f'_c(x)| = \left|\frac{-x}{\csc(x)}\csc(x)\cot(x)\right| = |-x\cot(x)| = \frac{|x\cos(x)|}{|\sin(x)|}$.

Now, we define the function $p(x) = |x\cos(x)| - |\sin(x)|$, the function $p(x)$ is continuous and has 3 zeros when $x = x_1$, x_2 and x_3. By the graph, we can conclude that $p(x)$ is decreasing function in the intervals $(0, x_3)$ and (x_2, x_1), as well as it is increasing for all $x \in (x_3, x_2)$ and $(x_1, 2\pi)$. Therefore $p(x)$ has maxima point at π and it has minima at x=-1. That is $p(x) > 0$; for all $x \in (x_3, x_2) \cup (x_1, 2\pi)$, $p(x) = 0$; at $x = x_1$, x_2 ,x_3 and $p(x) < 0$ at $x \in (0, x_3) \cup (x_2, x_1)$ see Fig(1). Therefore If $c = c_2$, the fixed point $r_1 \in (x_2, 2\pi)$ satisfies $|f'_c(r_1)| > 1$, then r_1 is unstable point. If $c \in (c_1, c_2)$, the fixed point $a_1 \in (x_2, x_1)$ satisfies $|f'_c(a_1)| < 1$, then a_1 is stable point. While, if the fixed point $r_2 \in (x_1, r_1)$ satisfies$|f'_c(r_2)| > 1$; then r_2 is unstable point. If $c \in (c_2, 0)$, the fixed points $r_3 \in (\pi, x_2)$, and $r_4 \in (r_1, 2\pi)$ such that $|f'_c(r_j)| > 1$, j=3,4, then r_j are unstable points for j=3,4. If $c \in (0, c_3)$, the fixed points $a_2 \in (0, x_3)$ and $r_5 \in (x_3, \pi)$ such that $|f'_c(a_2)| < 1$ and $|f'_c(r_5)| > 1$ coreespondigaly. Then a_2 is stable fixed point and r_5 is a unstable fixed point.



$p(x) = |x\cos(x)| - |\sin(x)|$, $\mu(x) = \frac{x}{\csc(x)}$ .Fig. (1)

3. The Proof of theorem 1

The two cases (2) and (3) are similar to case (1), thus their proofs are omitted.

Proof of the main results: Let $H_c(x) = f_c(x) - x$ then

(1) When $c = c_1$, f_c has an indifferent point x_1 by proposition (3).$H'_c(x_1) = 0$ and $H''_c(x_1) > 0$, then H_c has minimum at x_1. Because of $H_c(x_1) = 0$, so $H_c(x) > 0$ for all x in a set containing x_1.so the continuity of H_c leads that for very small $m_1 > 0$, $H_c(x) > 0$ in $(x_1 - m_1, x_1) \cup (x_1, x_1 + m_1)$. Graphicly analysis, $H_c(x) \neq 0$ in $(3\pi, x_1) \cup (x_1, 4\pi)$, $H_c(x)>0$ for all $x \in (3\pi, x_1) \cup (x_1, 4\pi)$ and $H_c(x) < 0$ for $x \in (2\pi, 3\pi)$.

The dynamics behaviour of f_c are studied as follows:

Case (1): For $x \in (b_1, x_1)$; $b_1 \in (3\pi, x_1)$. $x_1$is a minima point for H_k.$H_c(x) > 0$ in $x \in (\pi, x_1) \cup (x_1, 2\pi)$, then for all $x \in (b_1, x_1)$, $H'_c(x) < 0$, so $f'_c(x) - 1 < 0$, hence $f'_c(x) < 1$. therefore by the mean value theorem $|f_c(x) - f_c(x_1)| = f'_c(k)|x - x_1|$ for all $k \in (b_1, x_1)$. that is implies $|f_c(x) - f_c(x_1)| < |x - x_1|$ for all $x \in (b_1, x_1)$. because x_1 is a fixed point of f_c, so $f_c^n(x) \to x_1$ as $n \to \infty$, for which $x \in (b_1, x_1)$.

Case (2): For $x \in (3\pi, b_1) \cup (x_1, 4\pi)$, then $H_c(x) > 0$,hence $f_c(x) > x$, but $f_c(b_1) = x_1$, thus f_c maps the interval $(3\pi, b_1)$ into $(x_1, 4\pi)$, thus it is enough to prove that $f_c^n(x) \to \infty$ as $n \to \infty$ when $x \in (x_1, 4\pi)$, Since $f_c(x) > x$, then $\{f_c^n(x)\}$ is unbounded above and also increasing sequence for which $x \in (x_1, 4\pi)$, thus $f_c^n(x) \to \infty$ as $n \to \infty$, for all $x \in (x_1, 4\pi)$.

Case (3): For $x \in (2\pi, 3\pi)$, $H_c(x) < 0$ and $f_c(x) < x$, thus f_c is decreasing for $x \in (2\pi, 3\pi)$ then$\{f_c^n(x)\}$ is decreasing and it is unbounded below. then for $x \in (2\pi, 3\pi)$,$f_c^n(x) \to -\infty$ as $n \to \infty$)

4) For $c = c_4$, f_c has an indifferent point x_4 by proposition (3).$H'_c(x_4) = 0$, and $H''_c(x_4) > 0$, then H_c has minima at x_4. Because of $H'_c(x_4) = 0$, it

follows that $H_c(x) > 0$ for all x in a set containing x_4. Therefore the continuity of f_c for very little $m_1 > 0$, yields $H_c(x) > 0$ in $x \in (x_4 - m_1, x_4) \cup (x_4, x_4 + m_1)$. Graphicly analysis, $H_c(x) \neq 0$ in $x \in (2\pi, x_4) \cup (x_4, 3\pi)$, so $H_c(x)>0$ for all $x \in (2\pi, x_4) \cup (x_4, 3\pi)$, and $H_c(x) < 0$ for $x \in (3\pi, 4\pi)$.

Next, the dynamics behaver of f_k are obtained as bellow:
Case (1): For $x \in (b_4, x_4)$; $b_3 \in (2\pi, x_4)$,x_3 is a minima point forH_c. $H_c(x) > 0$ in $(2\pi, x_4) \cup (x_4, 3\pi)$, thus for all $x \in (b_4, x_4)$, $H'_c(x) < 0$, so $H_c(x) < 0$, then $f'_c(x) < 1$. For $x > x_4$, $f'_c(x) > 1$. So, by the mean value theorem $|f_c(x) - f_c(x_1)| = f'_c(k)|x - x_1|$ such that $k \in (b_4, x_4)$. Since x_3 is a fixed point of f_c, such that $|f_c(x) - f_c(x_4)| < |x - x_4|$ for all $x \in (b_4, x_4)$. Thus $f_c^n(x) \to x_4$as $n \to \infty$, for all $x \in (b_4, x_4)$.

Case (2): for $x \in (2\pi, b_4) \cup (x_4, 3\pi)$ then $H_c(x) > 0$ for all $x \in (x_4, 3\pi)$, hence $f_c(x) > x$, since f_c is increasing in this interval, and $0 < x < f_c(x) < f_c^2(x) < \cdots < f_c^n(x) <$.. Then the sequence $\{f_c^n(x)\}$ is increasing and unbounded above thus $f_c^n(x) \to \infty$as $n \to \infty$, when $x \in (x_4, 3\pi)$. Since $f_c(b_4) = x_4$ and $f_c$maps the interval $(2\pi, b_4)$ into $(x_4, 3\pi)$ hence in the similar way, we can prove $f_c^n(x) \to \infty$ as $n \to \infty$ when $x \in (2\pi, b_4)$.

Case (3): at $x \in (3\pi, 4\pi)$, $H_c(x) < 0$ then$f_c(x) < x$, therefore f_c is decreasing in above interval and $x_3 > f_c(x) > f_c^2(x) > \cdots > f_c^n(x) > \cdots$. Thus $\{f_c^n(x)\}$ is decreasing and it is unbounded below. So for $x \in (3\pi, 4\pi) f_c^n(x) \to -\infty$ as $n \to \infty$. 5) when $c \in (c_1, c_2)$, by proposition (3)f_c has stable fixed point $a_1 \in (x_2, x_3)$ and repelling fixed point $r_2 \in (x_1, r_1)$. Graphicly analysis, $H_c(x) \neq 0$ in $(3\pi, r_2) \cup (r_2, 4\pi)$,$T_c(x)>0$ for all $x \in (3\pi, a_1) \cup (r_2, 4\pi)$ and $H_c(x) < 0$ for $x \in (2\pi, 3\pi) \cup (a_2, r_2)$.

To consider the dynamics behavior of the functions f_c, we notice three cases:
Case (1) For $x \in (b_5, r_2)$. $f_c(x) < x$ for all $x \in (a_1, r_2)$and it is decreasing, so $x > f_c(x) > f_c^2(x) > \cdots > f_c^n(x) > a_1$. Thus, $\{f_c^n(x)\}$ is decreasing sequence and it is bounded below at a_1, and a_1 is the larger fixed point. Then $f_c^n(x) \to a_1$ as $n \to \infty$for each $x \in (b_5, r_2)$.

Case (2): For $x \in (3\pi, b_5) \cup (r_2, 4\pi)$, $f_c(x) > x$ for all $x \in (r_2, 4\pi)$. Since f_c is increasing functions and $\{f_c^n(x)\}$ continuing the forward iteration procedure, so, $0 < x < f_c(x) < f_c^2(x) < \cdots < f_c^n(x) < \cdots$ Then, the sequence $\{f_c^n(x)\}$ is increasing and r_2 is the larger fixed point, the orbit of r_2 must go to $\infty$when $n \to \infty$. Therefore $f_c^n(x) \to \infty$ when $n \to \infty$ for each $x \in (r_2, 4\pi)$. $f_c(b_5) = r_2$, f_c maps the interval (π, b_5)into $(r_2, 4\pi)$. Then by using the above arguments $f_c^n(x) \to \infty$ as $n \to \infty$ for all $x \in (3\pi, b_5)$.

Case (3): For all $x \in (2\pi, 3\pi)$, $H_c(x) < 0$, hence$f_c(x) < x$, therefore f_c is decreasing in this interval and $x > f_c(x) > f_c^2(x) > \cdots > f_c^n(x) > \cdots$. Then $\{f_c^n(x)\}$ is decreasing and unbounded below. Then, for each

$x \in (2\pi, 3\pi) f_c^n(x) \to -\infty$ as $n \to \infty$. 6) when $c \in (c_2, 0)$, f_c has two fixed points $r_3 \in (3\pi, x_2)$ and $r_4 \in (r_1, 4\pi)$ which are repelled by proposition (3). Graphicly analysis, $H_c(x) > 0$ for all $x \in (3\pi, b_6) \cup (r_4, 4\pi)$ and $H_c(x) < 0$ for $x \in (2\pi, 3\pi) \cup (r_3, r_4)$.

To evaluate the dynamics behavior of f_c, we can show following: -
Case (1): for each $x \in (r_4, 4\pi)$, $f_c(x) > x$. Hence f_c is strictly increasing in $(r_4, 4\pi)$, then $0 < r_4 < x < f_c(x) < f_c^2(x) < \cdots < f_c^n(x) < \cdots$. So, the sequence $\{f_c^n(x)\}$is an increasing that is unbounded above. So $f_c^n(x) \to \infty$as $n \to \infty$ for each $x \in (r_4, 4\pi)$. Then $f_c^n(x) \to \infty$ when $n \to \infty$ for each $x \in (3\pi, r_4)$. Because $f_c(b_6) = r_5$, andf_c maps the interval $(3\pi, r_4)$ into $(r_4, 4\pi)$. By using the above arguments $f_c^n(x) \to \infty$ as $n \to \infty$ when $x \in (3\pi, r_4)$.

Case (2): For all $x \in (r_3, r_4)$ the dynamics behavior of f_c has no attractors point. thus, the dynamic of the function will be chaotically, and $\{f_c^n(x)\}$ is chaotic for all $x \in (r_3, r_4)$.

Case (3): At $x \in (2\pi, 3\pi)$, $H_c(x) < 0$ then$f_c(x) < x$. Thus f_c is strictly decreasing in this interval and $x_1 > f_c(x) > f_c^2(x) > \cdots > f_c^n(x) > \cdots$ Then $\{f_c^n(x)\}$ is decreasing and unbounded below. So for $x \in (2\pi, 3\pi)$, $f_c^n(x) \to -\infty$ as $n \to \infty$. 7) when $c \in (0, c_3)$ this case is similar to the case (6). 8) when $c \in (c_4, c_3)$, this case is similar to case (5). 9) when $c < c_1$ $(c > c_4)$, by proposition (2) f_c has no fixed point. The functions H_c is continues and differentiable for each $x \in (2\pi, 4\pi)$ since f_c is continuous and differentiable for $x \in (2\pi, 4\pi)$. The sequence $\{f_C^n(x)\}$ is increasing and unbounded below for all $x \in (2\pi, 4\pi)$. Then $f_C^n(x) \to \infty$ when $n \to -\infty$ for all $x \in (2\pi, 4\pi)$. If $c > c_3$, we have the similar previous arguments.

References

[1] Devaney, R.L., "Dynamics Topology, and Bifurcations of Complex exponentials", *Topology Appl.* 110, 133-161 (2001).

[2] Al-Husseiny, H., F.," A Study of the Dynamics of the family $\lambda \frac{sinh^m z}{z^{2m}}$ ", *Iraqi Journal of Science*, 4(52), 494-503 (2011).

[3] Sajid, M., "Real and Complex Dynamics of One Parameter Family of Meromorphic Functions", *Far East.Dyn. Syst*., 19(2), 89-105 (2012).

[4] Akbari, M., Rabii, M., "Real Cubic Polynomials With a Fixed Point of Multiplicity Two", *Indagationes Mathematicate*, 26, 64-74 (2015).

[5] Magrenan, A., Gutierrez, J., "Real Dynamics for Damped Newtons Method Applied to Cubic Polynomials", *Comput. Appl. Math.*, 275, 527-538 (2015).

[6] Radwan, A. G., "On Some Generalized Discrete Logistic Maps", *J. Adv. Res.*, 4(2), 163-171 (2013).

[7] Faris, S. M., "Dynamics of Certain Families of Transcendental Meromorphic Functions", Ph.D. thesis University of Baghdad (2006).