Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 5, pp. 931–935
DOI : 10.47974/JIM-1531

New open set on topological space with generalized topology

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Abstract

We all know the vast number of studies that presented several generalizations of open sets in topological spaces. On the other hand, since 2002 when A. Csaszar introduced the definition of generalized topology, other generalizations of open sets have appeared in generalized topological spaces. Our study provides a new generalization for an open set, where the definition will be a mixture of ordinary topology and generalized topology. Our new open set is named *X*ω-open set. We give several characterizations, properties, and examples related to the new concept.

Subject Classification: *18B30, 28C15.*

Keywords: *Generalized topological Spaces, ω-open, ω-closed, Xω-open and Xω-closed set.*

1. Introduction

A pair $(Z,$ Ӽ$)$, where Z is a set and Ӽ is a family of subsets of Z, is denote a topological space. Members of Ӽ are said to be open in [1]. The complement of any open subset of Z is closed in Z [2]. We will denote the collection of all closed sets by Ӽ′. If Ꜧ is a subset of Z, the union of all open sets contained in Ꜧ is called the interior of Ꜧ and is symbolized by $int($Ꜧ$)$ whereas the intersection of all closed sets containing Ꜧ is the closure of Ꜧ and is symbolized by $cl($Ꜧ$)$[3].

A generalized topological space is a pair $(Z,$ ω$)$, where Z is a set, and ω is a collection of subsets of Z containing $\emptyset$ and closed under the arbitrary unions. Members of ω are ω -open sets. The complement of ω -open is ω – closed, and the collection of all ω − closed sets is symbolized by ω′[4]. ω −

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the interior of Ꜧ is the largest ѡ -open set in Ꜧ and is symbolized by $int_{ѡ}$(Ꜧ), while the smallest ѡ -closed set containing Ꜧ is the ѡ − closure of Ꜧ and is symbolized by $cl_{ѡ}$(Ꜧ)[5]. The union of all elements of ѡ will be symbolized by $M_{ѡ}$, and (Z, ѡ) is said to be strong if $M_{ѡ} = Z$[6].

2. Ꭓѡ − *open set*

Definition 2.1: Let (Z, Ꭓ) be a topological space and let ѡ be a generalized topology on Z. A subset Ꜧ of Z is said to be Ꭓѡ − *open* set if there exists an ѡ –open set Ŋ such that Ŋ ⊆ Ꜧ ⊆ cl(Ŋ).

The collection of all Ꭓѡ − *open* sets of Z will be denoted by Ꭓѡ$O(Z)$. The complement of Ꭓѡ − *open* is Ꭓѡ − *closed* set, the class of all Ꭓѡ − *closed* sets in Z will be denoted by Ꭓѡ$C(Z)$.

Theorem 2.2: *A subset* Ꜧ *of a space* (Z, Ꭓ) *with* GT, ѡ *is* Ꭓѡ − *open if and only if* Ꜧ ⊆ $cl\left(int_{ѡ}(Ꜧ)\right)$.

Proof:

The "if" part

Suppose that Ꜧ is Ꭓѡ − *open* set, then there exists an ѡ –open set Ŋ such that Ŋ ⊆ Ꜧ ⊆ cl(Ŋ), it follows that $int_{ѡ}$(Ŋ) ⊆ $int_{ѡ}$(Ꜧ), but $int_{ѡ}$(Ŋ) = Ŋ since Ŋ is ѡ –open, so Ŋ ⊆ $int_{ѡ}$(Ꜧ), this means that cl(Ŋ) ⊆ $cl\left(int_{ѡ}(Ꜧ)\right)$. Therefore, Ꜧ ⊆ $cl\left(int_{ѡ}(Ꜧ)\right)$.

The "only if" part

Suppose that Ꜧ ⊆ $cl\left(int_{ѡ}(Ꜧ)\right)$, put Ŋ = $int_{ѡ}$(Ꜧ). It follows that Ŋ ⊆ Ꜧ ⊆ cl(Ŋ) where Ŋ is an ѡ –open. Hence Ꜧ is Ꭓѡ − *open* set.

Remark 2.3: Every ѡ –open (ѡ - *closed*) set is Ꭓѡ − *open* (Ꭓѡ - *closed*) set.

The inverse of Remark 2.3 is not validated, as shown in the example below.

Example 2.4: Let Z = {Ǝ, Б, ȼ}, Ꭓ = {Z, ∅, {Ǝ}}, Ꭓ′ = {Z, ∅, {Б, ȼ}}, and let ѡ = {∅, {Ǝ}, {ȼ}, {Ǝ, ȼ}}, ѡ′ = {Z, {Б, ȼ}, {Ǝ, Б}, {Б}}, then Ꭓѡ$O(Z)$ = {Z, ∅, {Ǝ}, {ȼ}, {Ǝ, ȼ}, {Б, ȼ}, {Ǝ, Б} and Ꭓѡ$C(Z)$ = {Z, ∅, {Ǝ}, {ȼ}, {Б}, {Б, ȼ}, {Ǝ, Б}}. Not that, Z is Ꭓѡ − *open* set, but not ѡ –open, and {Ǝ} is Ꭓѡ - *closed*, but not ѡ - *closed*.

Remark 2.5: Open set and Ꭓѡ − *open* set are independent. As we show in the following example.

Example 2.6: Let $Z = \{ꝺ, Б, ȼ\}, Ӿ = \{Z, \emptyset, \{ꝺ\}\}, Ӿ' = \{Z, \emptyset, \{Б, ȼ\}\}$, and let $ῳ = \{\emptyset, \{ȼ\}, \{ꝺ, ȼ\}\}$, $ῳ' = \{Z, \{ꝺ, Б\}, \{Б\}\}$, then $ӾῳO(Z) = \{Z, \emptyset, \{ȼ\}, \{ꝺ, ȼ\}, \{Б, ȼ\}$ and $ӾῳC(Z) = \{Z, \emptyset, \{ꝺ\}, \{Б\}, \{ꝺ, Б\}\}$.

Not that, $\{ȼ\}$ is $Ӿῳ - open$ set, but not open, and $\{ꝺ\}$ is an open set, but not $Ӿῳ - open$.

Proposition 2.7: The union of any collection of $Ӿῳ - open$ sets is $Ӿῳ - open$ set.

Proof: Let $\{Ꜧ_\alpha : \alpha \in \Lambda\}$ be a family of $Ӿῳ - open$ sets, we have to show that $\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha$ is $Ӿῳ - open$ set. From Theorem 2.2 we have $Ꜧ_\alpha \subseteq cl\left(int_{ῳ}(Ꜧ_\alpha)\right)$ for each α, it follows that

$$\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha \subseteq \bigcup_{\alpha\in\Lambda} cl\left(int_{ῳ}(Ꜧ_\alpha)\right).$$

But $\bigcup_{\alpha\in\Lambda} cl\left(int_{ῳ}(Ꜧ_\alpha)\right) \subseteq cl\left(int_{ῳ}\left(\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha\right)\right)$ by using properties of the closure of a set and properties of the $ῳ -$ the interior of a set. Now we have $\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha \subseteq cl\left(int_{ῳ}\left(\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha\right)\right)$. Hence $\bigcup_{\alpha\in\Lambda} Ꜧ_\alpha$ is $Ӿῳ - open$ set by Theorem 2.2.

Remark 2.8: The intersection of two $Ӿῳ - open$ sets is not necessarily an $Ӿῳ - open$ set. As we show in the following example.

Example 2.9: In Example 2.4, note that $\{a, b\}$ and $\{b, c\}$ are $Ӿῳ - open$ sets, whereas $\{a, b\} \cap \{b, c\} = \{b\}$ is not $Ӿῳ - open$.

Theorem 2.10: For any topological space $(Z, Ӿ)$ with $GT, ῳ$, $ӾῳO(Z)$ is a generalized topology on Z.

Proof: Follows from Remark 2.3 and Proposition 2.7.

Remark 2.11: For any topological space $(Z, Ӿ)$ with $GT, ῳ$, the union of all elements of $ӾῳO(Z)$ will be denoted by $M_{Ӿῳ}$, and $(Z,\ ӾῳO(Z))$ is said to be strong if $M_{Ӿῳ} = Z$.

Remark 2.11 In any topological space $(Z, Ӿ)$ with $GT, ῳ$**,** $M_{ῳ} \subseteq M_{Ӿῳ}$, but not conversely.

For instance, see Example 2.6, $M_{Ӿῳ} = Z$, while $M_{ῳ} = \{a, c\}$.

Definition 2.12: Let $(Z, Ӿ)$ be a topological space and let $ῳ$ be a generalized topology on Z and let $Ꜧ$ be a subset of Z. $Ꜧ$ is an $Ӿῳ$ – neighborhood of a point $z \in Z$ if there is an $Ӿῳ$ –open subset $G\ of\ Z$ such that $z \in G \subseteq Ꜧ$. Every $Ӿῳ$ –

open set containing z is also Ꭓɷ – a neighborhood of z, which is called Ꭓɷ – an open neighborhood.

Definition 2.13: Let $(Z, Ꭓ)$ be a topological space and let ɷ be a generalized topology on Z.

1. **** Ꭓɷ $-$ the interior of a set Ꮒ is symbolized by $int_{Ꭓɷ}(Ꮒ)$, and is defined by $int_{Ꭓɷ}(Ꮒ) = \bigcup\{G: G\ is\ Ꭓɷ\ -open\ ,\ G \subseteq Ꮒ\}$.
2. Ꭓɷ $-$ closure of a set Ꮒ is symbolized by $cl_{Ꭓɷ}(Ꮒ)$, and is defined by $cl_{Ꭓɷ}(Ꮒ) = \bigcap\{F: F\ is\ Ꭓɷ\ -closed, Ꮒ \subseteq F\ \}$.

Example 2.14: Let's take Example2.6, if Ꮒ $= \{a, c\}$, then $int_{Ꭓɷ}(Ꮒ) = \{a, c\}$ and $cl_{Ꭓɷ}(Ꮒ) = Z$.

$$int_{ɷ}(Ꮒ) = \{a, c\} \text{ and } cl_{ɷ}(Ꮒ) = Z.$$

Theorem 2.15: *Let $(Z, Ꭓ)$ be a topological space and let ɷ be a generalized topology on Z and let Ꮒ $\subseteq Z$ and $z \in$ Ꮒ, then $z \in int_{Ꭓɷ}(Ꮒ)$ if and only if Ꮒ is an Ꭓɷ – neighborhood of z.*

Proof: $z \in int_{Ꭓɷ}(Ꮒ) = \bigcup\{G: G\ is\ Ꭓɷ\ -open\ , G \subseteq Ꮒ\} \Leftrightarrow \exists$ Ꭓɷ –open subset $G\ of\ Z \ni z \in G \subseteq Ꮒ \Leftrightarrow$ Ꮒ is an Ꭓɷ – neighborhood of z.

Theorem 2.16: *Let $(Z, Ꭓ)$ be a topological space and let ɷ be a generalized topology on Z and let Ꮒ $\subseteq Z$ and $z \in Z$, then $z \in cl_{Ꭓɷ}(Ꮒ)$ if and only if every Ꭓɷ –open neighborhood of z intersects Ꮒ.*

Proof:

The "if" part

Assume that $z \in cl_{Ꭓɷ}(Ꮒ) = \bigcap\{F: F\ is\ Ꭓɷ\ -closed, Ꮒ \subseteq F\ \}$ and suppose that there exists an Ꭓɷ –open neighborhood $G\ of\ z$ such that $G \cap Ꮒ = \emptyset$, it follows that Ꮒ $\subseteq G^c$ which is Ꭓɷ $-closed$ set with $z \notin G^c$, so $z \notin \bigcap\{F: F\ is\ Ꭓɷ\ -closed, Ꮒ \subseteq F\ \}$ which is a contradiction.

The "only if" part

Assume that every Ꭓɷ –open neighborhood of z intersects Ꮒ and suppose that $z \notin cl_{Ꭓɷ}(Ꮒ) = \bigcap\{F: F\ is\ Ꭓɷ\ -closed, Ꮒ \subseteq F\ \}$, then there exists an Ꭓɷ $-closed$ subset $F\ of\ Z$ with Ꮒ $\subseteq F$ such that $z \notin F$, so $z \in F^c$ which is Ꭓɷ –open, but $F^c \cap Ꮒ = \emptyset$, that implies to contradiction.

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Received June, 2022

Revised July, 2022