Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 26 (2023), No. 5, pp. 835–847
DOI : 10.47974/JIM-1509

New 2D numerical integration formula based on the Legendre wavelets

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Abstract

Based on Legendre wavelets, a new efficient numerical integration method is proposed to estimate double integrals. This new method is expressed in terms of the operational integration matrices. Those which were introduced by Parsian, are calculated in an approximate way. In this work, we also propose the exact computation of these matrices. Better precision of the new method is observed during a comparative numerical study with the composite Simpson method.

Subject Classification: *(2010) 65D30, 46B15, 42C10.*

Keywords: *Legendre wavelet, Operational matrix of integration, Numerical integration, Simpson composite method.*

1. Introduction

We know that the Legendre and the Jacobi wavelets have been widely used for solving variational equations. We can for example consult [11], [6] for the differential equations, [2], [1] for the partial differential equations, [4] for the integral equations and [10] for the integro-differential equations.

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Despite the emergence of efficient integration methods [8], the strength of wavelets methods lies in the operational matrices of integration. Indeed, solving a variational problem using wavelets reduces to solving a simple linear algebraic system. This transformation is mainly due to the introduction of these integration matrices.

In this work, we propose another application of operational matrices of integration. We will establish a new formula to numerically estimate double integrals. This formula is obtained by approaching the function to be integrated f by the discrete sum

$$\sum C_{n_x m_x n_y m_y} \psi_{n_x m_x n_y m_y}(x,y),$$

where $\psi_{n_x m_x n_y m_y}(x,y)$ are the two-dimensional Legendre wavelets and $C_{n_x m_x n_y m_y}$ are the coefficients of the decomposition of the function f.

Then the double integral

$$\iint f(x,y)\,dxdy$$

is estimated by

$$\iint \sum C_{n_x m_x n_y m_y} \psi_{n_x m_x n_y m_y}(x,y)dxdy. \tag{1}$$

The approximation 1 will give rise to a simple formula, called the integration formula of the Legendre wavelets **ILW** which is expressed as a function of the two matrices of integration associated with the two-dimensional Legendre wavelets.

Those that we know are the matrices introduced in 2005 by Parsian [9] which are denoted P_x and P_y and are calculated approximately.

The first objective of this work is the exact calculation of these integration matrices. We will denote them P_{xex} and P_{yex}.

The second and the main objective is, of course, to establish this new **ILW** formula. It is denoted **ILWN** when it is expressed as a function of the exact matrices P_{xex} and P_{yex} and is denoted **ILWP** when it is expressed as a function of the Parsian matrices P_x and P_y.

Several numerical tests were performed to validate the formula **ILW**. As the principle of obtaining the **ILW** formula is close to that of the Newton-Côtes formulas, the **ILW** and **ILWN** integration methods were compared with that of composite Simpson [5]. The results obtained by **ILWN** and **ILWP** are precise, satisfying and are better than of those given by the Simpson method.

2. Preliminaries

In this section, we present some definitions and basic concepts that will be used in the sequel.

Definition 1 (Legendre polynomials) : The Legendre polynomials L_m of order $m \geqslant 0$, are defined on the interval [0, 1] by the following recurrence formulas:

$$\begin{cases} L_0(x) = 1, L_1(x) = x \\ L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), m = 1, 2, \ldots \end{cases} \tag{2}$$

To define the two-dimensional Legendre wavelets on the square $[0,1]\times[0,1]$ (see [9]), we start by fixing two non-zero integers j_x and j_y. The square $[0,1]\times[0,1]$ is then subdivided into 2^{j_x-1} 2^{j_y-1} sub-domains

$$I_{n_x n_y} = \left[\frac{n_x-1}{2^{j_x-1}}, \frac{n_x}{2^{j_x-1}}\right[\times \left[\frac{n_y-1}{2^{j_y-1}}, \frac{n_y}{2^{j_y-1}}\right[; 1 \leqslant n_x \leqslant 2^{j_x-1} \text{ and } 1 \leqslant n_y \leqslant 2^{j_y-1},$$

in such a way that subdomains cover the original square domain $[0,1]\times[0,1]$, i.e.

$$[0,1]\times[0,1] = \bigcup_{1\leqslant n_x \leqslant 2^{j_x-1} \text{ and } 1 \leqslant n_y \leqslant 2^{j_y-1}} \overline{I_{n_x n_y}}.$$

Definition 2 : Two-dimensional Legendre wavelets are defined as

$$\psi_{n_x m_x n_y m_y}(x,y) = \begin{cases} 2^{\frac{jx}{2}} 2^{\frac{jy}{2}} \sqrt{\frac{2m_x+1}{2}} \sqrt{\frac{2m_y+1}{2}} L_{m_x}(2^{j_x} x - 2n_x + 1) L_{my}(2^{jy} y - 2n_y + 1) \\ \qquad\qquad \text{if } (x,y) \in I_{n_x n_y}, \\ 0 \text{ if } (x,y) \notin I_{n_x n_y}, \end{cases} \tag{3}$$

where $n_x = 1, 2, \ldots 2^{j_x-1}$, $n_y = 1, 2, \ldots 2^{j_y-1}$ and L_m, is Legendre polynomial defined in 2.

Theorem 3 [2] : The family $\{\psi_{n_x m_x n_y m_y}\}; x = 1, 2, \ldots 2^{j_x-1}$, $n_y = 1, 2, \ldots 2^{j_y-1}$, $m_x \geqslant 0$ and $m_y \geqslant 0$, is a Hilbert basis of $L^2(]0,1[\times]0,1[)$.

This theorem makes it possible to any function f of $L^2(]0,1[\times]0,1[)$ to be expanded as

$$f(x,y) \simeq \sum_{n_x=1,2^{j^{x-1}},\, m_x=0,M_x-1} \sum_{n_y=1,\,2^{j^{y-1}},\, m_y=0,\,M_y-1} C_{n_x m_x n_y m_y} \psi_{n_x m_x n_y m_y}(x,y), \tag{4}$$

where the coefficients of this decomposition are defined by

$$C_{n_x m_x n_y m_y} = \int_0^1 \int_0^1 f(x,y)\psi_{n_x m_x n_y m_y}(x,y)dxdy.$$

The truncated version of equation 4 can be expressed as the form

$$f(x,y) \simeq C^T \psi(x,y),$$

where C is coefficients vector and $\psi(x,y)$ is wavelets vector, given by

$$C = (C_{1010}, C_{1011} \ldots, C_{101M_y}-1, \ldots, C_{10j^{y-1}M_y-1}, \ldots, C_{2^{j_x-1}M_x-12^{j_y-1}M_y-1})^T,$$

$$\psi(x,y) = (\psi_{n_x m_x n_y m_y}(x,y))_{1\leqslant n_x \leqslant 2^{j_x-1};\, 0\leqslant m_x \leqslant M_x-1;\, 1\leqslant n_y \leqslant 2^{j_y-1};\, 0\leqslant m_y \leqslant M_y-1}. \tag{5}$$

3. Two-dimensional exact operational matrices of integration

The exact operational matrices of integration P_{xex} and P_{yex} are defined by the relations

$$\int_0^x \psi(\tau,y)d\tau = P_{xex}\psi(x,y), \tag{6}$$

and

$$\int_0^y \psi(x,\tau)d\tau = P_{yex}\psi(x,y). \tag{7}$$

3.1 *Determination of the matrix* P_{xex}

Theorem 4 : *The exact matrix of integration* P_{xex} *defined by the formula 6 is of order* $N = 2^{j_x-1}2^{j_y-1}M_xM_y$ *and is given by*

$$P_{xex} = \frac{1}{2^{j_x}} \begin{bmatrix} \mathbf{L}_x & F & F & \cdots & F \\ \mathbf{0} & \mathbf{L}_x & F & \cdots & F \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_x & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{L}_x \end{bmatrix}, \tag{8}$$

where $\mathbf{0}$ *is the zero matrix of order* $2^{j_y-1}M_xM_y$, F *is a matrix of dimension* $2^{j_y-1}M_xM_y$ *defined by*

$$F = 2 * I. \tag{9}$$

and $\mathbf{L}_x$ *is block tridiagonal, which is composed of* M_x *submatrices of order* $2^{j_y-1}M_y$

$$\mathbf{L}_x = \begin{bmatrix} I & \frac{1}{\sqrt{3}}I & \mathbf{0}' & \cdots & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \frac{-1}{\sqrt{3}}I & \mathbf{0}' & \frac{1}{\sqrt{3\times5}}I & \cdots & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0}' & \frac{-1}{\sqrt{5\times3}}I & \mathbf{0}' & \ddots & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \frac{-1}{\sqrt{7\times5}}I & \ddots & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \ddots & \frac{-1}{\sqrt{(2M_x-3)(2M_x-5)}}I & \mathbf{0}' & \frac{1}{\sqrt{(2M_x-1)(2M_x-3)}}I \\ \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \cdots & \mathbf{0}' & \frac{-(\sqrt{2M_x-1}}{M_x\sqrt{2M_x-3}}I & \frac{2^{jx}x-2n_x+1}{M_x}I \end{bmatrix}, \tag{10}$$

where I *is the* $2^{j_y-1}M_y \times 2^{j_y-1}M_y$ *unit matrix.*

Proof: In order to determine the matrix P_{xex}, we express $\int_0^x \psi_{n_xm_xn_ym_y}(\tau, y)d\tau$ in terms of the elements of the vector $\psi(x, y)$ given by 5.

During calculations, terms of the type $\psi_{n_xM_xn_ym_y}$ are written according to $\psi_{n_xM_x-1n_ym_y}$ and $\psi_{n_xM_x-2n_ym_y}$ using the definition 3 and the recurrence formula 2. □

3.2 *Determination of the matrix* P_{yex}

The same calculations as before are repeated but this time the integrals are done with respect to the second variable y.

Theorem 5 : *The exact matrix of integration* P_{yex} *defined by the formula 7 is of order* $N = 2^{j_x-1}2^{j_y-1}M_xM_y$ *(containing* $2^{j_x-1} \times 2^{j_x-1}$ *submatrices of order* $2^{j_y-1}M_xM_y$*) and is of the form*

$$P_{yex} = \frac{1}{2^{j_y}} \begin{bmatrix} D & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & D & & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & D \end{bmatrix}, \tag{11}$$

where $\mathbf{0}$ *is the zero matrix and* D *is a block diagonal matrix formed of* $M_x \times M_x$ *submatrices of order* $2^{j_y-1}M_y$ *and which is defined by*

$$D = \begin{bmatrix} Mat & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Mat & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & Mat \end{bmatrix} \text{ with } Mat = \begin{bmatrix} \mathbf{L}_y & F & \dots & & F \\ \mathbf{0} & \mathbf{L}_y & F & \dots & F \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{L}_y & F \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{L}_y \end{bmatrix}.$$

This matrix Mat *is formed of* $2^{j_y-1} \times 2^{j_y-1}$ *blocks of order* M_y *where*

$$F = 2 * I,$$

and

$$\mathbf{L}_y = \begin{bmatrix} I & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3\times5}} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{5\times3}} & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{7\times5}} & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \frac{-1}{\sqrt{(2M_y-3)(2M_y-5)}} & 0 & \frac{1}{\sqrt{(2M_y-1)(2M_y-3)}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{-\sqrt{2M_y-1}}{M_y\sqrt{2M_y-3}} & \frac{2^{jy}\, y-2n_y+1}{M_t} \end{bmatrix}.$$

4. Applications to double numerical integration

For a function f defined and integrable on a domain $[a_1, a_2]\times[b_1, b_2]$, we can numerically express the double integral $\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)dudv$, in terms of the matrices P_{xex} and P_{yex} defined above by 8 and 11 respectively.

Theorem 6 : *Let f be a function defined on $L^2(]a_1,a_2[\times]b_1,b_2[)$ and integrable on, then*

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)dudv \simeq (a_2-a_1)(b_2-b_1)C^T P_{xex} P_{yex}\psi(1,1). \tag{12}$$

The vector ψ is given by 5 and the vector C is composed of the decomposition coefficients of the function $f(a_1+(a_2-a_1)x, b_1+(b_2-b_1)y)$.

Proof: By changing variables

$$\begin{cases} u = a_1+(a_2-a_1)x \\ v = b_1+(b_2-b_1)y, \end{cases}$$

the integration on $]a_1,a_2[\times]b_1,b_2[$, is transformed to the integration on the unit square

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)dudv = (a_2-a_1)(b_2-b_1)\int_0^1\int_0^1 f(a_1+(a_2-a_1)x, b_1+(b_2-b_1)y)dxdy.$$

Recalling 4, we have the approximation

$$f(a_1+(a_2-a_1)x, b_1+(b_2-b_1)y) \simeq C^T\psi(x,y),$$

where the coffecients of the vector C are given by

$$C_{n_x m_x n_y m_y} = \int_0^1\int_0^1 f(a_1+(a_2-a_1)x, b_1+(b_2-b_1)y)\psi_{n_x m_x n_y m_y}(x,y)dxdy.$$

Thus, the double integral is approximated by

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)dudv \simeq (a_2-a_1)(b_2-b_1)\int_0^1\int_0^1 C^T\psi(x,y)dxdy. \tag{13}$$

By Fubini's theorem [7] applied to 13 and according to 7 with $y=1$, we have

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)dudv \simeq (a_2-a_1)(b_2-b_1)C^T P_{yex}\int_0^1 \psi(x,1)dx.$$

And for $x=1$, the relation 6 leads to

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} f(u,v)\,dudv \simeq (a_2-a_1)(b_2-b_1)C^T P_{yex} P_{xex}\psi(1,1).$$

□

5. Numerical results

To show the efficiency and the precision of the **ILW** method, three examples are presented in this section. We denote by **ILWN** and **ILWP** the integration formula expressed as a function of the exact matrices and Parsian matrices, respectively.

In other words, we estimate the double integral

$$I_{EX} = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)dxdy,$$

by the two formulas

$$ILWN = (a_2 - a_1)(b_2 - b_1)C^T P_{xex} P_{yex} \psi(1,1),$$

and

$$ILWP = (a_2 - a_1)(b_2 - b_1)C^T P_x P_y \psi(1,1).$$

The results obtained are compared with those given by the compound Simpson method, noted I_{SIMP}.

We consider the following absulte errors :

$$E_{ILWN} = \left|I_{EX} - ILWN\right|, \quad E_{ILWP} = \left|I_{EX} - ILWP\right|, \quad E_{SIMP} = \left|I_{EX} - I_{SIMP}\right|.$$

For each example, we fix the number of subdivisions of the domain of integration, which is $2^{j_x-1}2^{j_y-1}$, and this by fixing the non-zero integers j_x and j_y. Then, we vary the degrees of the polynomials M_x and M_y. For more details, see the definition 3 and the approximation formula 4.

Example 7 : For the function $f(x,y) = e^x + e^y$, we have

$$I_{EX} = \int_{-1}^{1}\int_{-1}^{1}(e^x + e^y)dxdy = 4(e^1 - e^{-1}).$$

The results are summarized in the following Tables 1 and 2.

Table 1

Calculation of the errors for $j_x = j_y = 2$.

	N	E_{ILWN}	E_{ILWP}	E_{SIMP}
$M_x = M_y = 3$	36	0.0012	0.0567	$6.3640e-04$
$M_x = M_y = 5$	100	$3.3532e-6$	$2.2289e-4$	$8.3173e-05$
$M_x = M_y = 8$	256	$3.7220e-9$	$1.6489e-8$	$1.2728e-05$
$M_x = M_y = 10$	400	$4.5101e-12$	$1.4384e-11$	$5.2169e-06$

Table 2

Calculation of the errors for $j_x = j_y = 3$.

	N	E_{ILWN}	E_{ILWP}	E_{SIMP}
$M_x = M_y = 2$	64	0.0244	0.2013	$2.02520e-04$
$M_x = M_y = 4$	256	$4.0766e-5$	$3.5641e-4$	$1.728e-05$
$M_x = M_y = 6$	576	$3.2269e-8$	$2.3144e-7$	$2.5168e-06$
$M_x = M_y = 8$	1024	$1.4923e-11$	$7.8740e-11$	$7.9661e-07$

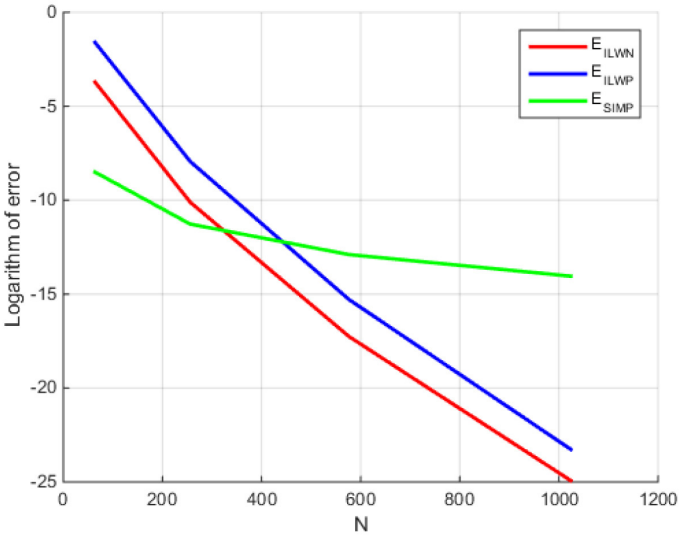


Figure 1

Logarithm of error for $j_x = j_y = 3$ and $f(x,y) = e^x + e^y$,.

Example 8 : For the function $f(x,y)=\frac{\sin y}{e^x+1}$, the exact integral on $]-1,1[\times]0,\pi[$ is

$$\int_{-1}^{1}\int_{0}^{\pi}\frac{\sin y}{e^x+1}dxdy=2\left(\ln(\frac{e^1}{1+e^1})-\ln(\frac{e^{-1}}{1+e^{-1}})\right).$$

The results are summarized in the Tables 3 and 4 below:

Table 3

Calculation of the errors for $j_x = j_y = 2$.

	N	E_{ILWN}	E_{ILWP}	E_{SIMP}
$M_x = M_y = 3$	36	0.0018	0.0196	$8.6319e-04$
$M_x = M_y = 5$	100	$1.2060e-5$	$2.0115e-4$	$1.0952e-04$
$M_x = M_y = 8$	256	$3.3219e-8$	$3.4284e-8$	$1.6591e-05$

Table 4

Calculation of the errors for $j_x = j_y = 3$.

	N	E_{ILWN}	E_{ILWP}	E_{SIMP}
$M_x = M_y = 3$	144	$1.0464e-4$	0.0012	$5.2624e-05$
$M_x = M_y = 5$	400	$1.7672e-7$	$3.3303e-6$	$6.7844e-6$
$M_x = M_y = 8$	1024	$1.2132e-10$	$3.7461e-10$	$1.0333e-6$
$M_x = M_y = 10$	1600	$9.0594e-14$	$2.0161e-13$	$4.2309e-7$

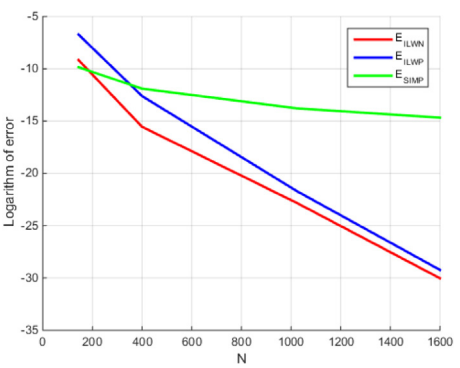


Figure 2

Logarithm of error for $j_x = j_y = 3$ and $f(x,y)=\frac{\sin y}{e^x+1}$.

Example 9 : It is clear that by changing variables

$$x = r\cos\theta, y = r\sin\theta,$$

the open disc $D(0_{\mathbb{R}^2}, R)$ centered at the origin of radius R transforms into $]0,R[\times]0,2\pi[$. So, for the function $f(x,y) = e^{-(x^2+y^2)}$, the exact integral on the disc $D(0_{\mathbb{R}^2}, R)$, is given by

$$\iint_{D(0_{\mathbb{R}^2},R)} f(x,y) = \pi(1-e^{-R^2}).$$

The numerical results obtained for the estimation of this integral for $R = 1$, $j_x = j_y = 2$, are given in the following Table 5.

Table 5

Calculation of the errors for $j_x = j_y = 2$.

	E_{ILWN}	E_{ILWP}	E_{SIMP}
$M_x = M_y = 3$	$5.5284e-4$	0.0098	$2.6935e-4$
$M_x = M_y = 5$	$3.7824e-6$	$5.1437e-5$	$3.4180e-5$
$M_x = M_y = 8$	$1.3170e-8$	$4.2761e-8$	$5.1785e-6$
$M_x = M_y = 10$	$5.9886e-11$	$1.5911e-10$	$2.1176e-6$

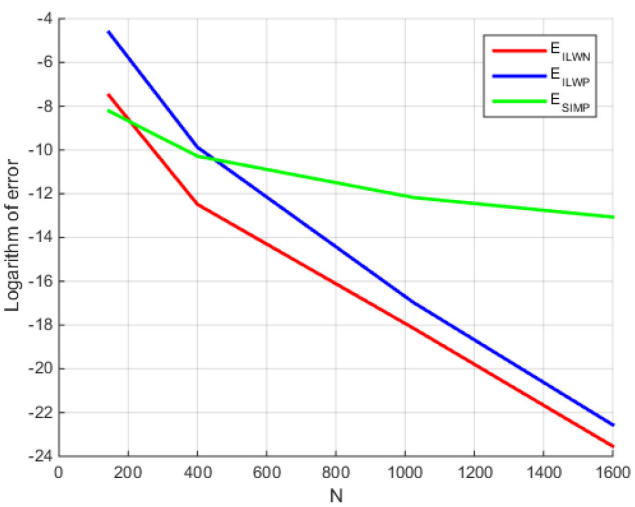


Figure 3

Logarithm of error for $j_x = 2_y = 2$ and $f(x,y) = e^{-(x^2+y^2)}$.

According to the previous Tables 1,2,3,4,5, we notice that the **ILW** and **ILWM** methods converge numerically because the error decreases when the number of levels (j_x, j_y) and the degree of Legendre polynomials (M_x, M_y) increase, (i.e when the number of points $N = 2^{j_x-1}2^{j_y-1}M_xM_y$ increases). This decrease in error is visible in the Figures. However, the **ILW** method is much more precise than the composite Simpson method. We notice that ILW reaches a precision of the order of 10^{-11} while the composite Simpson does not exceed the order of 10^{-7}.

6. Conclusion

In this work, we have established a numerical formula to estimate the double integrations. This one is based on the operational matrix of integration of Legendre wavelet. This integration formula has been tested on several examples. Satisfactory results have been obtained for a small number of subdivisions of integration domain.

The tests carried out show that the ***ILWN*** and ***ILWP*** methods are more precise and provide better results than those given by the Simpson method. Indeed, we see that the errors of the first two methods reach up to 10^{-10} while the Simpson error does not exceed 10^{-6}. However, with the **ILWN** method, we gain one decimal place 10^{-1} compared to ***ILWP***.

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Received July, 2021

Revised January, 2022