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Measure defined on Γ–algebra and some of their generalizations

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Abstract

This work aims to introduce concepts about Γ–algebra calculate relative to Γ–algebra and outer measure in relation to Γ–algebra are examples. After that, we establish a relationship between them. The notions of null-additive and weakly null-additive as two generalizations of measure and extreme measure are introduced and studied too.

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1. Introduction

Some types of classes of sets were studied by Ahmed et al. [1], [2], and [3], which are generalizations of σ–field such as α– σ–field, σ–ring, β– σ–field, δ–field and some important results of these concepts approved. Halmos [4] studied the notion of the ring, where a collection $\digamma \subseteq p(\Omega)$ is called a ring if whenever E, $\mathbf{F} \in \digamma$, then $\mathbf{E} \cup \mathbf{F} \in \digamma$ and $\mathbf{E} - \mathbf{F} \in \digamma$, where $\mathbf{E} \cup \mathbf{F}$ denotes the union of E and F, and E-F denotes the difference of E and F. **σ– field Concept** studied by Ash [5], where significant results in measure theory have been demonstrated. Mohaimen et al. [6] the idea of -algebra was presented, and its characteristics were investigated. M. M. Abbood et al. [7], introduced the concept of bi- Γ-algebra, an

Result Discussion

Definition 1: Let $\digamma$ be a group of subsets Ω of that is not empty. If the following conditions are met, then is called Γ –algebra [6]:

a. $\emptyset, \Omega \in \digamma$.
b. If $\mathrm{D} \in \digamma$ and there exist $\emptyset \neq \mathrm{E_i} \subset \mathrm{D} \subset \Omega$ if not more than one, then $\mathrm{E_i}'\mathrm{s} \in \digamma$.
c. If $\mathrm{A_1}, \mathrm{A_2}, \ldots \in \digamma$, then $\bigcup_{\mathrm{i}=1}^{\infty} \mathrm{A_i} \in \digamma$.

Definition 2: Let $\mathcal{X} = \{x : 0 \leq x \leq \infty\}$ and $\digamma$ defined as Γ –algebra of Ω. The set function $\not{f}: \digamma \to \mathcal{X}$ is called a measure on $\digamma$, if $\not{f}(\bigcup_{n=1}^{\infty} \mathcal{B}_n) = \sum_{n=1}^{\infty} \not{f}(\mathcal{B}_n)$ where $\emptyset \neq \mathcal{B}_1, \mathcal{B}_2, \ldots$ be a family of disjoint sets that can either be countably infinite or finite in $\digamma$. So, a triple $(\Omega, \digamma, \not{f})$ is called a measure space in comparison to the $\Gamma - \text{algebra}\ \digamma$.

Theorem 1: *If* $(\Omega, \digamma, \not{f}_j)$ *can be defined as a measure space the is proportional to* $\Gamma - algebra\ \digamma$ *and* $k_j \in \{x: 0 \leq x < \infty\}$ *for all* $j = 1,2, \ldots, r$. *If a set function* $\sum_{j=1}^{r} k_j \not{f}_j : \digamma \to \mathcal{X}$ *where* $\mathcal{X} = \{x : 0 \leq x \leq \infty\}$ *is defined as;* $(\sum_{j=1}^{r} k_j \not{f}_j)(\mathcal{B}) = \sum_{j=1}^{r} k_j . \not{f}_j(\mathcal{B})\ \forall\, \mathcal{B} \in \digamma$, *then* $(\Omega, \digamma, \sum_{j=1}^{r} k_j \not{f}_j)$ *is measure space relative to the* $\Gamma - algebra\ \digamma$.

Definition 3: Let $\digamma$ be a $\Gamma - \text{algebra}$ of Ω and $\not{f}$ is a measure on $\digamma$. A set $\mathcal{B} \in \digamma$ is called $\not{f}$ – null set, if $\not{f}(\mathcal{B}) = 0$.

Definition 4: A measure $\not{f}$ on a $\Gamma - \text{algebra}\ \digamma$ is called complete if for every $\not{f}$ – null set $\mathcal{S} \in \digamma$, we have $K \in \digamma\ \forall\, K \subset \mathcal{S}$.

Definition 5: Let $(\Omega, \digamma_1, \not{f}_1)$ and $(\Omega, \digamma_2, \not{f}_2)$ are two measure spaces proportional to the $\Gamma - \text{algebras}$ $\digamma_1$ and $\digamma_2$ on the same set Ω. The measure space $(\Omega, \digamma_2, \not{f}_2)$ is said to be an extension of $(\Omega, \digamma_1, \not{f}_1)$ iff $\digamma_1 \subseteq \digamma_2$ and $\not{f}_1(\mathcal{S}) = \not{f}_2(\mathcal{S})\ \forall\, \mathcal{S} \in \digamma_1$.

Proposition 1: Suppose that $(\Omega, \digamma_1, \not{f}_1)$ can be defined as measuring proportional space to the $\Gamma-$algebra $\digamma_1$. There exist the smallest complete extension of $(\Omega, \digamma_1, \not{f}_1)$ which is unique.

Definition 6: Let $\Omega \neq \emptyset$ be a set and $\digamma$ be a $\Gamma-$algebra of Ω. Let $\mathcal{X} = \{x : 0 \leq x \leq \infty\}$. A set function $\not{f}: \digamma \to \mathcal{X}$ is called outer measure (Out-M) on $\Gamma-$algebra $\digamma$, if all of the following are true:

1. $\not{f}(\emptyset) = 0$.
2. If $A, B \in \digamma$ such that $A \subset B$, then $\not{f}(A) \leq \not{f}(B)$.
3. If $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \digamma$, then $\not{f}(\bigcup_{n=1}^{\infty} \mathcal{S}_n) \leq \sum_{n=1}^{\infty} \not{f}(\mathcal{S}_n)$.

Definition 7: Let $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ is $\Gamma-$algebra set of Ω A $\not{f}: \digamma \to \mathcal{X}^*$ is called null-additive (N-Ad) on $\Gamma-$algebra $\digamma$ if whenever $\mathcal{S}, K \in \digamma$ such that $\mathcal{S} \cap K = \emptyset$ and $\not{f}(\mathcal{S}) = 0$, we have $\not{f}(\mathcal{S} \cup K) = \not{f}(K)$.

Proposition 2: Every measure on a $\Gamma-$algebra $\digamma$ is N-Ad on $\digamma$.

Proposition 3: Suppose $\mathcal{X} = \{x : 0 \leq x \leq \infty\}$ and $\digamma$ is a $\Gamma-$algebra of Ω. If $\not{f}: \digamma \to \mathcal{X}$ be an Out-M on $\digamma$, then $\not{f}$ is an N-Ad on $\digamma$.

Proof: Let $\not{f}$ be an Out-M on $\digamma$ and $\mathcal{S}_1, \mathcal{S}_2 \in \digamma$ such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and $\not{f}(\mathcal{S}_1) = 0$. Since $\digamma$ is $\Gamma-$algebra of Ω then $\mathcal{S}_1 \cup \mathcal{S}_2 \in \digamma$. Also, $\mathcal{S}_1 \subset \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_2 \subset \mathcal{S}_1 \cup \mathcal{S}_2$ and $\not{f}$ is an Out-M on $\Gamma-$algebra $\digamma$, then $\not{f}(\mathcal{S}_2) \leq \not{f}(\mathcal{S}_1 \cup \mathcal{S}_2)$. Let $\mathcal{S}_{\mathrm{r}} = \emptyset \ \forall r > 2$, then $\bigcup_{r=3}^{\infty} \mathcal{S}_{\mathrm{r}} = \emptyset$.

Now, since $\not{f}$ is an Out-M on $\digamma$, then $\not{f}(\mathcal{S}_{\mathrm{r}}) = 0 \ \forall r > 2$ and $\sum_{r=3}^{\infty} \not{f}(\mathcal{S}_{\mathrm{r}}) = 0$.

Hence $\not{f}(\bigcup_{r=1}^{\infty} \mathcal{S}_{\mathrm{r}}) \leq \sum_{r=1}^{\infty} \not{f}(\mathcal{S}_{\mathrm{r}})$. Also, $\not{f}(\mathcal{S}_1 \cup \mathcal{S}_2 \cup [\bigcup_{r=3}^{\infty} \mathcal{S}_{\mathrm{r}}]) \leq \not{f}(\mathcal{S}_1) + \not{f}(\mathcal{S}_2) + \sum_{r=3}^{\infty} \not{f}(\mathcal{S}_{\mathrm{r}})$. This means that $\not{f}(\mathcal{S}_1 \cup \mathcal{S}_2) \leq \not{f}(\mathcal{S}_1) + \not{f}(\mathcal{S}_2)$. Then $\not{f}(\mathcal{S}_1 \cup \mathcal{S}_2) = \not{f}(\mathcal{S}_2)$ since $\not{f}(\mathcal{S}_1) = 0$, and hence $\not{f}$ is an N-Ad on $\digamma$.

Definition 7: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and let $\digamma$ be a $\Gamma-$algebra of Ω. A set function $\not{f}: \digamma \to \mathcal{X}^*$ is called additive if and only if $\mathcal{S}, K \in \digamma$ such that $\mathcal{S} \cap K = \emptyset$, then $\not{f}(\mathcal{S} \cup K) = \not{f}(\mathcal{S}) + \not{f}(K)$.

Remark 1: If $\not{f}$ is a measure on $\Gamma-$algebra $\digamma$. Then $\not{f}$ is an additive on $\Gamma-$algebra $\digamma$.

Proposition 4: Every additive on $\Gamma-$algebra $\digamma$ is an N-Ad on $\Gamma-$algebra $\digamma$.

Definition 8: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma-$algebra of Ω. A set function $\not{f}: \digamma \to \mathcal{X}^*$ is called weakly null-additive (WN-Ad) on $\Gamma-$

algebra if and only if $\mathcal{S}, K \in \digamma$ such that $\mathcal{S} \cap K = \emptyset$ and $\not{f}(\mathcal{S}) = \not{f}(K) = 0$, then $\not{f}(\mathcal{S} \cup K) = 0$.

Proposition 5: If $\not{f}$ is an N-Ad on $\Gamma - \text{algebra}\,\digamma$, then it is a WN-Ad on $\Gamma - \text{algebra}\,\digamma$.

Lemma 1: *Every F-Ad on $\Gamma - algebra\,\digamma$ is WN-Ad on $\digamma$.*

Proposition 5: Every measure on $\Gamma - \text{algebra}\,\digamma$ is a WN-Ad on $\digamma$.

Proposition 6: Every additive on $\Gamma - \text{algebra}\,\digamma$ is a WN-Ad on $\digamma$.

Definition 9: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma - \text{algebra}$ of Ω. A set function $\not{f}: \digamma \to \mathcal{X}^*$ is called countably null-additive (CN-Ad) on $\digamma$; if whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \digamma$, $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset\ \forall i \neq j$ and $K \in \digamma$ such that $K \cap \mathcal{S}_r = \emptyset$ and $\not{f}(\mathcal{S}_r) = 0\ \forall r = 1,2,3, \ldots$, then $\not{f}(K \cup \bigcup_{r=1}^{\infty} \mathcal{S}_r) = \not{f}(K)$.

Definition 10: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma - \text{algebra}$ of Ω. A set function $\not{f}: \digamma \to \mathcal{X}^*$ is called finitely null-additive (FN-Ad) on $\digamma$; if whenever $\mathcal{S}_1, \mathcal{S}_2, \ldots \in \digamma$, $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset\ \forall i \neq j$ and $K \in \digamma$ such that $K \cap \mathcal{S}_r = \emptyset$ and $\not{f}(\mathcal{S}_r) = 0\ \forall r = 1,2,3, \ldots k$, then $\not{f}(K \cup \bigcup_{r=1}^{k} \mathcal{S}_r) = \not{f}(K)$.

Example 1: Let Ω be a countable infinite set, and $\digamma$ be the power set of Ω. Define $\not{f}(K) = \begin{cases} 0 & K \text{ finite}, \\ \infty & K \text{ infinte}. \end{cases}$

Then $\not{f}$ is an FN-Ad on $\Gamma - \text{algebra}\,\digamma$. Let $\mathcal{S}_1, \mathcal{S}_2, \ldots, , \mathcal{S}_{\mathrm{r}} \in \digamma$, $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset\ \forall i \neq j$ and $K \in \digamma$ such that $K \cap \mathcal{S}_n = \emptyset$ and $\not{f}(\mathcal{S}_n) = 0\ \forall n = 1,2,3, \ldots r$; then $\mathcal{S}_n$ is finite $\forall n = 1,2,3, \ldots, r$. Hence $\bigcup_{n=1}^{r} \mathcal{S}_n$ is finite. Now, we have two conditions:

Condition 1: if K is a finite set, then $K \cup \bigcup_{n=1}^{r} \mathcal{S}_n$ is finite.

Condition 2: if K is an infinite set, then $K \cup \bigcup_{n=1}^{r} \mathcal{S}_n$ is infinite.

Therefore; $\not{f}(K \cup \bigcup_{n=1}^{r} \mathcal{S}_n) = \not{f}(K) = \begin{cases} 0 & in\ case\ 1, \\ \infty & in\ case\ 2. \end{cases}$

Consequently, $\not{f}$ is a FN-Ad on $\Gamma - \text{algebra}\,\digamma$.

Theorem 2: *Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$ and $k_j \in \{x : 0 \leq x < \infty\}\ \forall j = 1,2,3, \ldots, q$.*

Let $\not{f}_1, \not{f}_2, \ldots, \not{f}_q: \digamma \to \mathcal{X}^*$ be a FN-Ad on $\Gamma - algebra\,\digamma$ of Ω. If $\sum_{j=1}^{q} k_j \not{f}_j: \digamma \to \mathcal{X}^*$ is defined as;

$\left(\sum_{j=1}^{q} k_j \, \not{f}_j\right)(K) = \sum_{j=1}^{q} k_j \, . \not{f}_j(K) \; \forall K \in \mathrm{F}$, *then* $\sum_{j=1}^{q} k_j \not{f}_j$ *is a FN-Ad on* $-algebra \; \mathrm{F}$.

Proposition 7: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and F be a $\Gamma -$ algebra of Ω. Let $\not{f}_1, \not{f}_2$ be a CN-Ad on F and $\not{f}_1(\emptyset) = \not{f}_2(\emptyset) = 0$. Assume that $\not{f}_1$ or $\not{f}_2$ is finite and define $\not{f}_1 - \not{f}_2 \colon \mathrm{F} \to \mathcal{X}^*$ as:

$(\not{f}_1 - \not{f}_2)(K) = \not{f}_1(K) - \not{f}_2(K) \; \forall K \in \mathrm{F}$. Then $(\not{f}_1 - \not{f}_2)$ is a FN-Ad on $\Gamma -$ algebra F of Ω.

Proof: Suppose $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \in \mathrm{F}$ are disjoint sets, $K \in \mathrm{F}$ such that $K \cap \mathcal{S}_n = \emptyset$ and $\not{f}(\mathcal{S}_n) = 0 \; \forall n = 1,2,3, \dots r$. Consider $\mathcal{S}_n = \emptyset \; \forall n > r$, then $\bigcup_{n=1}^{\infty} \mathcal{S}_n = \bigcup_{n=1}^{r} \mathcal{S}_n \; \forall \, n > r$. Note that $\not{f}_i(K \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n) = \not{f}_i(K)$ because $\not{f}_i$, $i = 1,2$ is CN-Ad on F, then $\not{f}_i(K \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n) = \not{f}_i(K)$. Then;

$$
\begin{aligned}
(\not{f}_1 - \not{f}_2)(K \cup \textstyle\bigcup_{n=1}^{r} \mathcal{S}_n) = \; & (\not{f}_1 - \not{f}_2)(K \cup \textstyle\bigcup_{n=1}^{\infty} \mathcal{S}_n) \\
& = \not{f}_1(K \cup \textstyle\bigcup_{n=1}^{\infty} \mathcal{S}_n) - \not{f}_2(K \cup \textstyle\bigcup_{n=1}^{\infty} \mathcal{S}_n) \\
& = \not{f}_1(K) - \not{f}_1(K) = (\not{f}_1 - \not{f}_2)(K).
\end{aligned}
$$

Therefore $(\not{f}_1 - \not{f}_2)$ is a FN-Ad on $\Gamma -$ algebra F.

Proposition 8: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and F be a $\Gamma -$ algebra of Ω. If $\not{f} \colon \mathrm{F} \to \mathcal{X}^*$ be a CN-Ad on $\Gamma -$ algebra F such that $\not{f}(\emptyset) = 0$, then $\not{f}$ is a FN-Ad on F.

Proof: Let $\not{f}$ be a CN-Ad on F , and let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \in \mathrm{F}$ are disjoint sets, $K \in \mathrm{F}$ such that $K \cap \mathcal{S}_n = \emptyset$ and $\not{f}(\mathcal{S}_n) = 0 \; \forall n = 1,2,3, \dots r$. Consider $\mathcal{S}_n = \emptyset \; \forall n > r$, then $\bigcup_{n=1}^{\infty} \mathcal{S}_n = \bigcup_{n=1}^{r} \mathcal{S}_n \; \forall \, n > r$ and $\not{f}(\mathcal{S}_n) = 0, \forall n$. Then, $\not{f}(K \cup \bigcup_{n=1}^{r} \mathcal{S}_n) = \not{f}(K \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n) = \not{f}(K)$ .Therefore $\not{f}$ is a FN-Ad on F.

Clearly that the reverse of Proposition 2.54 not true.

Example 2: Let Ω be a countable infinite set, and F is the power set of Ω. Define $\not{f}\,(K) = \begin{cases} 0 & K \text{ finite}, \\ \infty & K \text{ infinte}. \end{cases}$

Then $\not{f}$ is a FN-Ad on $\Gamma -$ algebra F , but it is not a CN-Ad since;

Proposition 9: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and F be a $\Gamma -$ algebra of Ω. If $\not{f} \colon \mathrm{F} \to \mathcal{X}^*$ be a FN-Ad on F such that $\not{f}(\emptyset) = 0$, then $\not{f}$ is a N-Ad .

Proof: Let $\emptyset \neq \mathcal{S}, K \in \mathrm{F}$ such that $\mathcal{S} \cap K = \emptyset$, $\not{f}(K) = 0$, and $\not{f}$ be a FN-Ad on F. Assume $K = \mathcal{S}_1$ and $\mathcal{S}_n = \emptyset \; \forall n = 2,3, \dots, r$, then $\mathcal{S} \cup K = \mathcal{S} \cup \bigcup_{n=1}^{r} \mathcal{S}_n$ and $\not{f}(\mathcal{S}_n) = 0 \; \forall n = 1,2, \dots, r$. Thus; $\not{f}(\mathcal{S} \cup K) = \not{f}(\mathcal{S} \cup \bigcup_{n=1}^{r} \mathcal{S}_n) = \not{f}(\mathcal{S})$ since $\not{f}$ is a FN-Ad and $\not{f}(\mathcal{S}_n) = 0, \forall n = 1,2, \dots, r$. Thus, $\not{f}$ is a N-Ad on F.

Proposition 10: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma-$algebra of Ω. If $\not{f}: \digamma \to \mathcal{X}^*$ be a FN-Ad on $\digamma$ such that $\not{f}(\emptyset) = 0$, then $\not{f}$ is WN-Ad.

Proposition 11: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma-$algebra of Ω. If $\not{f}: \digamma \to \mathcal{X}^*$ be a CN-Ad on $\digamma$ such that $\not{f}(\emptyset) = 0$, then $\not{f}$ is a N-Ad on $\digamma$.

Proposition 12: Suppose $\mathcal{X}^* = \{x : -\infty \leq x \leq \infty\}$, and $\digamma$ be a $\Gamma-$algebra of Ω. If $\not{f}: \digamma \to \mathcal{X}^*$ be a CN-Ad on $\digamma$ such that $\not{f}(\emptyset) = 0$, then $\not{f}$ is a WN-Ad on $\digamma$.

Proposition 13: Every measure is a CN-Ad.

Proof: Assume that $\digamma$ is a $\Gamma-$algebra of a set. Suppose that $\not{f}$ is a measure and $\emptyset \neq \mathcal{S}_1, \mathcal{S}_2, \dots, \in \digamma$ are disjoint sets, $K \in \digamma$ such that $K \cap \mathcal{S}_n = \emptyset$ and $\not{f}(\mathcal{S}_n) = 0 \ \forall n = 1,2,3, \dots$. We show that $\not{f}(K \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n) = \not{f}(K)$.

Since $K \cap \mathcal{S}_n = \emptyset \ \forall n = 1,2, \dots$, then $K \cap \bigcup_{n=1}^{\infty} \mathcal{S}_n = \emptyset \ \forall n = 1,2, \dots$. Let $\bigcup_{n=1}^{\infty} \mathcal{S}_n = \mathcal{S}$ then $K \cap \mathcal{S} = \emptyset$. Since $\not{f}$ is a measure then $\not{f}(K \cup \mathcal{S}) = \not{f}(K) + \not{f}(\mathcal{S})$. But $\not{f}(\mathcal{S}) = \not{f}(\bigcup_{n=1}^{\infty} \mathcal{S}_n) = \sum_{n=1}^{\infty} \not{f}(\mathcal{S}_n) = 0$. Therefore $\not{f}(K \cup \mathcal{S}) = \not{f}(K)$, and hence $\not{f}$ is a CN-Ad.

Proposition 14: Every measure is a FN-Ad.

3. Conclusion

In this work, some important concepts related to the relatively new structure of the collection of sets called Γ–algebra was introduced and discussed. The measure, the outer measure, the finitely additive, countably additive, null additive, finitely null additive, and countably null additive are the four types of additives, and the weakly null additive relative to the Γ –algebra were studied. It is noted that the null-additive and the weakly null-additive are two generalizations of the measure and outer measure notions. In conclusion, the relationships between the previously presented concepts were investigated and demonstrated, and some suitable instances were also taken in order to make the concepts that were studied more easily consumable.

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