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Fully fuzzy visible modules with distinguishing results

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Abstract

In this paper, the concept of fully fuzzy visible modules has been introduced. Many of the properties that distinguish this concept have been proven. Also, the direct sum of two fully fuzzy visible modules have been studied. Some results have been given that clarify of this study.

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1. Introduction

Consider $\mathbb{M}$ is a unitary я -module. Fuzzy modules of $\mathbb{M}$ were defined by C.V. Negoita and D.A. Ralescu [1]. A study on the idea of the fuzzy direct sum of modules was done by H. Rabi [2]. The concept of fuzzy visible modules has been introduced by K.M.Sajda and N.S.Buthyna [3]. The purpose of this paper is to introduce the concept of fully fuzzy visible modules and study some of the basic characteristics. In addition, the direct sum of two fuzzy fully visible modules are presented and showed that the direct sum of two fully visible will

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be fully visible and the converse is true with a special condition. Several other outcomes have been found. Through this paper я a commutative ring with identity and for each fuzzy ideal J of я, $J(0)=1$.

2. Preliminaries

This part contains some concepts that will be needed in our work.

Definition 2.1: Let я $\neq \emptyset$ be a set and Ɨ = [0,1] of the real line (real numbers). Let ɦ: я → Ɨ is a function. Then ɦ is known as a fuzzy set in я (a fuzzy subset of я). [4] . Let FFS(я) = {ɦ: ɦ: я → Ɨ is a function }.

Definition 2.2: Let Ç $\neq \emptyset$ be a set, and Đ ∈ FFS(Ç), for all, m ∈[0,1] the level set of Ç concerning Đ is a set $Đ_m$={z∈ Ç : Đ $(z) \geq m$}, [5, 6, 7] .

Definition 2.3: Let $\mathbb{M}$ be an я -module and † ∈ $FFS(\mathbb{M})$ such that † satisfied the following condition :

1- †(ê − â) ≥ min{†(ê), †(â)} ∀ ê, â ∈ $\mathbb{M}$,

2- †(r̂ê) ≥ †(ê) ∀ ê ∈ $\mathbb{M}$, r̂ ∈ я,

3- † (0)=1. Then † known to as a fuzzy module of an я -module $\mathbb{M}$,[8,9].

Let $FMM(\mathbb{M})$ = {†: † is a fuzzy module of an я − module $\mathbb{M}$}.

Definition 2.4: Let Đ, † ∈ $FMM(\mathbb{M})$ and Đ ⊆ †. Then Đ is called a fuzzy submodule of †, [10,11].

Let FSM(†) = {Đ: Đ is a fuzzy submodule of a fuzzy module † }

Definition 2.5: Let Ɨ ⊆ FFS(я) such that the following holds :

1- Ɨ($\ell - \hbar$) ≥ min{Ɨ(ℓ), Ɨ($\hbar$)},

2- Ɨ($\ell . \hbar$) ≥ max{Ɨ(ℓ), Ɨ($\hbar$)}, [12].

FFI(я) will be used to represent the set of all fuzzy ideals of я.

Definition 2.6: Let † ∈ $FMM(\mathbb{M})$ *and* †≠ Đ∈*FSM*(†).Then Đ is known to as a fuzzy visible submodule if Đ=ƗĐ ∀ non-empty Ɨ ∈ FFI(я). A fuzzy ideal of a ring я is defined as visible if it is visible of a fuzzy я -module я, [3] .

Let FMV(†) = {† ≠ Đ ∈ FSM(†): Đ is a fuzzy visible of † }.

Definition 2.7: Let †∈*FMM*($\mathbb{M}_1$) and Ł ∈*FMM*($\mathbb{M}_2$). Define †⊕Ł: $\mathbb{M}_1 \oplus \mathbb{M}_2 \rightarrow$ [0,1] by: (†⊕Ł)(ɗ, ɋ) = min{†(ɗ), Ł(ɋ) $for\ all$ (ɗ, ɋ) ∈ $\mathbb{M}_1 \oplus \mathbb{M}_2$}. †⊕Ł is termed a fuzzy external direct sum of † and Ł. If $\mathfrak{S}$∈*FSM*(†) and $\mathbb{D}$∈*FSM*(Ł).Define $\mathfrak{S} \oplus \mathbb{D}$: $\mathbb{M}_1 \oplus \mathbb{M}_2 \rightarrow$ [0,1]by:
($\mathfrak{S} \oplus \mathbb{D}$)(ɗ, ɋ) = min{$\mathfrak{S}$(ɗ), $\mathbb{D}$(ɋ) $for\ all$ (ɗ, ɋ) ∈ $\mathbb{M}_1 \oplus \mathbb{M}_2$},[2]. It follows that $\mathfrak{S} \oplus \mathbb{D}$ is a fuzzy submodule of †⊕Ł,[13]. Note that, if † = $\mathfrak{S} + \mathbb{D}$ and $\mathfrak{S} \cap \mathbb{D}$ =

0, then † is called the internal direct sum of $\mathfrak{S}$ and $\mathbb{D}$ which is denoted by $\mathfrak{S}\oplus\mathbb{D}$. Moreover, $\mathfrak{S}$ and $\mathbb{D}$ are called direct summand of †, [2].

Definition 2.8: Let $\mathfrak{X}, \mathfrak{C} \in FSM(\mathbb{P})$. The $\mathfrak{X} + \mathfrak{C}$ defined by

$((\mathfrak{X} + \mathfrak{C})(d) = sup\{\inf\{\mathfrak{X}(\hat{a}), \mathfrak{C}(\hat{e}) : d = \hat{a} + \hat{e}\ for\ all\ d, \hat{a}, \hat{e} \in \mathbb{M}\}$, [8,14].

Definition 2.9: Let $† \in FMM(\mathbb{M}_1)$ and $Ł \in FMM(\mathbb{M}_2)$. $f: † \to Ł$ is defined by a fuzzy homomorphism if $f: \mathbb{M}_1 \to \mathbb{M}_2$ is я -homomorphism and $Ł(f(†) = †(x)\ \forall\ x \in \mathbb{M}_1$, [15] .

Remark 2.10: For $Đ \in FMM(\mathbb{M})$ we let $Đ_* = \{\hat{a} \in \mathbb{M}: Đ(\hat{a}) = 1\}$, [8,14] .

3. Fully Fuzzy Visible Modules with Their Important Features

The concept of fully fuzzy visible is displayed in this section. In addition, the concept has truly been characterized by some conditions required to be †, is fully fuzzy and visible.

Definition 3.1: Let $† \in FMM(\mathbb{M})$. Then † is said to be a fully fuzzy visible module (FFV-module) if $\forall\ † \neq \mathfrak{A} \in FSM(†)$ is fuzzy visible.

The following proposition gives an FFV- module in terms of its level module.

Proposition 3.2: Let $† \in FMM(\mathbb{M})$. Then † is an FFV -module if and only if $†_t$ is fully visible module, $\forall\ t \in (0,1]$.

Proof: Suppose that $† \in FMM(\mathbb{M})$, Ḥ is a submodule of $†_t$, $t \in (0,1]$ and $I \neq \emptyset$ is an ideal of я. Let ɨ: я → [0,1] such that;

$ɨ(x) = \begin{cases} 1 & if\ x \in I \\ t & o.w \end{cases}$,clear that $ɨ \in FFI(я)$. Let $\mathfrak{W}: \mathbb{M} \to [0,1]$ such that ;

$\mathfrak{W}(x) = \begin{cases} 1 & if\ x \in Ḥ \\ 0 & o.w \end{cases}$ $\mathfrak{W} \in FSM(†)$ and $\mathfrak{W}_t = Ḥ$, $ɨ_t = I\ \ \forall\ t \in (0,1]$.

Since † is an FFV-module, then $ɨ\mathfrak{W} = \mathfrak{W}$, then $(ɨ\mathfrak{W})_t = \mathfrak{W}_t$ by [17], hence $ɨ_t\mathfrak{W}_t = \mathfrak{W}_t$ by[15], then $\mathfrak{W}_t = Ḥ$ is a visible submodule of $†_t$. Then $†_t$ is fully visible. Conversely, let $†_t$ be an FFV- module. Let $\mathfrak{W} \in FSM(†)$ and $ɨ \in FFI(я)$be nonempty. Because $\mathfrak{W}_t$ is a submodule of $†_t$, $ɨ_t$ which is an ideal of я and $†_t$ is fully visible, then $ɨ_t\mathfrak{W}_t = \mathfrak{W}_t$, hence $(ɨ\mathfrak{W})_t = \mathfrak{W}_t$ by [15]. Then $ɨ\mathfrak{W} = \mathfrak{W}$. Hence † is an FFV- module.

3.3 *Remark and Examples:*

1- Let $\mathbb{M} = Z_p$, $я = Z_{p^n}$ and p is a prime number, $n \geq 1$. Define $\mathfrak{C}: Z_p \to [0,1]$ as follows : $\mathfrak{C}(x) = \begin{cases} 1 & if\ \ x \in Z_p \\ 0 & o.w. \end{cases}$, then $\mathfrak{C} \in FMM(\mathbb{M})$, hence

$\mathfrak{C}_t = Z_P = \mathbb{M}$. But $\mathfrak{C}_t$ is a fully visible module by [18, remarks and examples 2.2,(1)]. From prop.3.2, we have $\mathfrak{C}$ is an FFV- module.

2- Let $\mathbb{M} = Z_P, я = Z$ and p be a prime number. Define $\mathfrak{C}$ in a similar way to (1) above, then $\mathfrak{C} \in FMM(\mathbb{M})$ and hence $\mathfrak{C}_t = Z_P$ is a fully visible module by [18, remarks and examples 2.2,(2)]. Therefore $\mathfrak{C}$ holds what is required by prop. 3.2.

3- If $\mathbb{M} = Z = я$. Define $N: Z \to [0,1]$ by $N(x) = \begin{cases} 1 & if \quad x \in Z \\ 0.5 & o.w. \end{cases}$, define $\mathfrak{W}: Z \to [0,1]$ by $\mathfrak{W}(x) = \begin{cases} 1 & if \quad x \in 2Z \\ 0.25 & o.w. \end{cases}$, then $N \in FMM(\mathbb{M})$ and $\mathfrak{W} \in FSM(N)$. Let $J: Z \to [0,1]$ define by $J(x) = \begin{cases} 1 & if \quad x \in 3Z \\ 0 & o.w. \end{cases}$, then $J \in FFI(я)$. Note that $N_t = \mathbb{M} = Z$, $\mathfrak{W} = 2Z$ and $J_t = 3Z$, then by [18, remarks and examples 2.2(3)], we have $\mathfrak{W}_t$ which is not visible of N_t, then $\mathfrak{W}$ is not fuzzy visible of *N* by [3, prop. 3.2]. Therefore *N* is not FFV–module.

Proposition 3.4: Let ϯ is an FFV-module. If ϯ $\neq \mathfrak{W} \in FSM($ϯ$)$, then $\mathfrak{W}$ is also an FFV module.

Proof: Let ϯ $\neq \mathfrak{W} \in FSM($ϯ$)$, indicate $\mathfrak{W}$ is an FFV-module. Let $\mathfrak{W} \neq \beta \in FSM(\mathfrak{W})$, but ϯ is an FFV-module, then by definition 3.1, β is a fuzzy visible. Therefore $\mathfrak{W}$ is an FFV.

Corollary 3.5: *Let* $ɦ:$ϯ $\to$ Ł *be an epimorphism, where* ϯ $\in FMM(\mathbb{M})$ *and* Ł $\in FMM(\mathbb{M}')$ *where* ϯ *is an* ɦ *– invariant. If* ϯ *is an FFV- module then* Ł *is an FFV-module.*

Proof: Let $\mathfrak{W} \in FSM($Ł$)$, then $ɦ^{-1}(\mathfrak{W}) \in FSM($ϯ$)$ by[19], since ϯ is an FFV-module, then $ɦ^{-1}(\mathfrak{W}) \in FMV($ϯ$)$, hence $ɦ^{-1}(\mathfrak{W}) = Jɦ^{-1}(\mathfrak{W})$ for any non-empty fuzzy ideal of я, then $ɦ^{-1}(\mathfrak{W}) = ɦ^{-1}(J\mathfrak{W})$ by[13,lemma 2.3.4] then $ɦ\left(ɦ^{-1}(\mathfrak{W})\right) = ɦ\left(ɦ^{-1}(J\mathfrak{W})\right)$ by [12], then $\mathfrak{W} = J\mathfrak{W}$. Hence $\mathfrak{W} \in FMV($Ł$)$. As a result, the task is correct.

Definition 3.6: Let ϯ $\in FMM(\mathbb{M})$. If $\mathfrak{W} \in FSM($ϯ$)$, then $\mathfrak{W}$ is called an F-fully invariant submodule if $ɦ(\mathfrak{W}) \subseteq \mathfrak{W}$ for each fuzzy я $-$ homomorphism $ɦ:$ϯ $\to$ ϯ.

Definition 3.7: let ϯ $\in FMM(\mathbb{M})$. We say that ϯ is a fuzzy duo module if every fuzzy submodule of ϯ is F-fully invariant.

Proposition 3.8: Let ϯ $\in FMM(\mathbb{M})$ and $ɦ:$ϯ $\to$ ϯ is a fuzzy я $-$ epimorphism where each fuzzy submodule of ϯ is constant on *ker* ɦ. If ϯ is a fuzzy duo module, then ϯ$_*$ will be duo module and conversely is true, provided that $\forall\ \mathfrak{A}, \mathfrak{Z} \in FSM($ϯ$)$ if $\mathfrak{A}_* \subseteq \mathfrak{Z}_*$ then $\mathfrak{A} \subseteq \mathfrak{Z}$.

Proof: Let † be a fuzzy duo module. Let H be a submodule of $†_*$. Now define $\mathfrak{A}: \mathbb{M} \to [0,1]$ as follows: $\mathfrak{A}(x) = \begin{cases} 1 & if\ x \in H \\ 0.1 & o.w. \end{cases}$, then $\mathfrak{A} \in FSM(†)$ and $\mathfrak{A}_* = H$. But $(\mathfrak{A}) \subseteq \mathfrak{A}$, since † is a fuzzy duo module and hence $f(\mathfrak{A}_*) \subseteq \mathfrak{A}_*$by [17, Theorem 3.2.7]. Then $\mathfrak{A}_* = H$ is a fully invariant and hence $†_*$ is duo. Conversely, let $\mathfrak{W} \in FSM(†)$, then $\mathfrak{A}_*$ is a submodule of $†_*$, hence $f(\mathfrak{A}_*) \subseteq \mathfrak{A}_*$, because $†_*$ is a duo module and hence $f(\mathfrak{A})_* \subseteq \mathfrak{A}_*$by[17, Theorem 3.2.7]. Now, from the hypothesis, we get $f(\mathfrak{A}) \subseteq \mathfrak{A}$. So $\mathfrak{A}$ is F-fully invariant and hence † is a duo.

Proposition 3.9: Let Ł be an FFV-module and fuzzy duo *я* -module such that Ł=$\mathfrak{A}_1 \oplus \mathfrak{A}_2$where, $\mathfrak{A}_1, \mathfrak{A}_2 \in FSM(Ł)$. Then $\forall$ Ł $\neq A \in FSM(Ł)$,

A=($\mathfrak{A}_1 \cap A$)⊕ ($\mathfrak{A}_2 \cap A$), where each fuzzy submodule of Ł is constant on *kerf.*

Proof: $Ł_*$ is fully visible and duo by prop. 3.2 and prop.3.8 . If $\mathfrak{A}_1, \mathfrak{A}_2 \in FSM(Ł)$ then for each $A \in FSM(Ł)$, $\mathfrak{A}_{1_*}$, $\mathfrak{A}_{2_*}$ and A_* are submodules of $Ł_*$. Now, by [15 ,prop. 2.5], we have A_* =($\mathfrak{A}_{1_*} \cap A_*$) ⊕ ($\mathfrak{A}_{2_*} \cap A_*$)= $((\mathfrak{A}_1 \cap A) \oplus (\mathfrak{A}_2 \cap A))_*$ by [2], then A=($\mathfrak{A}_1 \cap A$) ⊕ ($\mathfrak{A}_2 \cap A$) by [17].

4. Fuzzy Direct Sum of Fully Fuzzy Visible Modules

In this part, we look at the fuzzy direct sum of two fully fuzzy visible modules, and we show that the sum of two fully fuzzy visible modules is also fully fuzzy visible, and opposite under certain conditions. In this section, we have indicated a number of results.

Proposition 4.1: Let Ł $= \omega \oplus \mathfrak{B} \in FMM(\mathbb{M})$, where $\omega, \mathfrak{B} \in FSM(Ł)$ such that $ann\omega_t + ann\mathfrak{B}_t =$ я $\forall$ t∈ [0,1]. Then ω and $\mathfrak{B}$ are FFV-modules fully if and only if Ł is an FFV-module.

Proof: $\Longrightarrow$ Since ω and $\mathfrak{B}$ are FFV- modules, ω_t and $\mathfrak{B}_t$ are fully visible modules by prop. 3.2, then $\omega_t \oplus \mathfrak{B}_t = Ł_t$ is fully visible by[18, proposition 3.1]. But $\omega_t \oplus \mathfrak{B}_t = (\omega \oplus \mathfrak{B})_t$ by [2] and hence $\omega \oplus \mathfrak{B}$ is an FFV-module by prop. 3.2, that means Ł is an FFV- module.

$\Longleftarrow$ We know that $\omega \subseteq \omega \oplus \mathfrak{B}$ and $\mathfrak{B} \subseteq \omega \oplus \mathfrak{B}$, but Ł is an FFV-module, then by prop. 2.3, we have that ω and $\mathfrak{B}$ are FFV-modules.

Lemma 4.2: *If* $\omega \in FMV(Y_1)$ *and* $\mathfrak{B} \in FMV(Y_2)$, *so* $\omega \oplus \mathfrak{B} \in FMV(Y_1 \oplus Y_2)$.

Proof: By visibility of ω and $\mathfrak{B}$ we have $\omega =$ ɟω and $\mathfrak{B} =$ ɟ$\mathfrak{B}$ $\forall$ non − empty ɟ ∈ FFI(я), then$(\omega \oplus \mathfrak{B})_t = ($ɟ$\omega \oplus$ ɟ$\mathfrak{B})_t = ($ɟ$\omega)_t \oplus ($ɟ$\mathfrak{B})_t$by [15]. But $($ɟ$\omega)_t \oplus ($ɟ$\mathfrak{B})_t =$ ɟ$_t\omega_t \oplus$ɟ$_t\mathfrak{B}_t$ =ɟ$_t(\omega_t \oplus \mathfrak{B}_t)$= (ɟ$(\omega \oplus \mathfrak{B}))_t$, and hence $\omega \oplus \mathfrak{B} =$ ɟ($\mathfrak{W} \oplus \mathfrak{B}$). Therefore the result is true.

Proposition 4.3: Let Y_1 , $Y_2 \in FMM(\mathbb{M})$. If $Y_1 \oplus Y_2$ is an FFV-module then Y_1 and Y_2 are FFV- modules and the converse is true if $annY_{1_t} + annY_{2_t} =$ я $\forall t \in [0,1]$.

Proof: Let $\mathbb{P} = Y_1 \oplus Y_2$ be an FFV- module. Then $(Y_1 \oplus Y_2)_t$ is fully visible by prop.3.2.But $(Y_1 \oplus Y_2)_t = Y_{1_t} \oplus Y_{2_t}$ and hence Y_{1_t}, Y_{2_t} are fully visible by [20, prop. 2.2.2].Therefore Y_1 and Y_2 are FFV-modules. Conversely, in a similar way, the result is true.

Proposition 4.4: Let ω and ǯ be two F- fully invariant submodules of the F-regular fuzzy module Ł. Let Ł $=$ ω⊕ǯ. Then ω and ǯ are fuzzy visible if and only if Ł is an FFV-module, where $\forall$ ω $\in FSM($Ł$)$ is constant on the ker*f*.

Proof: Let ω and ǯ be F-fully invariant visible submodules and Ł $\neq H \in FSM($Ł$)$. Then $H = (H \cap \omega) \oplus (H \cap ǯ)$ by prop. 3.9. Let $J \in FFI($я$)$ be a nonzero. To prove that $H = JH$, we know that $JH \subseteq H$, we need to prove that $H \subseteq JH$. Let $x_t \subseteq (H \cap \omega) \oplus (H \cap ǯ)$, where $x_t = a_t + b_t$ and $t \in (0,1]$, therefore $a_t, b_t \subseteq H$ which implies $x_t \in H$, also $a_t \subseteq \omega = J\omega$ and $b_t \subseteq ǯ = Jǯ$, $\forall$ nonzero , $J \in FFI($я$)$ (since ω, ǯ ∈ FMV(Ł)). Therefore $a_t \subseteq \omega = \omega \cap J$Ł and $b_t \subseteq ǯ = ǯ \cap J$Ł (since Ł be a F – regular fuzzy, and hence $x_t \subseteq H \cap (\omega \cap J$Ł$) \oplus (ǯ \cap J$Ł$) = H \cap (\omega \cap J$Ł$) \oplus H \cap (ǯ \cap J$Ł$) = (H \cap \omega) \cap J$Ł $\oplus (H \cap ǯ) \cap J$Ł $= J(H \cap \omega) \oplus J(H \cap ǯ) = J\left((H \cap \omega) \oplus (H \cap ǯ)\right) = JH$.
Therefore $x_t \subseteq JH$ and hence $H \subseteq JH$. Thus Ł is an FFV – module. Conversely, it is clear.

Proposition 4.5: Let Ł$_1$, Ł$_2 \in FMM(\mathbb{M})$ and $\omega_1 \in FSM($Ł$_1)$, $\omega_2 \in FSM($Ł$_2)$. Then ω_1 and ω_2 are FFV – modules if and only if $\omega_1 \oplus \omega_2$ is an FFV – submodule of Ł$_1 \oplus$ Ł$_2$, where annŁ$_{1_t} + ann$Ł$_{2_t} =$ я $\forall t \in [0,1]$.

Proof: Let $\omega_1 \oplus \omega_2 \neq \mathfrak{B}_1 \oplus \mathfrak{B}_2 \in FSM(\omega_1 \oplus \omega_2)$, where $\mathfrak{B}_1 \in FSM(\omega_1)$ and $\mathfrak{B}_2 \in FSM(\omega_2)$, then $\forall$ nonzero $J \in FFI($я$)$ we must prove that $\mathfrak{B}_1 \oplus \mathfrak{B}_2 = J(\mathfrak{B}_1 \oplus \mathfrak{B}_2)$. It is clear that $J(\mathfrak{B}_1 \oplus \mathfrak{B}_2) = J\mathfrak{B}_1 \oplus J\mathfrak{B}_2$, but $\mathfrak{B}_1 \subseteq \omega_1$ and ω_1 is an FFV- module, then $\mathfrak{B}_1$ is fuzzy visible submodule and $J\mathfrak{B}_1 = \mathfrak{B}_1$, similarly $J\mathfrak{B}_2 = \mathfrak{B}_2$. Therefore $J(\mathfrak{B}_1 \oplus \mathfrak{B}_2) = \mathfrak{B}_1 \oplus \mathfrak{B}_2$, and hence $\omega_1 \oplus \omega_2$ is FFV-module. Conversely, let $\omega_1 \oplus \omega_2$ be an FFV-module then $(\omega_1 \oplus \omega_2)_t$ is fully visible, but $(\omega_1 \oplus \omega_2)_t = \omega_{1_t} \oplus \omega_{2_t}$, then ω_{1_t} and ω_{2_t} are fully visible by [18, prop. 3.3], and hence ω_1 and ω_2 are FFV-modules.

Remark 4.6: A fuzzy direct summand of fully fuzzy visible я – module is also fully visible.

Proof: Let $\mathfrak{X}$ be a direct summand of Ł, hence ω is a submodule of Ł. We have $\mathfrak{X}$ which is fully visible by prop. 3.4.

Remark 4.7: If a direct summand ω of a fuzzy я – module *Ł*, is an FFV-module, then Ł doesn't need to become fully visible. We can see that in the following example :

Define $Ł: Z \oplus 0 \longrightarrow [0,1]$ by $Ł(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in Z \oplus 0 \\ 0.2 & \text{o.w.} \end{cases}$.

Define $\omega: Z \oplus 0 \rightarrow [0,1]$ *as* $\omega(\hat{a}) = \begin{cases} 1 & if \quad \hat{a} \in Z \\ 0.1 & o.w. \end{cases}$.

Define $\mathfrak{B}: Z \oplus 0 \rightarrow [0,1]$ *as* $\mathfrak{B}(\hat{a}) = \begin{cases} 1 & if \quad \hat{a} = 0 \\ 0 & o.w. \end{cases}$.

It is clear that $Ł \in FMM(\mathbb{M})$ and $\omega, \mathfrak{B} \in FSM(Ł)$. Then $Ł_t = Z \oplus 0 = Z$, $\omega_t = Z$ and $\mathfrak{B}_t = \{0\}$, then Ł is not FFV- module by example and remarks 3.3. But $\mathfrak{B}_t$ is a fully visible module and hence $\mathfrak{B}$ is an FFV module.

Remark 4.8: Let $† \in FMM(\mathbb{M})$ and $\check{З} \in FMV(†)$, then $\check{З}^2 \in \text{FMV}(† \oplus † = †^2)$, where $ann†_t + ann†_t = я \ \forall t \in [0,1]$.

Proof: Let $\check{З} \in \text{FMV}(†)$, then $\check{З} = J\check{З}$, ∀ nonzero $J \in FFI(я)$ and $\check{З}^2 = \check{З} \oplus \check{З}$ is a fuzzy of $† \oplus † = †^2$, then $J\check{З}^2 = J(\check{З} \oplus \check{З}) = J\mathfrak{W} \oplus J\check{З} = \check{З} \oplus \check{З} = \check{З}^2$.

Therefore $\check{З}^2 \in \text{FMV}(†^2)$.

Remark 4.9: If † is FFV-module, then it is not necessary that $†^2$ is an FFV-module.

Proof: Let $†: Z_3 \rightarrow [0,1]$ be defined by $†(a) = \{\begin{matrix} 1 & \text{if} & a \in Z_3 \\ 0.5 & & \text{o.w.} \end{matrix}\}$. It is clear that $† \in FFM(Z_3)$, then † is an FFV-module by remark and examples 3.2. But if we define $†^2: Z_3 \oplus Z_3 \rightarrow\rightarrow [0,1]$ by $†^2(\hat{e}) = \{\begin{matrix} 1 & if & \hat{e} \in Z_3 \oplus Z_3 \\ 0.5 & & o.w. \end{matrix}\}$ and

$£: (\bar{0}) \oplus Z_3 \rightarrow [0,1]$ by $£(\hat{e}) = \{\begin{matrix} 1 & if & \hat{e} \in (\bar{0}) \oplus Z_3 \\ 0.2 & & o.w. \end{matrix}\}$.

Then $†^2 \in FMM(Z_3 \oplus Z_3)$ and $£ \in FSM(†^2)$, then ${†^2}_t = Z_3 \oplus Z_3$, $£_t = (\bar{0}) \oplus Z_3$ and hence $£_t$ is not visible in ${†^2}_t$ by [*18*,remark 3.4]. Then $£ \notin FMV(†)$. Therefore $†^2$ is not FFV-module.

5. Conclusion

1. The concept of a fully fuzzy visible module achieves the hereditary property (the relationships between the module and its submodules) as a result of the above. It was found that the homomorphic image of the fully fuzzy visible module is also fully fuzzy visible.
2. It is not necessary that the direct sum of two fully fuzzy visible modules is also fully fuzzy visible, but under particular conditions, the relationships are realized.

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