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D_J-coessential submodules

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Abstract

Let R be an individual left R-module of the same type as W, with W being a ring containing one. W's submodules N and K should be referred to as N and K, respectively that K ⊆ N ⊆ W if N/K ≪_J (D_j (W)+K)/K, Then K is known as the D J-coessential submodule of Nin W as K⊆_ (Rce) N. Coessential submodule is a generalization of this idea. These submodules have certain interesting qualities, such that if a certain condition is met, the homomorphic image of D J- N has a coessential submodule called D J-coessential submodule.

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1. Introduction

To simplify, we'll consider W to be a unitary left-R module, where R is any ring. when W = N + C for some sub-module C of W implies that c =w, N is called a small sub-module of W, [3]. Jacobson –small sub-modules of W were developed as a generalization of a small submodule,[3] where Jacobson –small sub-module defined as follows: if W = C + N with J (W/(N)) =W/(N) implies that W = N for short denoted by J –small, [3], 'R 〖ad〗 _j (W) is defined to be the sum of all J –small sub-modules. [1]. In this paper for short, we will refer to R 〖ad〗 _j (W) by D_j (W)Let ⊆K⊆W, if K/N≪W/N, Then Nis developed a generalization of the coessential submodule, which he named a J-coessential submodule. W's submodules K and N. Such that K ⊆N⊆W, as a result, K is referred to as the J-coessential submodule of N in W K⊆_ (Rce) N in W) if

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N/K ≪_J W/K. B-coessential submodules are defined in this study. B ⊂_Rce A in D_J (w)+B) if A/B ≪_J (D_j (W)+B)/B. In this paper, we investigate the characterizations and properties of the D_J-coessential submodule we prove that under certain conditions that the homomorphic image of the D_J-coessential subcomponent of N is the D_J-coessential subcomponent of the image of N.

2. Definition

Submodules A and B are part of W, an R-module. such that B⊆A ⊆W In that case, B is referred to as the D J-coessential submodule of A. D_J (W) +B. (Denoted by B⊂_Rce A in D_J (w)+B) if A/B ≪_J (D_j (W)+B)/B.

3. Examples

1) As a Z-module, Z6 is Z6. D J-coessential submodule of Z6 {0,3}, it is evident.
2) As a Z-module, Z8 is Z8. There is no doubt that {0,4} ⊆_Rce {0,2,4,6} in Z8
3) In the form of a Z-module in Z. 4Z is D_J- 2Z's coessential submodule in Z.
4) Theorize W = Z ⊕Z_ (p^∞) as if it were a Z-module It is obvious that 0 is not a valid number D_J- subcomponent of coessential Z_ (p^∞) in W. {0} ⊇Z_ (p^∞)⊇ W, but Z_ (p^∞)/ ({0}) not D_ (J)- tiny in size W/ ({0}).
5) Assume W is an R-module and N is one of W's submodules. in this example. Then N≪_J D_J (W) if and only if {0} ⊆_ (Rce) N in D_J (W) + {0}.
6) D J-coessential submodules may be seen in all coessential submodules in this example. However, in most cases, the opposite is true. Z6 is D J- 0, 3 is coessential in Z6, but it is not coessential in Z6, for example.

4. Theorem

Let A ⊆ B ⊆ N ⊆ W, then B⊂_Rce N in D_J (w)+Biff B/A ⊂_Rce N/A in D_J (W/B)

Proof: let B⊆_Rce N in D_J (w)+B then N/B ≪_J (D_J (W)+B)/B⊆D_J (W/B) hence N/B ≪_J D_J (W/B). Since (N/A)/(B/A)≃N/B ≪_J D_J (W/B) ≃ D_J ((W/A)/(B/A)). N/B ≪_J (D_J (W)+B)/B≃((D_J (W)+B)/A)/(B/A) thus, we get the result

Conversely, let B/A ⊂_

5. Lemma

Assume that N and X are submodules of W, and therefore N⊆ X, if X/N ≪_J W/N and N≪_J W , then X≪_J W.

Proof: Let $X + L = W$ with $J(W/L) = W/L$ Now $(X+L)/N = W/N$, then $X/N+(L+N)/N = W/N$ since $J(W/L) = W/L$, so by [2] $J(W/(L+N)) = W/(L+N)$ but $X/N \ll_J W/N$, then $W = L + N$. Now $J(W/L) = W/L$ and $\ll_J W$, then $W = L$., therefore, $X \ll_J W$.

6. Proposition

Let $A \subseteq B \subseteq N \subseteq W$, then $A \subseteq_{Rce} N$ in $D_J(W)+A$ iff $A \subseteq_{Rce} B$ in $D_J(W)+A$ and $B \subseteq_{Rce} N$ in $D_J(W)+B$.

Proof: $\Longrightarrow$ let$A \subseteq_{Rce} N$ in $D_J(w)+A$hen $N/A \ll_J (D_J(W)+A)/A \subseteq D_J(W/A)$ but $B/A \subseteq N/A$, then $B/A \ll_J (D_J(W)+A)/A$ .therefore by [2],then $B \subseteq_{Rce}$ Ain $D_J(w)+A$ Define $f:(D_J(W)+A)/A \to (D_J(W)+B)/B$ by $f(W+A)=W+B$ $\forall$ $W \in D_J(w)+A$. f an epimorphism since. $N/A \ll_J (D_J(W)+A)/A$, then $f(N/A)=N/B \ll_J (D_J(W)+B)/B$. By [2] Conversely $N/B \simeq (N/A)/(B/A)$, but $N/B \ll_J (D_J(W)+B)/B$ and $B/A \ll_J (D_J(W)+A)/A$ therefore by lemma 2.4 $N/A \ll_J (D_J(W)+A)/A$

7. Proposition

If $B \subseteq_{Rce} N$ in $D_J(W)+B$ and $A \subseteq W$ then $B+A \subset_{Rce} N+A$ in $D_J(W)+B+A$. The converse is true if $A \ll_J D_J(W)$ and $J((D_J(W)+A+B)/X) = (D - J(W)+A+B)/X$. For each $X \subseteq D_J(W)+B$

Proof: Since $B \subseteq_{Rce} N$ in $D_J(W)+B$ then $N/B \ll_J (D_J(W)+B)/B$. To show that $(N+A)/(B+A) \ll_J (D_J(W)+A+B)/(A+B) \subseteq D_J(w/(A+B))$. Let $(N+A)/(B+A) + X/(B+A) = (D_J(W)+A+B)/(A+B)$ with $J((D_J(W)+A+B)/X) = (D_J(W)+A+B)/X$. Now $D_J(W)+A+B=N+A+X$,

$A \subseteq A+B \subseteq X$ then $(D_J(W)+A+B)/B=(N+X)/B=N/B+X/B$ but $J((D_J(W)+A+B)/X)=(D_J(W)+A+B)/X$ & $N/B \ll_J (D_J(W)+B)/B \subseteq (D_J(W)+A+B)/B$ then $(D_J(W)+A+B)/B=X/B$, thus $D_J(W)+A+B=X$ Suppose that $B+A \subseteq_{Rce} N+A$ in $D_J(W)+A+B$ thus $(N+A)/(B+A) \ll_J (D_J(W)+B+A)/(B+A)$ to show that $B \subseteq_{Rce} N$ in $D_J(W)+B$ we need to prove that $N/B \ll_J (D_J(W)+B)/B$ let $N/B+X/B=(D_J(W)+B)/B$ with $J((D_J(W)+B)/X) = (D_J(W)+B)/X$ $N + X = D_J(W)+B$, hence $(D_J(W)+A+B)/(A+B)=(N+A+B+X)/(A+B)=(N+A+X)/(A+B)$ (since $B \subseteq N$) $(D_J(W)+A+B)/(B+A) = (N+A)/(A+B) + (X+A)/(A+B)$ But $J((D_J(W)+A+B)/(X+A))=(D_J(W)+A+B)/(X+A)$ and $(N+A)/(A+B) \ll_J (D_J(W)+A+B)/(A+B)$ then $D_J(W)+A+B=X+A$ and $J((D_J(W)+A+B)/X)=(D_J(W)+A+B)/X$ then $D_J(W)+A+B=X=D_J(W)+B$ Since $A \ll_J D_J(W)$. Hence $B \subseteq_{Rce} N$ in $D_J(W)+B$

8. Prposition

Let A and B submodules Let $A \ll_J D_J(W)$, if $B \subseteq_{Rce} N$ in $D_J(W)$, B then $B \subseteq_{Rce} N+A$ in $D_J(W)+B$

Proof: Let $(D\_J\ (W)+B)/B=(N+A)/B+X/B$ with $J((D\_J\ (W)+B)/X)=(D\_J\ (W)+B)/X$, then $D\_J\ (W)+B=N+A+X$ but $J((D\_J\ (W)+B)/X)=(D\_J\ (W)+B)/X$, then $J((D\_J\ (W)+B)/(X+N))=(D\_J\ (W)+B)/(X+N)$ by[2] and since $A\ll\_J\ D\_J\ (W)$, then $A\ll\_J\ D\_J\ (W)+B$, hence $D\_J\ (W)+B=X+N$ and $(D\_J\ (W)+B)/B=(X+N)/B=X/B+N/B$ but $N/B \ll\_J\ (D\_J\ (W)+B)/B$ and $J((D\_J\ (W)+B)/X)=(D\_J\ (W)+B)/X$ then $D\_J\ (W)+B=X$. Hence $(D\_J\ (W)+B)/B=X/B$.

9. Proposition

Let W, W\` be R-module and let $f: W\rightarrow W\`$ be R-homomorphism, if$B\subseteq_Rce\ N$ in $D\_J\ (W)+B$, then $f(B)\subseteq\_Rce\ f(N)$ in $D\_J\ (f(W)+f(K)$ provided that $D\_J\ (f(X))=f(X)$, for each $X \subseteq W$.

***Proof**:* Suppose that $B\subseteq\_Rce\ N$ in $D\_J\ (W)+B$ thus $N/B \ll\_J\ (D\_J\ (W)+B)/B$ Define φ: $W/B\longrightarrow(f(W))/(f(B)$ by $\varphi(W+B)=f(W)+f(B)$. for each $w\in W$ since $N/B \ll\_J\ (D\_J\ (W)+B)/B \subseteq W/B$ then $\varphi(N/B)=(f(N))/(f(B)) \ll\_J\ (f(W))/(f(B))$ but $(f(W))/(f(B))$ by assumption thus $(f(N))/(f(B)) \ll_J\ D_J\ ((f(W))/(f(B)))$to show that $(f(W))/(f(B)) = D\_J\ ((f(W))/(f(B)))$

Let $g: f(W) \longrightarrow (f(W))/(f(B)$ be the natural epimorphism, then $g(D_J\ f(W))\subseteq D_J\ ((f(W))/(f(B)))$but $D\_J\ (f(X))=f(X)$, for each $X \subseteq W$. , then $(f(W))/(f(B) = D\_J\ ((f(W))/(f(B)$. therefore $(f(N))/(f(B)) \ll\_J\ D\_J\ ((f(W))/(f(B)$.

10. Proposition

Let $A\subseteq N\subseteq W$ if $N = A + S$ and $S\ll\_J\ D\_J\ (W)$, then $A\subseteq\_Rce\ N$ in $D\_J\ (W)+A$.

Proof: Let $(D\_J\ (W)+A)/A=N/A+X/A$ with $J((D\_J\ (W)+A)/X)=(D\_J\ (W)+A)/X$ then $D\_J\ (W)+A= N + X = A + S + X = S + X$ but Let $J((D\_J\ (W)+A)/X)=(D\_J\ (W)+A)/X$ and $S\ll\_J\ D\_J\ (W)\subseteq D\_J\ (W)+A$, then $S\ll\_J\ D\_J\ (W)+A$ .so $D\_J\ (W)+A=X$ .and $N/A \ll\_J\ (D\_J\ (W)+A)/A$.

11. Proposition

Let W be R-module let $A\subseteq N\subseteq W$ if $D\_J\ (W)=A+N$, $A\subseteq X\subseteq W$, and $\cap X\ll\_J\ D\_J\ (W)$, then $A\subseteq\_Rce\ X$ in $D\_J\ (W)+A$.

***Proof**:* Let $(D\_J\ (W)+A)/A=X/A+S/A$, where $S/A= (D\_J\ (W)+A)/A$ With $J((D\_J\ (W)+A)/S) = (D\_J\ (W)+A)/S$ then $D\_J\ (W)+A=X+S$. 〖X∩ D〗 _J (W)=X ∩(A+N) =A+X ∩ N modular law. then $X \cap (D\_J\ (W)+A) =A+ (X \cap N)$

So, $X =A + (X \cap N)$ but $D\_J\ (W)+A=X+S=A+ (X \cap N) +S$

So $D\_J\ (W)+A= (X \cap N) +S$ but $N\cap S\ll_J\ D_J\ (W)\subseteq D_J\ (W)+A$hence $N\cap S\ll\_J\ D\_J\ (W)+A$ and $J(D\_J\ (W)+A)/S)=(D\_J\ (W)+A)/S$ So $D\_J\ (W)+A=S$ thus $X/A \ll\_J\ (D\_J\ (W)+A)/A$.

12. Proposition

Let $A \subseteq B \subseteq N \subseteq W$ with ((D_J (W)+B)/B) = (D_J (W)+B)/B, then the following are equivalent

1. $B \subseteq$_Rce N in D_J (W)+B
2. for any sub-module $X \subseteq W$, with X + N = D_J (W)+B implies that B + X = D_J (W)+B

Proof: 1→ 2) let D_J (W)+B= X + N for $X \subseteq W$. then (D_J (W)+B)/B=(X+N)/B=N/B+(X+B)/B but J (D_J (W)+B)/B) = (D_J (W)+B)/B, then J (D_J (W)+B)/(X+B) = (D_J (W)+B)/(X+B) and N/B ≪_J (D_J (W)+B)/B by (1) So D_J (W)+B=X+B

2→ 1) Let (D_J (W)+B)/B=N/B+S/B for $S/B \subseteq$(D_J (W)+B)/B with J((D_J (W)+B)/S) = (D_J (W)+B)/S, then D_J (W)+B = N + S by (2) we get D_J (W)+B=B+ S but $B \subseteq S$ thus D_J (W)+B=S hence N/B ≪_J (D_J (W)+B)/B .

13. Proposition

Let $A \subseteq$_Rce Bin D_J (W)+A and $X \subseteq$_Rce N in D_J (W)+X. then $A+X \subseteq$_Rce B+N in D_J (W/(A+X). provided that D_J (W)=W

Proof: Suppose that $A \subseteq$_Rce B in D_J (W)+A and $X \subseteq$_Rce N in D_J (W)+X Let f: W/A ⟶W/(A+X) defined by f (W+A) = w+ (A + X), ∀ W ∈ W. and g: W/X ⟶W/(A+X) defined by g (w + X) = w+ (A + X), ∀ w ∈ W clearly f & g are epimorphisms. since B/A ≪_J (D_J (W)+A)/A ⊆D_J (W/A), then B/A ≪_J D_J (W/A) so, f(B/A) = (B+X)/(A+X) ≪_J W/(A+X), hence (B+X)/(A+X) ⊆ D_J (W/(A+X) = W/ (A+X) and (B+X)/(A+X) ≪_J D_J (W/(A+X) also N/X ≪_J (D_J (W)+X)/X⊆D_J (W/X) so N/X ≪_J D_J (W/X). Therefore g(N/X) =(N+X)/(A+X) ≪_J W/(A+X) then (N+X)/(A+X) ⊆ D_J (W/(A+X) by [1] but D_J (W)=W thus D_J (W/(A+X) =W/(A+X) by [1] hence (N+X)/(A+X) ≪_J D_J (W/(A+X) and (B+X)/(A+X) +(N+X)/(A+X) ≪_J D_J (W/(A+X) by [2].

14. Proposition

Let A, B, N, X be submodules of W the following are equivalent:

1. IfA⊆_Rce A+B in D_J (W)+A then A∩B⊆_Rce B in D_J (W)+A∩B
2. A⊆_Rce B in D_J (W)+A and Y⊆W, then A∩Y⊆_Rce B∩Y in D_J (W)+A∩Y
3. A⊆_Rce B in D_J (W)+A, and X⊆_Rce N in D_J (W)+X, then X∩A⊆_Rce B∩N in D_J (W)+X∩A.

Proof: 1→2) let A⊆_Rce Bin D_J (W)+A and Y⊆WA+(B∩Y) ⊆B, then A⊆_Rce A+(B∩Y) in D_J (W)+A. by (1) we get A∩(B∩Y) ⊆_Rce B∩Y in D_J (W)+(A∩B∩Y).

2→ 3 let A⊆_Rce B in D_J (W)+Aand X⊆_Rce N in D_J (W)+X, then by (2) A⋂X⊆_Rce B⋂X in D_J (W)+A⋂X also X⊆_Rce N in D_J (W)+X and

$B \subseteq W$ then $B \cap X \subseteq\_Rce\ B \cap N$ in $D\_J\ (W)+B \cap X$ thus $A \cap X \subseteq\_Rce\ B \cap N$ in $D\_J\ (W)+A \cap X$

$3 \rightarrow 1)$ let $A \subseteq\_Rce\ A+B$ in $D\_J\ (W)+A$ Since $B \subseteq\_Rce\ B$ in $D\_J\ (W)+B$, then by (3) $A \cap B \subseteq\_Rce\ (A+B) \cap B$ in $D\_J\ (W)+A \cap B$

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