

CMC and CGC surfaces in Lorentz Heisenberg group $\mathcal{H}_3$

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Abstract

The aim of this paper is to study the characterization of some constant mean curvature (CMC) and constant Gauss curvature (CGC) A-net surfaces in the 3 – dimensional Lorentz-Heisenberg group $\mathcal{H}_3$ endowed with a left invariant Lorentzian metric g_i in the three following cases :

1. $g_1 = -dx^2 + dy^2 + (xdy + dz)^2$,
2. $g_2 = dx^2 + dy^2 - (xdy + dz)^2$.
3. $g_3 = dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2$.

Using Levi-Civita connection, we will calculate the mean curvature H and the Gaussian curvature K in order to study CMC and CGC A-net surfaces in $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$.

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1. Introduction

Since the $\overline{XX}^{th}$ century, geometry of surfaces has become an interesting field of differential geometry, more exactly from 1920s, the introduction of the notion connection by Tullio Levi-Civita progressed the theory of surfaces in a more powerful conceptual frame.

This theory disposes effective ways to classify the geometrical structures (curves and surfaces) in three dimensional Euclidean spaces,

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particularity those which are endowed with Riemaniann and Lorentzian metrics. Among these ways, we quote :

1. The mean curvature $H = \frac{k_1+k_2}{2}$ given by the two fundamentals forms I and II . It is null for all minimal surface and invariant in the case of a surface with constant curvature.
2. The Gauss curvature $K = k_1 k_2$ studied in depth by Carl Friedrich Gauss (articles of 1825 and 1827). It allows to classify points on a surface (parabolic, elliptic, hyperbolic, and planar points).

We recall that, $k_1 = k_1(m)$ and $k_2 = k_2(m)$ are called the principal curvatures of a surface S at the point m . They serve to determine the local shape of a point m on a surface S .

In this last decade, there has been an intensive effort to develop the theory of surfaces in homogeneous Riemannian 3-spaces see [7, 8, 12, 13, 19], partly in the direction of classification of surfaces with constant curvature (mean curvature and Gauss curvature) in Lorentz Heisenberg group $\mathcal{H}_3$ endowed with different metrics. This group has been the subject of several recent research works, we quote some of them :

Graphs in Heisenberg groups : Pankaj, Gunjan and Manju Pruthi (see [14] studied unit graphs in Heisenberg groups and they proved that the subgroups of a group, normal subgroups of a group are carried out using the unit graph of the group.

Biharmonic curves in Heisenberg groups : T. Körpınar and E. Turhan [9] defined the biharmonic curves with regards to their curvatures in order to examine inextensible flows associated biharmonic curves depending to Sabban frame in Heisenberg 3 dimensional group. They also discovered equations of one parameter family of biharmonic curves in $\mathcal{H}_3$.

It is shown in [2] and [11] that any unit speed spacelike biharmonic curve according to the metrics g_i , $i = 1, 2, 3$ in Lorentz Heisenberg group $\mathcal{H}_3$ is a pseudo helix.

Discrete Heisenberg groups : In 2019, V. Vibitha Kochamani and P. L. Lilly [9] used discrete heisenberg group to evaluate the useful characteristics of a hash function which is a Pseudo-Randomness of another hash function, and which can assigned uniformly over the range $[0, p)$ to reduce the chance of occurring collisions. They also verified the security properties of the hash function using different attacks.

Maximal and minimal surfaces in Heisenberg groups : In 2011, H. Lee in his paper [10] entitled “Maximal surfaces in Lorentzian Heisenberg space” built

the twin correspondence between surfaces of constant mean curvature in $\mathbb{R}^3$ and maximal surfaces in the Lorentzian Heisenberg space $Nil_1^3(\tau)$ with $\tau \neq 0$ defined by $\mathbb{R}^3$ endowed with the metric

$$dS_\tau^2 = dx^2 + dy^2 + [\tau(ydx - xdy) + dz]^2.$$

He also solved the Calabi-Bernstein problem for maximal surfaces in $Nil_1^3(\tau)$.

In [17], E. Turhan and G. Altay characterized minimal surfaces in three dimensional Lorentzian Heisenberg group $Heis_3$ which is endowed with left invariant Lorentz metrics g_1 , g_2 and g_3 mentioned above.

D. Bensikaddour and L. Belarbi classified in their papers [3, 4, 5] some types of minimal translation surfaces in three dimensional Lorentzian Heisenberg group $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$ respectively.

In [1], G. Altay Suroglu, M. Ergut and T. Körpınar studied constant mean curvature A-net surfaces in three dimensional Heisenberg group $\mathcal{H}_3$ endowed with the metric

$$g = ds^2 = dx^2 + dy^2 + (xdy + dz)^2.$$

In this work, we will base on the Levi-Civita connection, the first and the second fundamental forms to calculate curvatures (mean curvature and Gauss curvature) of A-net surfaces in the Lorentzian Heisenberg group $\mathcal{H}_3$ endowed with left invariant Lorentz metrics g_1 , g_2 and g_3 . Practically, the most of shapes like planes, cylinders, spheres, have constant or zero curvature, so we want to focus on minimal surfaces and CMC A-net surfaces.

2. Preliminaries

Definition 2.1 : Lorentzian Heisenberg group $\mathcal{H}_3$ is a three dimensional Cartesian space with respect to the following not commutative product

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' - xy' + x'y), \quad (2.1)$$

for any (x, y, z) and (x', y', z') of $\mathcal{H}_3$. The identity of this group is $(0, 0, 0)$ and the inverse of each element $(x, y, z) \in \mathcal{H}_3$ is $(-x, -y, -z)$.

N. Rahmani and S. Rahmani showed in their papers [[15], [16]] that modulo an automorphism of the Lie algebra of the Lorentzian Heisenberg group $\mathcal{H}_3$ has three classes of left invariant Lorentzian metrics :

$$1. \quad g_1 = -dx^2 + dy^2 + (xdy + dz)^2 \quad (2.2)$$

$$2. \quad g_2 = dx^2 + dy^2 - (xdy + dz)^2 \quad (2.3)$$

$$3. \quad g_3 = dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2 \quad (2.4)$$

They also gave the Levi-Cevita connection in the three cases of Lorentzian Heisenberg group $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$.

In all that follows, we consider a surface Σ in $(\mathcal{H}_3, g_i)_{i=1,2,3}$ which is parameterized by

$$X(x, y) = f(x)e_1 + g(y)e_2 + h(x, y)e_3 \quad (2.5)$$

where $f, g \in \mathcal{C}^n(\mathbb{R})$, $h \in \mathcal{C}^n(\mathbb{R}^2)$ with $n \geq 2$ are arbitrary not constant functions and $B_i = (e_1, e_2, e_3)$ is the pseudo-orthonormal basis of $(\mathcal{H}_3, g_i)$, for all $i \in \{1, 2, 3\}$ satisfying

$$g_i(e_1, e_1) = g_i(e_2, e_2) = 1, \quad g_i(e_3, e_3) = -1.$$

The coefficients of the first fundamental form of the surface Σ are :

$$E = g_i(X_x(x, y), X_x(x, y)), F = g_i(X_x(x, y), X_y(x, y)), G = g_i(X_y(x, y), X_y(x, y)), \quad (2.6)$$

and its coefficients of the second fundamental form are :

$$L = g_i(\nabla_{X_x(x, y)} X_x(x, y), n), M = g_i(\nabla_{X_y(x, y)} X_x(x, y), n), N = g_i(\nabla_{X_y(x, y)} X_y(x, y), n). \quad (2.7)$$

Where ∇ is the Levi-civita connection and n is the unit normal vector satisfying the following system :

$$\begin{cases} g_i(X_x, n) = 0, \\ g_i(X_y, n) = 0, \\ g_i(n, n) = 1. \end{cases} \quad (2.8)$$

Using the coefficients of the first and the second fundamental forms, we define the mean curvature and the Gauss curvature of a surface Σ .

Definition 2.2 : The mean curvature of a surface Σ is given by the formula

$$H = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2}. \quad (2.9)$$

Definition 2.3 : The Gauss curvature K of a surface Σ is

$$K = \frac{LN - M^2}{EG - F^2}. \quad (2.10)$$

3. CMC and CGC A-net surfaces in Lorentzian Heisenberg group $\mathcal{H}_3$

Since the Lorentzian Heisenberg group $\mathcal{H}_3$ has three classes of left invariant Lorentzian metrics, we study minimal surfaces and CMC A-net surfaces in the three cases $(\mathcal{H}_3, g_1)$, $(\mathcal{H}_3, g_2)$ and $(\mathcal{H}_3, g_3)$.

3.1 CMC and CGC A-net surfaces in Lorentzian Heisenberg group $(\mathcal{H}_3, g_1)$

The space $(\mathcal{H}_3, g_1)$ is a Lorentzian Heisenberg group given with the left invariant lorentzian metric (2.2), its pseudo-orthonormal basis $B_1 = (e_1, e_2, e_3)$ such as

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}. \quad (3.1)$$

The Levi-civita connection ∇ of g_1

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ \nabla_{e_2} e_1 = \frac{1}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{1}{2} e_1, \\ \nabla_{e_3} e_1 = \frac{1}{2} e_2, & \nabla_{e_3} e_2 = -\frac{1}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{cases} \quad (3.1)$$

The components of the tangent vector of the surface Σ defined by its parametrization (2.5) are the following derivatives

$$\begin{cases} X_x(x, y) = f'(x)e_1 + h'_x(x, y)e_3, \\ X_y(x, y) = g'(y)e_2 + h'_y(x, y)e_3. \end{cases} \quad (3.3)$$

Using (2.5), we get the coefficients of the first fundamental form of Σ

$$\begin{cases} E = f'^2(x) - h_x'^2(x, y), \\ F = -h'_x(x, y)h'_y(x, y), \\ G = g'^2(y) - h_y'^2(x, y). \end{cases} \quad (3.4)$$

We use the Levi-Civita connection (3.2) to calculate directional derivatives

$$\begin{cases} \nabla_{X_x} X_x = f''(x)e_1 + f'(x)h'_x(x, y)e_2 + h''_x(x, y)e_3, \\ \nabla_{X_x} X_y = -\frac{1}{2}g'(y)h'_x(x, y)e_1 + \frac{1}{2}f'(x)h'_y(x, y)e_2 + (h''_{xy}(x, y) + \frac{1}{2}f'(x)g'(y))e_3, \\ \nabla_{X_y} X_x = \frac{1}{2}g'(y)h'_x(x, y)e_1 + \frac{1}{2}f'(x)h'_y(x, y)e_2 + (h''_{xy}(x, y) + \frac{1}{2}f'(x)g'(y))e_3, \\ \nabla_{X_y} X_y = g''(y)e_2 + h''_2(x, y)e_3. \end{cases} \quad (3.5)$$

Note that $\nabla_{X_x} X_y = \nabla_{X_y} X_x$, if $h'_x(x, y) = 0$. Therefore, we take $h(x, y) = \varphi(y)$.

The coefficients of the first fundamental form (3.4) of Σ become

$$\begin{cases} E &= f'^2(x), \\ F &= 0, \\ G &= g'^2(y) - \varphi'^2(y). \end{cases} \quad (3.6)$$

The unit normal vector of the surface Σ is

$$\frac{1}{w} [f'(x)\varphi'(y)e_2 + f'(x)g'(y)e_3] \quad (3.7)$$

where

$$w = \sqrt{f'^2(x)\varphi'^2(y) - f'^2(x)g'^2(y)}.$$

Using (2.7), (3.2) and (3.7), we obtain the coefficients of the second fundamental form of Σ

$$\begin{cases} L &= 0, \\ M &= \frac{1}{2w} [f'^2(x)\varphi'^2(y) - f'^2(x)g'^2(y)], \\ N &= \frac{1}{w} [g''^2(y)f'(x)\varphi'(y) - f'(x)g'(y)\varphi''(y)]. \end{cases} \quad (3.8)$$

From (3.6) et (3.8) we get the mean curvature of the surface Σ

$$H = \frac{1}{2} \frac{g''(y)\varphi'(y) - g'(y)\varphi''(y)}{[g'^2(y) - \varphi'^2(y)]^{3/2}} \quad (3.9)$$

Theorem 3.1 : *The surface Σ in $(\mathcal{H}_3, g_1)$ parameterized by*

$$X(x, y) = f(x)e_1 + g(y)e_2 + \varphi(y)e_3$$

is minimal if and only if $\varphi(y) = Cg(y) + c_0$, $(C, c_0) \in \mathbb{R}^{*+} \times \mathbb{R}$.

Proof: From (3.9) we have

$$\begin{aligned} H = 0 &\Leftrightarrow g''(y)\varphi'(y) - g'(y)\varphi''(y) = 0 \\ &\Leftrightarrow \frac{g''(y)}{g'(y)} = \frac{\varphi''(y)}{\varphi'(y)} \\ &\Leftrightarrow \ln|g'(y)| + c = \ln|\varphi'(y)|, c \in \mathbb{R}, c \end{aligned}$$

So, we obtain $\varphi'(y) = Cg'(y)$, $C = \exp(\pm c) \in \mathbb{R}^{*+}$.

By integrating this last equation, we obtain

$$\varphi(y) = Cg(y) + c_o,$$

where $(C, c_o) \in \mathbb{R}^{*+} \times \mathbb{R}$. □

Corollary 3.1 : If $M = 0$ then the surface Σ is minimal in $(\mathcal{H}_3, g_1)$.

Theorem 3.2 : Let Σ defined above be an A-net surface in $(\mathcal{H}_3, g_1)$. If Σ is a constant mean curvature A-net surface, then $f(x) = \sqrt{\lambda}x + x_0$ and $\varphi(y) = -\frac{1}{2}\lambda \cosh(-2Cy + y_0)$, where λ, x_0, C, y_0 are real constants with $\lambda > 0$ and $C \neq 0$.

Proof: Since Σ is an A-net surface in $(\mathcal{H}_3, g_1)$, then

$$E = G \Rightarrow g'^2(y) - \varphi'^2(y) = f'^2(x)$$

The right hand side of this last equation depends only on x and the left one depends only on y , so there exists a constant $\lambda \in \mathbb{R}^{*+}$ such as

$$g'^2(y) - \varphi'^2(y) = f'^2(x) = \lambda \quad (3.10)$$

From (3.10), we find $f(x) = \sqrt{\lambda}x + x_0$, where $x_0 \in \mathbb{R}$ and $g'(y) = \sqrt{\varphi'^2(y) + \lambda}$.

Replacing $g'(y)$ and $g''(y)$ in (3.9), we obtain the main curvature of the a-net surface Σ

$$H = -\frac{1}{2\sqrt{\lambda}} \frac{\varphi''(y)}{\sqrt{\varphi'^2(y) + \lambda}} \quad (3.11)$$

which is constant if

$$-\frac{1}{2\sqrt{\lambda}} \frac{\varphi''(y)}{\sqrt{\varphi'^2(y) + \lambda}} = C, \quad (3.12)$$

with $C \in \mathbb{R}^*$.

Solving (3.12), we find

$$\varphi(y) = -\frac{1}{2} \lambda \cosh(-2Cy + y_0),$$

where y_0 is a real constant.

Finally, the constant mean curvature A-net surface Σ is parameterized by

$$X(x, y) = (\sqrt{\lambda}x + x_0)e_1 + g(y)e_2 - \left(\frac{1}{2} \lambda \cosh(-2Cy + y_0)\right)e_3.$$

□

Corollary 3.2 : *The Gauss curvature of the surface Σ is a non-zero negative constant K if $\varphi(y) = g(y) + c$, $c \in \mathbb{R}$ is an integration constant.*

Proof: From the definition (2.3), we get

$$K = \frac{-1}{4} [f'^2(x)(\varphi'^2(y) - g'^2(y))]^2, \quad (3.13)$$

which implies

$$2\sqrt{-K} = f'^2(x)(\varphi'^2(y) - g'^2(y)). \quad (3.14)$$

We divide the equation (3.14) by $f'^2(x)$, then we take derivative with respect to y , we obtain

$$2\varphi'(y)\varphi''(y) - 2g'(y)g''(y) = 0.$$

Therefore, $\varphi(y) = g(y) + c$ with c is a real integration constant. □

Corollary 3.3 : *The Gauss curvature of the A-net surface Σ is a non-zero negative constant $K < 0$ if*

$$\begin{cases} \varphi'(y) = \sqrt{g'^2(y) - \lambda} & \text{where } l \in \mathbb{R}^{+,+} \\ \text{and} \\ f(x) = \sqrt{\lambda}x + x_0, & \text{where } x_0 \in \mathbb{R}. \end{cases}$$

Proof: Following the same steps of the proof of theorem (3.2), we find the Gauss curvature of the A-net surface Σ

$$\begin{aligned} K &= \frac{-1}{4}(\lambda f'^2(x))^2 \\ &= \frac{-1}{4}\lambda^4 \end{aligned}$$

It's clear that $K < 0$ when the A-net surface Σ is parameterized by

$$X(x, y) = (\sqrt{\lambda}x + x_0)e_1 + g(y)e_2 + \varphi(y)e_3,$$

such as $g'^2(y) - \varphi'^2(y) = \lambda$ □

3.2 CMC and CGC A-net surfaces in Lorentzian Heisenberg group $(\mathcal{H}_3, g_2)$

In this case, we consider the left invariant lorentzian metric (2.3) such as the vectors of its the pseudo-orthonormal basis B_2 are

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial x}. \quad (3.15)$$

The Levi-civita connection ∇ of g_2

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{1}{2}e_3, & \nabla_{e_1} e_3 = \frac{1}{2}e_2, \\ \nabla_{e_2} e_1 = \frac{1}{2}e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3} e_1 = \frac{1}{2}e_2, & \nabla_{e_3} e_2 = -\frac{1}{2}e_1, & \nabla_{e_3} e_3 = 0. \end{cases} \quad (3.16)$$

The components of the tangent vector of the surface Σ defined by its parametrization (2.5) are the following derivatives

$$\begin{cases} X_x(x, y) = f'(x)e_1 + h'_x(x, y)e_3, \\ X_y(x, y) = g'(y)e_2 + h'_y(x, y)e_3. \end{cases} \quad (3.17)$$

We know that $h \in \mathcal{C}^n(\mathbb{R}^2)$ with $n \geq 2$, then using (2.5), we get the coefficients of the first fundamental form of Σ

$$\begin{cases} E &= & f'^2(x), \\ F &= & 0, \\ G &= & g'^2(y) - \varphi'^2(y). \end{cases} \quad (3.18)$$

The unit normal vector of the surface Σ is the same found in the previous case and the coefficients of the second fundamental form of Σ

$$\begin{cases} L &= 0, \\ M &= \frac{1}{2w} [f'^2(x)\varphi'^2(y) + f'^2(x)g'^2(y)], \\ N &= \frac{1}{w} [g''^2(y)f'(x)\varphi'(y) - f'(x)g'(y)\varphi''(y)]. \end{cases} \quad (3.19)$$

From (3.18) et (3.19) we get the mean curvature of the surface Σ

$$H = \frac{1}{2} \frac{g''(y)\varphi'(y) - g'(y)\varphi''(y)}{[g'^2(y) - \varphi'^2(y)]^{3/2}} \quad (3.20)$$

which is minimal if the surface Σ in $(\mathcal{H}_3, g_2)$ is parameterized by

$$X(x, y) = f(x)e_1 + g(y)e_2 + \varphi(y)e_3,$$

where $\varphi(y) = Cg(y) + c$, $(C, c) \in \mathbb{R}^{*+} \times \mathbb{R}$.

In addition, H given in (3.20) is constant for all Σ A-net surface in $(\mathcal{H}_3, g_2)$ defined by its parametrization

$$X(x, y) = (\sqrt{\lambda}x + x_0)e_1 + g(y)e_2 - \frac{1}{2} \lambda \cosh(-2Cy + y_0)e_3,$$

where λ, x_0, C, y_0 are real constants with $\lambda > 0$ and $C \neq 0$.

Corollary 3.4 : *The Gauss curvature of the surface Σ is a non-zero constant if $\varphi(y) = -g(y) + c$, $c \in \mathbb{R}$ is an integration constant.*

Proof: From the definition (2.3), we get

$$2\sqrt{-K} = f'^2(x)(\varphi'^2(y) + g'^2(y)) \quad (3.21)$$

We divide this last equation (3.21) by $f'^2(x)$, then we take derivative with respect to y , we obtain

$$2\varphi'(y)\varphi''(y) + 2g'(y)g''(y) = 0.$$

Therefore, $\varphi(y) = -g(y) + c$ with c is a real integration constant. $\square$

Corollary 3.5 : *The Gauss curvature of the A-net surface Σ is a non-zero negative constant K if*

$$\begin{cases} \varphi(y) = \sqrt{\frac{2\sqrt{-K} - \lambda^2}{2\lambda}} y + y_0, & \text{where } \lambda \in \mathbb{R}^{*+} \text{ and } y_0 \in \mathbb{R}, \\ f(x) = \sqrt{\lambda}x + x_0, & \text{where } x_0 \in \mathbb{R}. \end{cases}$$

Proof: Since $E = G$ then there exists a positive constant λ such as

$$g'^2(y) - \varphi'^2(y) = \lambda$$

and

$$f(x) = \sqrt{\lambda}x + x_0,$$

where $x_0 \in \mathbb{R}$.

Consequently, the Gauss curvature of the A-net surface Σ is a non-zero negative constant which satisfies

$$2\sqrt{-K} = \lambda(2\varphi'^2(y) + \lambda)$$

Solving this differential equation, we find

$$\varphi(y) = \sqrt{\frac{2\sqrt{-K} - \lambda^2}{2\lambda}}y + y_0,$$

where $y_0 \in \mathbb{R}$. Finally, the constant Gauss curvature A-net surface Σ is parameterized by

$$X(x, y) = (\sqrt{\lambda}x + x_0)e_1 + g(y)e_2 + \left(\sqrt{\frac{2\sqrt{-K} - \lambda^2}{2\lambda}}y + y_0 \right) e_3,$$

with $g'^2(y) - \varphi'^2(y) = \lambda$

3.3 CMC and CGC A-net surfaces in Lorentzian Heisenberg group $(\mathcal{H}_3, g_3)$

The Lorentzian Heisenberg group $(\mathcal{H}_3, g_3)$ endowed with the left invariant lorentzian metric (2.4) has a pseudo-orthonormal basis B_3 :

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}. \quad (3.22)$$

The Levi-civita connection ∇ of g_2

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = e_2 - e_3, & \nabla_{e_2} e_2 = -e_1, & \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 = e_2 - e_3, & \nabla_{e_3} e_2 = -e_1, & \nabla_{e_3} e_3 = -e_1. \end{cases} \quad (3.23)$$

The components of the tangent vector of the surface Σ defined by its parametrization (2.5) are the following derivatives

$$\begin{cases} X_x(x, y) = f'(x)e_1 + h'_x(x, y)e_3, \\ X_y(x, y) = g'(y)e_2 + h'_y(x, y)e_3. \end{cases} \quad (3.24)$$

Using the Levi-Civita connection (3.23), we obtain the following directional derivatives

$$\begin{cases} \nabla_{X_x} X_x &= -f''(x)h_x'^2(x, y)e_1 + f'(x)h'_x(x, y)e_2 + (h'_x(x, y) - f'(x)h'_x(x, y))e_3, \\ \nabla_{X_x} X_y &= -(g'(y) + 1)h'_x(x, y)e_1 + h''_{xy}(x, y)e_3, \\ \nabla_{X_y} X_x &= -(g'(y) + h'_y(x, y))h'_x(x, y)e_1 + f'(x)(h'_y(x, y) + g'(y))e_2 + (h''_{xy}(x, y) - f'(x) - g'(y))e_3, \\ \nabla_{X_y} X_y &= -(g'(y) + h'_y(x, y))^2e_1 + g''(y)e_2 + h''_{yy}(x, y)e_3. \end{cases} \quad (3.25)$$

Since $\nabla_{X_x} X_y = \nabla_{X_y} X_x$, then

$$\begin{cases} f(x) &= \text{constant}, \\ h'_y(x, y) &= g'(y) = 1. \end{cases} \quad (3.26)$$

Consequently, we get

$$\begin{cases} f(x) &= x_0, \\ g(y) &= y + y_0, \\ h(x, y) &= y + \alpha(x) + c, \end{cases} \quad (3.27)$$

where x_0 , y_0 and c are real constants and $\alpha(x) \in C^n(\mathbb{R})$, with $n \geq 2$.

The coefficients of the first fundamental form of the surface Σ in $(\mathcal{H}_3, g_3)$ are

$$\begin{cases} E &= \alpha'^2(x), \\ F &= -\alpha(x), \\ G &= 0. \end{cases} \quad (3.28)$$

The unit normal vector of the surface Σ is $\frac{1}{w}[\alpha'(x)e_1]$, where $w = \alpha'(x)$.

The coefficients of the second fundamental form of Σ are :

$$\begin{cases} L &= 0, \\ M &= -\frac{2}{w}\alpha'^2(x), \\ N &= -\frac{4}{w}\alpha(x). \end{cases} \quad (3.29)$$

from (3.28) and (3.29), we find the main curvature H and the Gauss curvature K of the surface Σ

$$\begin{cases} H &= 2, \\ K &= \frac{4}{\alpha^2(x)}. \end{cases} \quad (3.30)$$

Finally, we summarize with the following theorems :

Theorem 3.3 : *The surface Σ parameterized in $(\mathcal{H}_3, g_3)$ by*

$$X(y) = x_0 e_1 + (y + y_0) e_2 + (y + \alpha x + c) e_3$$

with x_0, y_0 and c are real constants and $\alpha(x) \in C^n(\mathbb{R})$, $(n \geq 2)$ is a constant mean curvature surface.

Theorem 3.4 : *The surface Σ parameterized in $(\mathcal{H}_3, g_3)$ by*

$$X(x, y) = x_0 e_1 + (y + y_0) e_2 + \left(y + \frac{2}{\sqrt{K}} x + x_1 \right) e_3,$$

with x_0, y_0, x_1 and K are real constants such as $K > 0$ and $\alpha(x) \in C^n(\mathbb{R})$, $(n \geq 2)$ is a constant Gauss curvature surface.

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