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Some properties of subordination differential and superordination for univalent functions associated with the convolution operators

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Abstract

In this paper, we define the Hadamard product (or convolution) $\big(\phi * \psi([\eta])\big)(\xi)$ of the analytic function in the unit disk $\mathfrak{A} = [\xi : |\xi| < 1 , \xi \in \mathbb{C}]$ with the non-zero parameter η, satisfactory to the relationship $h_\eta\big(\phi * \psi([\eta])\big)(\xi) = \big(h_\eta - 1\big)\big(\phi * \psi([\eta+1])\big)(\xi) + \xi\big(\phi * \psi([\eta+1])\big)'(\xi)$, to derive subordination, superordination and sandwich results for functions of the form $\big(\phi * \psi([\eta])\big)(\xi)$ by using some properties of Subordination and Superordination concepts. Our results serve to generalize and improve some previous studies where the results of this research were applied to linear operator $\mathfrak{B}_d^b$ band the multiplier operator $\mathfrak{I}_{\ell,\mu}^i$. This work can be generalized to other linear operators.

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1. Introduction

Let $\mathfrak{N}$ be the class of function $\phi(\xi)$ in the form:

$$\phi(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots = \xi + \sum_{k=2}^{\infty} a_k \xi^k , \tag{1.1}$$

here $\phi(\xi)$ analytic in open unit disk $\mathfrak{A} = [\xi : |\xi| < 1 , \xi \in \mathbb{C}]$, and normalized with $\phi(0)=0$, $\phi'(0)=1$. Also, consider $\mathcal{S}$ be the class of univalent function in $\mathfrak{A}$.

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Let $\mathcal{S}^*(\varrho), \mathcal{C}(\varrho)$ and $\mathcal{K}$ be the subclasses of $\mathfrak{N}$ such that:

$\left\{\phi \in \mathcal{S}^*: \mathcal{R}e\left\{\frac{\xi\phi'(\xi)}{\phi(\xi)}\right\} > \varrho\right\}, \xi \in \mathfrak{U}, \quad (0 < \varrho < 1)$. Then ϕ is starlike function.

$\left\{\phi \in \mathcal{C}: \mathcal{R}e\left\{1 + \frac{\xi\phi''(\xi)}{\phi'(\xi)}\right\} > \varrho\right\}, \xi \in \mathfrak{U}, (0 < \varrho < 1)$. Then ϕ is convex function.

$\left\{\phi_1 \in \mathcal{K}: \mathcal{R}e\left\{\frac{{\phi_1}'(\xi)}{{\phi_2}'(\xi)}\right\} > 0: \phi_2 \in \mathcal{C}, \xi \in \mathfrak{U}\right\}$. Then ϕ is close-to-convex function.

If the functions ϕ_1 and ϕ_2 are analytic in $\mathfrak{U}$. We say ϕ_1 is subordinate to ϕ_2 or ϕ_2 is said to be superordinate to ϕ_1 in $\mathfrak{U}$, written as $\phi_1 \prec \phi_2$ or $\phi_1(\xi) \prec \phi_2(\xi)$ if there a Schwarz function $\upsilon(\xi)$ analytic in $\mathfrak{U}$, with $|\upsilon(\xi)| < 1$. So $\phi_1(\xi)$ is equal to $\phi_2(\upsilon(\xi)), \xi \in \mathfrak{U}$. In particular, if the function ϕ_2 is univalent in $\mathfrak{U}$, then subordination $\phi_1 \prec \phi_2$ is equivalent to $\phi_1(0) = \phi_2(0)$ and $\phi_1(\mathfrak{U}) \subset \phi_2(\mathfrak{U})$, (see [8], [5]).

Further, if $\phi_1(\xi) = \xi + \sum_{k=2}^{\infty} a_k\xi^k$ and $\phi_2(\xi) = \xi + \sum_{k=2}^{\infty} b_k\xi^k$, then we define the Hadamard product (or convolution) by:

$$\phi_1(\xi) * \phi_2(\xi) = \xi + \sum_{k=2}^{\infty} a_k b_k \xi^k = (\phi_1 * \phi_2)(\xi).$$

Suppose $\psi([\eta]) \in \mathfrak{N}$ be the form $\psi([\eta])(\xi) = \xi + \sum_{k=2}^{\infty} c_k([\eta])\xi^k$, where $c_k([\eta])$ affected by the presence of a non-zero parameter $\eta \in \mathbb{C}$, satisfying for some $h_\eta \in \mathbb{C}$ .Then the recurrent relationship: [10]

$$h_\eta\big(\phi * \psi([\eta])\big)(\xi) = \big(h_\eta - 1\big)\big(\phi * \psi([\eta + 1])\big)(\xi) + \xi\big(\phi * \psi([\eta + 1])\big)'(\xi). \tag{1.2}$$

Recently, in [3],a linear operator is $\mathfrak{B}_d^b: \mathfrak{N} \to \mathfrak{N}$ known to use a generalized Bessel function $\mathcal{W}_{q,t,b}(\xi)$ [4] which is the first sort of order q, as defined by:

$$\mathfrak{B}_d^b\phi(\xi) = \mathcal{M}_{d,b}(\xi) * \phi(\xi) , \tag{1.3}$$

where $\mathcal{M}_{d,b}(\xi) = \mathcal{M}_{q,d,b}(\xi) = 2^q\mathcal{T}\left(q + \frac{r+1}{2}\right)\xi^{\frac{1-q}{2}}\mathcal{W}_{q,t,b}(\sqrt{\xi})$

$= \xi + \sum_{k=2}^{\infty}\frac{(-b)^{k-1}}{4^{k-1}(d)_{k-1}(K-1)!}\xi^k$, $d = q + \frac{r+1}{2} \in \mathbb{C}(\neq 0, -1, . . .), b, q, r, \xi \in \mathbb{C}$.

The multiplier operator $\mathfrak{I}_{\ell,\mu}^i: \mathfrak{N} \to \mathfrak{N}$ studied in [10], is defined for $i \in \mathbb{Z}$, $\mu > -1, \ell > 0$, by:

$$\mathfrak{I}_{\ell,\mu}^i\,\phi(\xi) = \phi(\xi), \qquad i = 0,$$

$$\Im_{\ell,\mu}^{i}\,\phi(\xi)=\frac{\mu+1}{\ell}\xi^{1-\frac{\mu+1}{\ell}}\int_0^z t^{\frac{\mu+1}{\ell}-2}\Im_{\ell,\eta}^{i+1}\phi(t)dt, \qquad i\in\mathbb{Z}^-,$$
$$\Im_{\ell,\mu}^{i}\,\phi(\xi)=\frac{\ell}{\mu+1}\xi^{2-\frac{\mu+1}{\ell}}\frac{d}{dt}\left(\xi^{\frac{\mu+1}{\ell}-1}\Im_{\ell,\mu}^{i-1}\phi(t)\right), \qquad i\in\mathbb{Z}^+.$$

The series expansion of $\Im_{\ell,\eta}^{i}\,\phi(z)$ is for $i\in\mathbb{Z},\mu>-1,\ \ell>0$ for the function ϕ of the form (1.1) given by:

$$\Im_{\ell,\mu}^{i}\,\phi(\xi)=\xi+\sum_{k=2}^{\infty}\left(1+\frac{\ell(k-1)}{\mu+1}\right)^{i}a_k\xi^k. \tag{1.4}$$

Bulboaca in 2005 [5], used results of Miller and Mocanu [9],studied some classes of first order differential superordination. Ali in 2004 and others [2], they use results of Bulboaca to get conditions for some normalized analytic functions to satisfy $b_1\prec\frac{\xi\phi'(\xi)}{\phi(\xi)}\prec b_2$,where b_1,b_2univalent in $\mathfrak{A}$ with $b_1(0)=b_2(0)=1$. Recently, Goyal and others [7], used certain results by sandwich theorem on some categories of analytical functions. Further,Cotˆırla and Al-khafaji ([9],[10]) they got conditions for some normalized analytic functions. We are working on generalizing and improving these results using the concept of convolution.The main objective of this research is to identify sufficient conditions for the analytical function $\phi*\psi([\eta])$ to obtaining and proof several subordination, superordination results and certain results by using sandwich theorem.

2. Preliminaries

Definition 2.1: [11] Let $\Upsilon:\mathbb{C}^3\times\mathfrak{A}\to\mathbb{C}$, and $h(\xi)$ be univalent in $\mathfrak{A}$. If $\mathcal{F}(\xi)$ is analytic in $\mathfrak{A}$ that fulfills the second-order differential subordination:

$$\Upsilon\left(\mathcal{F}(\xi),\xi\mathcal{F}'(\xi),\xi^2\mathcal{F}''(\xi);\xi\right)\prec h(\xi), \tag{2.1}$$

then $\mathcal{F}(\xi)$ is differential dependency solving of (2.1) .

Definition 2.2: [11] Let $\Upsilon_1:\mathbb{C}^3\times\mathfrak{A}\to\mathbb{C}$, and $h(\xi)$ be univalent in $\mathfrak{A}$. If $\mathcal{F}(\xi)$ and $\Upsilon_1\left(\mathcal{F}(\xi),\xi\mathcal{F}'(\xi),\xi^2\mathcal{F}''(\xi);\xi\right)$ are univalent in $\mathfrak{A}$ and $\mathcal{F}(\xi)$ that fulfills the second-order differential superordination:

$$h(\xi)\prec\Upsilon_1\left(\mathcal{F}(\xi),\xi\mathcal{F}'(\xi),\xi^2\mathcal{F}''(\xi);\xi\right), \tag{2.2}$$

then $\mathcal{F}(\xi)$ differential superordination solving of (2.2).

Definition 2.3 [11] Let Q be the collections of functions ϕ which are analytic and injective on $\overline{\mathfrak{A}}\setminus E(\phi)$,when $E(\phi)=\{\varsigma\in\partial\mathfrak{A}:\lim_{\xi\to\varsigma}\ \phi(\xi)=\infty\}$, and $\phi'(\xi)\neq 0$ for $\varsigma\in\ \partial\mathfrak{A}\setminus E(\phi)$.

Lemma 2.4: [9] *Let $p_1(\xi)$ be univalent function in $\mathfrak{A}$ and let Σ and ϑ be analytic in a domain $\mathcal{D}$, $p_1(\mathfrak{A}) \subset \mathcal{D}$, with $\vartheta(\omega) \neq 0$ when $\omega \in p_1(\mathfrak{A})$. Set $\mathcal{O}(\xi) = \xi\, p_1'(\xi)\vartheta(p_1(\xi))$ and $\hbar(\xi) = \Sigma(p_1(\xi) + \mathcal{O}(\xi)$. Suppose that*

(i) $\mathcal{O}$ is starlike in $\mathfrak{A}$.

(ii) $\mathcal{R}e\left(\frac{\xi\hbar'(\xi)}{\mathcal{O}(\xi)}\right) > 0, \xi \in \mathfrak{A}$.

If $p_2(\xi)$ is analytic in $\mathfrak{A}$ with $p_2(0) = p_1(0), p_2(\mathfrak{A}) \subset \mathcal{D}$ and $\Sigma\big(p_2(\xi) + \xi p_2'(\xi)\phi(p_2(\xi))\big) \prec \Sigma\big(p_1(\xi) + \xi p_1'(\xi)\vartheta(p_1(\xi))\big)$, then $p_2(\xi) \prec p_1(\xi)$.

Lemma 2.5: [11] *Let $p_1(\xi)$ be convex in $\mathfrak{A}$ and $\theta_1 \in \mathbb{C}, \theta_2 \in \mathbb{C}^*$ with $\mathcal{R}e\left(1 + \frac{p_1''(\xi)}{p_1'(\xi)}\right) > max \left\{0, -\mathcal{R}e\frac{\theta_1}{\theta_2}\right\}$. If $p_2(\xi)$ is analytic in $\mathfrak{A}$ and $\theta_1 p_2(\xi) + \theta_2\xi p_2'(\xi) \prec \theta_1 p_1(\xi) + \theta_2 \xi p_1'(\xi)$, then $p_2(\xi) \prec p_1(\xi)$.*

Lemma 2.6: [9] *Let $p_1(\xi)$ be convex univalent in $\mathfrak{A}$ and let and let Σ and ϑ be analytic in a domain $\mathcal{D}$, $p_1(\mathfrak{A}) \subset \mathcal{D}$. Suppose that*

(i) $\xi p_1'(\xi)\vartheta(p_1(\xi)$ is starlike univalent in $\mathfrak{A}$.

(ii) $\mathcal{R}e\left(\frac{\Sigma'(p_1(\xi))}{\vartheta(p_1(\xi))}\right) > 0, \xi \in \mathfrak{A}$.

If $p_2(\xi) \in \mathfrak{N}[p_1(0), 1] \cap Q$, with $p_2(\mathfrak{A}) \subset \mathcal{D}$, and $\Sigma\big(p_2(\xi) + \xi p_2'(\xi)\vartheta(p_2(\xi))\big)$ is univalent in $\mathfrak{A}$, and $\vartheta\big(p_1(\xi) + \xi p_1'(\xi)\vartheta(p_1(\xi))\big) \prec \vartheta(p_2(\xi) + \xi p_2'(\xi)\vartheta(p_2(\xi))$ then $p_1(\xi) \prec p_2(\xi)$.

Lemma 2.7: [9] *Let $p_1(\xi)$ be convex in $\mathfrak{A}$ and $\mathcal{R}e(\theta_2) > 0$. If $p_2(\xi) \in \mathfrak{N}[p_1(0), 1] \cap Q$, $p_2(\xi) + \theta_2\xi p_2'(\xi)$ is univalent in $\mathfrak{A}$ and $p_1(\xi) + \theta_2 \xi p_1' \prec p_2(\xi) + \theta_2\xi p_2'(\xi)$, then $p_1(\xi) \prec p_2(\xi)$.*

3. Subordination Results

This part introduces some subordination results for analytic function $\big(\phi * \psi([\mu])\big)(\xi)$

Theorem 3.1: *Let $b(\xi)$ be convex univalent in $\mathfrak{A}$, with* $b(0) = 1, \mathfrak{a}_1 > 0, 0 \neq \mathfrak{a}_2 \in \mathbb{C}$ *and suppose* $\mathcal{R}e\left(1 + \frac{b''(\xi)}{b'(\xi)}\right) > max \left\{0, -\mathcal{R}e\frac{\mathfrak{a}_1}{\mathfrak{a}_2}\right\}$,

If $\phi \in \mathfrak{N}$, satisfies the subordination:

$$\left(1 - \mathfrak{a}_2 h_\eta\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1} + \mathfrak{a}_2 h_\eta\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right) \prec$$
$$b(\xi) + \frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'(\xi), \text{ then } \left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1} \prec b(\xi). \tag{3.1}$$

Proof: Let $d(\xi) = \left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\alpha_1}, \xi \in \mathfrak{A}.$ (3.2)

By derivation of (3.1) and using the recurrent relationship (1.2) we get

$$\frac{\xi d'(\xi)}{d(\xi)} = \alpha_1\left(\frac{h_\eta(\phi*\psi([\eta]))(\xi)+(\phi*\psi([\eta+1]))(\xi)-h_\eta(\phi*\psi([\eta+1]))(\xi)}{(\phi*\psi([\lambda+1]))(\xi)} - 1\right)$$

$$= \alpha_1 h_\mu\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)} - 1\right),$$

and $\frac{\xi d'(\xi)}{\alpha_1} = h_\mu\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\alpha_1} - h_\eta\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\alpha_1}.$

by using the hypothesis (3.1) we get $d(\xi) + \frac{\alpha_2}{\alpha_1}\xi d'(\xi) \prec b(\xi) + \frac{\alpha_2}{\alpha_1}\xi b'(\xi)$,

apply Lemma 2.5, when $\beta_1 = 1$ and $\beta_2 = \frac{\alpha_2}{\alpha_1}$,

then $$\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\alpha_1} \prec b(\xi)\ .$$

Theorem 3.2: *Let b be convex univalent in* $\mathfrak{A}$, $b(0) = 1$, *and* $b(\xi) \neq 0$ *for all* $\xi \in \mathfrak{A}$, *and suppose that b satisfies:*

$$\mathcal{R}e\left\{1 + \frac{t\sigma}{\alpha_2} + \frac{\varepsilon(\sigma+1)}{\alpha_2}(\xi) + (\sigma-1)\frac{\xi b'(\xi)}{b(\xi)} + \frac{\xi b''(\xi)}{b'(\xi)}\right\} > 0\ , \tag{3.3}$$

where $\sigma, \varepsilon, t \in \mathbb{C}, \alpha_1 > 0, 0 \neq \alpha_2 \in \mathbb{C}$ and $\xi \in \mathfrak{A}$, suppose that $\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi)$ is starlike univalent in $\mathfrak{A}$. If $\phi \in \mathfrak{N}$, satisfies the subordination:

$$\mathcal{M}(\sigma, t, \varepsilon, h_\eta, \eta, \alpha_1, \alpha_2; \xi) \prec \big(t + \varepsilon\, b(\xi)\big)\big(b(\xi)\big)^{\sigma} + \alpha_2\big(b(\xi)\big)^{\sigma-1}b'(\xi), \tag{3.4}$$

where $\mathcal{M}\big(\sigma, t, \varepsilon, h_\eta, \eta, \alpha_1, \alpha_2; \xi\big) = t\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\alpha_1\sigma}$

$$+\varepsilon\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\alpha_1(\sigma+1)} +$$

$$\alpha_2\alpha_1\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\alpha_1\sigma}$$

$$\times\left(\frac{\xi h_\eta[(\phi*\psi([\eta-1]))(\xi)]'+\xi(1-h_\eta)[(\phi*\psi([\eta]))(\xi)]'}{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)} - 1\right),$$

then $$\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\alpha_1} \prec b(\xi), \xi \in \mathfrak{A}.$$

Proof: Let $\mho(\beta) = (t + \varepsilon\beta)\beta^{\sigma}$ and $\mathcal{L}(\beta) = \alpha_2(\beta)^{\sigma-1}$, $0 \neq \beta \in \mathbb{C}$,
when $\mho(\beta)$ and $\mathcal{L}(\beta)$ are analytic in $\mathbb{C}$. Then we get
$\mathcal{G}(\xi) = \xi b'(\xi)\mathcal{L}\big(b(\xi)\big) = \alpha_2\xi(b(\xi))^{\sigma-1}b'(\xi)$ and

$$y(\xi) = \mho\big(b(\xi)\big) + \mathcal{G}(\xi) = \Big(t + \varepsilon\big(b(\xi)\big)\Big)\big(b(\xi)\big)^{\sigma} + \alpha_2\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi).$$

Since $\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi)$ starlike, then $\mathcal{G}(\xi)$ is starlike in $\mathfrak{A}$, and

$$\mathcal{R}e\left(\frac{y'(\xi)}{\mathcal{G}(\xi)}\right)$$

$$= \mathcal{R}e\left(\begin{array}{c}\dfrac{\xi t\sigma\big(b(\xi)\big)^{\sigma-1}b'(\xi)+\xi\sigma\varepsilon b(\xi)\big(b(\xi)\big)^{\sigma-1}b'(\xi)+\varepsilon\xi b(\xi)\big(b(\xi)\big)^{\sigma-1}b'(\xi)}{\mathfrak{a}_2\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi)}+\\ \dfrac{\xi^2\mathfrak{a}_2(\sigma-1)b'(\xi)\big(b(\xi)\big)^{\sigma-2}b'(\xi)+\xi^2\mathfrak{a}_2\big(b(\xi)\big)^{\sigma-1}b''(\xi)+\mathfrak{a}_2\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi)}{\mathfrak{a}_2\xi\big(b(\xi)\big)^{\sigma-1}b'(\xi)}\end{array}\right)$$

$$= \mathcal{R}e\left\{1+\frac{t\sigma}{\mathfrak{a}_2}+\frac{\varepsilon(\sigma+1)}{\mathfrak{a}_2}b(\xi)+(\sigma-1)\frac{\xi b'(\xi)}{b(\xi)}+\frac{\xi b''(\xi)}{b'(\xi)}\right\}>0.$$

Also, consider

$$d(\xi)=\left(\frac{h_\eta\big(\phi*\psi([\eta-1])\big)(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)(\xi)}{\xi}\right)^{\mathfrak{a}_1}, \text{ then}$$

$$d'(\xi)=\mathfrak{a}_1\left(\frac{h_\eta\big(\phi*\psi([\eta-1])\big)(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)(\xi)}{\xi}\right)^{\mathfrak{a}_1}$$
$$\left[\frac{h_\eta\big(\phi*\psi([\eta-1])\big)'(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)'(\xi)}{h_\eta\big(\phi*\psi([\eta-1])\big)(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)(\xi)}-\frac{1}{\xi}\right].$$

And

$$t\left(\frac{h_\eta\big(\phi*\psi([\eta-1])\big)(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)(\xi)}{\xi}\right)^{\mathfrak{a}_1\sigma}$$
$$+\varepsilon\left(\frac{h_\eta\big(\phi*\psi([\eta-1])\big)(\xi)+\big(1-h_\eta\big)\big(\phi*\psi([\eta])\big)(\xi)}{\xi}\right)^{\mathfrak{a}_1(\sigma+1)}$$
$$=t\big(d(\xi)\big)^{\sigma}+\varepsilon\big[\big(d(\xi)\big)^{\sigma}d(\xi)\big]=\big(t+\varepsilon d(\xi)\big)\big(d(\xi)\big)^{\sigma}.$$

We have $\mathfrak{a}_1\left(\frac{\xi h_\eta[(\phi*\psi([\eta-1]))(\xi)]'+\xi(1-h_\eta)[(\phi*\psi([\eta]))(\xi)]}{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}-1\right)=\frac{\xi d'(\xi)}{d(\xi)}$,

That is $\mathfrak{a}_2\mathfrak{a}_1\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1\sigma}\times$
$\left(\frac{\xi h_\eta[(\phi*\psi([\eta-1]))(\xi)]'+\xi(1-h_\eta)[(\phi*\psi([\eta]))(\xi)]}{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}-1\right)$

$$=\mathfrak{a}_2\,\xi\big(d(\xi)\big)^{\sigma-1}d'(\xi).$$

From (3.4) we get $\big(t+\varepsilon d(\xi)\big)\big(d(\xi)\big)^{\sigma}+\mathfrak{a}_2\,\xi\big(d(\xi)\big)^{\sigma-1}d'(\xi)\prec$
$$\big(t+\varepsilon b(\xi)\big)\big(b(\xi)\big)^{\sigma}+\mathfrak{a}_2\big(b(\xi)\big)^{\sigma-1}b'(\xi),$$

by using (3.3) and Lemma 2.4 we obtain $d(\xi)\prec b(\xi)$.

4. Super Ordinations Results

Theorem 4.1: *Let* $b(\xi)$ *be convex in* $\mathfrak{A}$, *with* $b(0) = 1$, $\mathfrak{a}_1 > 0, \mathcal{R}e\mathfrak{a}_2 > 0$, *If* $\phi \in \mathfrak{N}$, $\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1} \in \mathfrak{N}[d(0),1] \cap Q$ *and*

$$\left(1-\mathfrak{a}_2 h_\eta\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}+\mathfrak{a}_2 h_\eta\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right)$$

be univalent in $\mathfrak{A}$and satisfies the superordination:

$$b(\xi)+\frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'(\xi) \prec$$

$$\left(1-\mathfrak{a}_2 h_\mu\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}+\mathfrak{a}_2 h_\mu\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right), \quad (4.1)$$

then $b(\xi) \prec \left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}$.

Proof: Let $d(\xi) = \left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}, \xi \in \mathfrak{A}$.

Then $d'(\xi) = \mathfrak{a}_1\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1-1}\left(\frac{[(\phi*\psi([\eta+1]))(\xi)]'}{\xi}-\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi^2}\right)$.

We have $\frac{d'(\xi)}{d(\xi)} = \mathfrak{a}_1\left(\frac{[(\phi*\psi([\eta+1]))(\xi)]'}{(\phi*\psi([\eta+1]))(\xi)}-\frac{1}{\xi}\right)$,

with the same steps of theorem (3.1) and using the hypothesis (4.1), we get

$b(\xi)+\frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'(\xi) \prec b(\xi)+\frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'(\xi)$.

by apply Lemma 2.7 we obtain $b(\xi) \prec \left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}$.

Theorem 4.2*:* *Let* b *be convex univalent in* $\mathfrak{A}$, $b(0) = 1$, *and* $b(\xi) \neq 0$ *for all* $\xi \in \mathfrak{A}$, *and suppose that* b *satisfies:* $\mathcal{R}e\left\{\frac{t\sigma}{\mathfrak{a}_2}b'(\xi)+\frac{\varepsilon(\sigma+1)}{\mathfrak{a}_2}b(\xi)b'(\xi)\right\} > 0$,

where $\sigma, \varepsilon, t \in \mathbb{C}, 0 \neq \mathfrak{a}_2 \in \mathbb{C}^*$, $\xi \in \mathfrak{A}$ *and* $\xi\left(b(\xi)\right)^{\sigma-1}b'(\xi)$ *is starlike univalent in* $\mathfrak{A}$. *If* $\phi \in \mathfrak{N}$, *satisfies the condition :*

$\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1} \in \mathfrak{N}[b(0),1] \cap Q$, and

$\mathcal{M}\left(\sigma,t,\varepsilon,h_\eta,\eta,\mathfrak{a}_1,\mathfrak{a}_2;\xi\right)$ is univalent in $\mathfrak{A}$.

If $\left(t+\varepsilon\, b(\xi)\right)\left(b(\xi)\right)^{\sigma}+\mathfrak{a}_2\left(b(\xi)\right)^{\sigma-1}b'(\xi) \prec \mathcal{M}\left(\sigma,t,\varepsilon,h_\eta,\eta,\mathfrak{a}_1,\mathfrak{a}_2;\xi\right)$, (4.2)

then $b(\xi) \prec \left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}$.

Proof: Let $\mho(\beta) = (t+\varepsilon\beta)\beta^{\sigma}$ and $\mathcal{L}(\beta) = \mathfrak{a}_2(\beta)^{\sigma-1}$, $0 \neq \beta \in \mathbb{C}$,
when $\mho(\beta)$ is analytic in $\mathbb{C}$ and $\mathcal{L}(\beta) \neq 0$ is analytic in $\mathbb{C}$ / 0 .Then we get

$$\mathcal{G}(\xi) = \xi b'(\xi)\mathcal{L}\left(b(\xi)\right) = \mathfrak{a}_2\xi\left(b(\xi)\right)^{\sigma-1}b'(\xi).$$

Since $\xi\left(b(\xi)\right)^{\sigma-1}b'(\xi)$ starlike, then $\mathcal{G}(\xi)$ is starlike in $\mathfrak{A}$, and

$$\mathcal{R}e\left(\frac{\mho'(b(\xi))}{\mathcal{L}(b(\xi))}\right)=\mathcal{R}e\left(\frac{\left(t+\varepsilon(b(\xi))\right)(b(\xi))^{\sigma}}{\mathfrak{a}_2(b(\xi))^{\sigma-1}}\right)=\mathcal{R}e\left\{\frac{t\sigma}{\mathfrak{a}_2}b'(\xi)+\frac{\varepsilon(\sigma+1)}{\mathfrak{a}_2}b(\xi)b'(\xi)\right\}>0$$

Now let $d(\xi)=\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}$.

In a similar way to theorem (3.2), we get:

$$\begin{aligned}&\left(t+\varepsilon\, b(\xi)\right)(b(\xi))^{\sigma}+\mathfrak{a}_2(b(\xi))^{\sigma-1}b'(\xi)\\ &\prec\left(t+\varepsilon d(\xi)\right)(d(\xi))^{\sigma}+\mathfrak{a}_2\,\xi(d(\xi))^{\sigma-1}d'(\xi).\end{aligned}$$

by Lemma 2.6 we obtain $b(\xi)\prec d(\xi)$.

5. Sandwich Results

By combining the above theories, we get the following two sandwich theories.

Theorem 5.1: *Let b_1, b_2 be convex univalent in $\mathfrak{A}$, with $b_1(0)=b_2(0)=1$ $\mathcal{R}e\mathfrak{a}_2>0$ and $\mathcal{R}e\left(1+\frac{d''(\xi)}{d'(\xi)}\right)>max\ \left\{0,-\mathcal{R}e\frac{\mathfrak{a}_1}{\mathfrak{a}_2}\right\}$, where $\mathfrak{a}_1>0, 0\neq\mathfrak{a}_2\in₵$. If $\phi\in\mathfrak{N}$ and $\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\in\mathfrak{N}[1,1]\cap Q$, and $\left(1-\mathfrak{a}_2h_\eta\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}+\mathfrak{a}_2h_\mu\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right)$ is univalent in $\mathfrak{A}$ satisfies: $b_1(\xi)+\frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'_1(\xi)\prec\left(1-\mathfrak{a}_2h_\mu\right)\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}+\mathfrak{a}_2h_\eta\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\left(\frac{(\phi*\psi([\eta]))(\xi)}{(\phi*\psi([\eta+1]))(\xi)}\right)\prec b_2(\xi)+\frac{\mathfrak{a}_2}{\mathfrak{a}_1}\xi b'_2(\xi)$, then $b_1(\xi)\prec\left(\frac{(\phi*\psi([\eta+1]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\prec b_2(\xi)$.*

Theorem 5.2: *Let b_1, b_2 be convex univalent in $\mathfrak{A}$, with $b_1(0)=b_2(0)=1$, and let $\phi\in\mathfrak{N}$, satisfies the condition:*

$\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\in\mathfrak{N}[1,1]\cap Q$, and

$\mathcal{M}\left(\sigma,t,\varepsilon,h_\eta,\eta,\mathfrak{a}_1,\mathfrak{a}_2;\xi\right)$, is univalent in $\mathfrak{A}$.

If
$$\begin{aligned}&\left(t+\varepsilon\, b_1(\xi)\right)(b_1(\xi))^{\sigma}+\mathfrak{a}_2(b_1(\xi))^{\sigma-1}b_1{}'(\xi)\prec\mathcal{M}\left(\sigma,t,\varepsilon,h_\eta,\eta,\mathfrak{a}_1,\mathfrak{a}_2;\xi\right)\\ &\prec\left(t+\varepsilon\, b_2(\xi)\right)(b_2(\xi))^{\sigma}+\mathfrak{a}_2(b_2(\xi))^{\sigma-1}b_2{}'(\xi),\end{aligned}$$
then $b_1(\xi)\prec\left(\frac{h_\eta(\phi*\psi([\eta-1]))(\xi)+(1-h_\eta)(\phi*\psi([\eta]))(\xi)}{\xi}\right)^{\mathfrak{a}_1}\prec b_2(\xi)$.

6. Discussion and Conclusion

This paper studies the sufficient conditions for the analytic functions of the form $(\phi*\psi([\eta]))(\xi)$, defined by Hadamard product (or convolution) to obtain subordination and superordination results for those functions. Some sandwich results were also obtained by merging subordination and superordination results.

This research can be considered a generalization and improvement of some previous studies.

Now the above theories can be applied according to certain conditions, as follows:

(1) Let $h_\eta = d$ and $(\phi * \psi([\eta + j]))(\xi) = \mathfrak{B}^{b}_{d+j}\phi(\xi), j = 0,1,\text{-}1$. Then

$$d\mathfrak{B}^{b}_{d}\phi(\xi) = (d-1)\mathfrak{B}^{b}_{d+1}\phi(\xi) + \xi\left(\mathfrak{B}^{b}_{d+1}\phi(\xi)\right)'(\xi), \tag{6.1}$$

by using (6.1). All results above mentioned, satisfying with linear operator $\mathfrak{B}^{b}_{d+j}\phi(\xi)$.

(2) Let $h_\eta = \frac{\mu+1}{\ell}$ and $(\phi * \psi([\mu + j]))(\xi) = \mathfrak{I}^{i+j}_{\ell,\mu}\phi(\xi), j = 0,1,\text{-}1$. Then

$$\frac{\mu+1}{\ell}\left(\mathfrak{I}^{i}_{\ell,\mu}\phi(\xi)\right)(\xi) = \left(\frac{\mu+1}{\ell} - 1\right)\mathfrak{I}^{i+1}_{\ell,\mu}\phi(\xi) + \xi\left(\mathfrak{I}^{i+1}_{\ell,\mu}\phi(\xi)\right)'(\xi), \tag{6.2}$$

by using (6.2) all results above mentioned, satisfying with the multiplier operator.

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