

Some properties of semi (p, ε) -convexity

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Abstract

In this paper, we present a new class of functions called semi (p, ε) -convex functions which represent a combination of p -convex and semi ε -convex functions. We discuss some important properties and characteristics of this class. For applying this class of new functions to optimize programming, we present some optimality properties such that the cost function of the optimization programming is semi (p, ε) -convex and the set of constraints is (p, ε) -convex.

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1. Introduction and Preliminaries

Research on convexity is considered an important topic in the fields of applied and pure mathematics. Various attempts have been widely and deeply made in the literature by many researchers in different areas to extend and generalize convex sets (functions) into non-convex sets (functions) by altering the convexity assumptions (see, e.g., 1-8). In this regard, Youness [8] presented ε -convex sets (resp., functions) with applications to optimization problems. The results on ε -convexity were studied and improved further by many researchers. Chen [2], for example, defined and studied new ε -convex functions called semi ε -convex. Another important type of non-convex sets (functions) is the class p -convex sets (resp., functions) recently introduced by Bayoumi and Fathy [1] and

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Sezer et. al. [6] by making the effect of the real number $p \in (0,1]$ on the new defined sets and functions. Very recently Hazim and Majeed [3] presented a class of (p, ε) -convex sets and (p, ε) -convex functions by combining ε -convex sets (resp., functions) and p -convex sets (resp., functions).

In this paper, motivated by p -convexity [1, 6], semi ε -convexity [2], and (p, ε) -convexity [3], we introduce a new class named semi (p, ε) -convex functions. We analyze the main properties that semi (p, ε) -convex functions satisfy and study some optimality properties of the optimization problem defined in terms of this class of functions.

Throughout this paper, let $p \in [0,1]$ and $\mathbb{R}^n$ is the n -dimensional Euclidean space. Assume that A is a non-empty subset of $\mathbb{R}^n$, $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and $\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given mapping. In what follows, the related concepts needed in this work are provided.

Definition 1.1: [8] A is named as ε -convex set if $\forall x, y \in A$, $r \in [0,1]$, then $r\varepsilon(x) + (1-r)\varepsilon(y) \in A$.

Definition 1.2: [1], [6] The set A is named as p -convex if for an arbitrary $x, y \in A$ and all $r, s \in [0,1]$ such that $r^p + s^p = 1$, then $rx + sy \in A$.

Definition 1.3: For an arbitrary $x, y \in A$, and $r, s \in [0,1]$, the function f is named as:

1. ε -convex if A is ε -convex and

$$f(r\varepsilon(x) + (1-r)\varepsilon(y)) \leq rf(\varepsilon(x)) + (1-r)f(\varepsilon(y)). \quad [8]$$
2. semi ε -convex if A is ε -convex and

$$f(r\varepsilon(x) + (1-r)\varepsilon(y)) \leq rf(x) + (1-r)f(y). \quad [2]$$
3. p -convex if A is p -convex set and

$$f(rx + sy) \leq rf(x) + sf(y), \quad r^p + s^p = 1. \quad [1]$$

Definition 1.4: [3] A is named as (p, ε) -convex if $\forall x, y \in A$, and $s, r \in [0,1]$ such that $r^p + s^p = 1$ we have $r\varepsilon(x) + s\varepsilon(y) \in A$.

Definition 1.5: [3] f is named as (p, ε) -convex on the set A if A is (p, ε) -convex set and for arbitrary $x, y \in A$, $r, s \in [0,1]$ such that $r^p + s^p = 1$ we have $f(r\varepsilon(x) + s\varepsilon(y)) \leq rf(\varepsilon(x)) + sf(\varepsilon(y))$.

2. Semi (p, ε) –Convexity with Properties

In this section, semi (p, ε) -convex function is introduced. This class generalizes semi ε -convex functions [2] and p -convex functions [1, 6]. Some properties and related examples are established.

Definition 2.1 Let A be (p, ε) -convex set. Then for each $x, y \in A$, $r, s \in [0, 1]$, and $p \in (0, 1]$ such that $r^p + s^p = 1$, f is named as semi (p, ε) -convex if $f(r\varepsilon(x) + s\varepsilon(y)) \leq rf(x) + sf(y)$.

Next, we show an example of semi (p, ε) -convex function.

Example 2.2 let $A = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^2 < 1\}$ and $f: A \rightarrow \mathbb{R}$ such that $f(x) = \left(\sum_{i=1}^n |x_i|^{\frac{1}{3}}\right)^3$ for all $x \in A$. Assume that $\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\varepsilon(x) = (\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_n}{2})$. From [3, Example 2.4], A is $(\frac{1}{2}, \varepsilon)$ convex set. Now, let $x, y \in A$, $p = \frac{1}{2}$ and $r, s \in [0, 1]$ such that $r^p + s^p = 1$, then we have $r\varepsilon x + s\varepsilon y \in A$ and $f(r\varepsilon x + s\varepsilon y) = f\left(r\frac{x_1}{2} + s\frac{y_1}{2} + \dots, r\frac{x_n}{2} + s\frac{y_n}{2}\right)$

$$\begin{aligned} &= \left(\sum_{i=1}^n \left|r\frac{x_i}{2} + s\frac{y_i}{2}\right|^3\right)^{\frac{1}{3}} \leq \left(\sum_{i=1}^n \left|r\frac{x_i}{2}\right|^3\right)^{\frac{1}{3}} + \left(\sum_{i=1}^n \left|s\frac{y_i}{2}\right|^3\right)^{\frac{1}{3}} \\ &\leq r \left(\sum_{i=1}^n |x_i|^3\right)^{\frac{1}{3}} + s \left(\sum_{i=1}^n |y_i|^3\right)^{\frac{1}{3}}. \end{aligned}$$

Hence, f is $(\frac{1}{2}, \varepsilon)$ -semi convex function.

The following remark is needed in the sequel.

Remark 2.3 Throughout the paper and unless otherwise specified.

- i. The mapping $\varepsilon(x)$ will be written as εx in the proofs and the calculations.
- ii. The set A is (p, ε) -convex set.
- iii. From Definitions 1.4 and 2.1, we always have $r, s \in [0, 1]$, and $p \in (0, 1]$ such that $r^p + s^p = 1$.

The next result represents a relationship between (p, ε) -convex and semi (p, ε) -convex functions.

Proposition 2.4: Let f is (p, ε) -convex on A . Then $f(x) \geq f(\varepsilon x)$ for any $x \in A$ if and only if f is semi (p, ε) -convex on A .

Proof: Let $x, y \in A$, and f be semi (p, ε) -convex on the (p, ε) -convex set A . Thus, $r\varepsilon x + s\varepsilon y \in A$ and $f(r\varepsilon x + s\varepsilon y) \leq rf(x) + sf(y)$. For $r = 1$ and $s = 0$ we get $f(\varepsilon x) \leq f(x)$ as required. Conversely, if f is (p, ε) -convex on A and $f(\varepsilon x) \leq f(x)$ for each $x \in A$. Then $f(r\varepsilon x + s\varepsilon y) \leq rf(\varepsilon x) + sf(\varepsilon y) \leq rf(x) + sf(y)$. Therefore, f is semi (p, ε) -convex on A .

Proposition 2.5: If f is semi (p, ε) -convex function then the level set $K_\alpha = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is (p, ε) -convex for any $\alpha \in \mathbb{R}$.

Proof: Let $x, y \in K_\alpha$ then $f(x) \leq \alpha$ and $f(y) \leq \alpha$. From the semi (p, ε) -convexity of f , we get $f(r\varepsilon x + s\varepsilon y) \leq r f(x) + s f(y) \leq r\alpha + s\alpha \leq r^p\alpha + s^p\alpha = (r^p + s^p)\alpha = \alpha$. In conclusion, $f(r\varepsilon x + s\varepsilon y) \leq \alpha$. Thus, K_α is (p, ε) -convex set.

Remark 2.6: Proposition 2.5 does not satisfy if the function f is (p, ε) -convex. To confirm this observation, let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f(x_1, \dots, x_n) = 2x_1$ and $\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\varepsilon(x_1, \dots, x_n) = (x_1 + 1, x_2, \dots, x_n)$. Then, f is (p, ε) -convex and not semi (p, ε) -convex function. Indeed, let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ then $r\varepsilon x + s\varepsilon y \in \mathbb{R}^n$ and

$$\begin{aligned} f(r\varepsilon x + s\varepsilon y) &= f(rx_1 + sy_1 + r + s, rx_2 + sy_2 + \dots + rx_n + sy_n) = \\ 2(rx_1 + sy_1 + r + s) &= 2r(x_1 + 1) + 2s(y_1 + 1) \\ &= rf(\varepsilon x) + sf(\varepsilon y), \end{aligned}$$

which yields the (p, ε) -convexity of f . To show that f is not (p, ε) -semi convex function, let $\omega = (0, \dots, 0) \in \mathbb{R}^n$ be a zero vector in $\mathbb{R}^n$ then $f(\varepsilon(\omega)) = 2 > f(\omega) = 0$. Thus, from Proposition 2.4, f is not semi (p, ε) -semi convex function. Next, we show that k_α is not necessary (p, ε) -convex set for all $\alpha \in \mathbb{R}$. Take $x = (\frac{1}{4}, 0, \dots, 0)$, $y = (\frac{1}{10}, 0, \dots, 0) \in \mathbb{R}^n$, $\alpha = p = \frac{1}{2}$ and $r = s = \frac{1}{4}$. Then $x, y \in k_{\frac{1}{2}}$ such that $r\varepsilon x + s\varepsilon y = (\frac{47}{80}, 0, \dots, 0) \in \mathbb{R}^n$ with $f(r\varepsilon x + s\varepsilon y) = \frac{47}{40} > \alpha = \frac{1}{2}$. Then, $r\varepsilon x + s\varepsilon y \notin k_{\frac{1}{2}}$ as required to show.

Now, we discuss some properties of semi (p, ε) -convex functions.

Proposition 2.7: Let $f, g: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be two semis (p, ε) -convex functions on A . Then, $\alpha f + \beta g$ is semi (p, ε) -convex on A for all $\alpha, \beta \geq 0$.

Proof: Let $x, y \in A$ and let $z = r\varepsilon x + s\varepsilon y \in A$. Using the semi (p, ε) -convexity of f and g and Remark 2.3(ii), we obtain

$$\begin{aligned} (\alpha f + \beta g)(z) &= \alpha f(z) + \beta g(z) \leq \alpha rf(x) + \alpha sf(y) + \beta rg(x) + \beta sg(y), \\ &= r(\alpha f + \beta g)(x) + s(\alpha f + \beta g)(y). \end{aligned}$$

Hence, $\alpha f + \beta g$ is semi (p, ε) -convex on A .

Proposition 2.8: Let $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and $G: \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that f is semi (p, ε) -convex on A and G is an increasing linear function. Then $gof: A \rightarrow \mathbb{R}$ is semi (p, ε) -convex function.

Proof: Let $x, y \in A$. From semi (p, ε) -convexity of f on A and Remark 2.3(ii), $r\varepsilon x + s\varepsilon y \in A$ and $f(r\varepsilon x + s\varepsilon y) \leq rf(x) + sf(y)$. Thus, $G(f(r\varepsilon x + s\varepsilon y)) \leq$

$G(rf(x) + sf(y))$. The last inequality is satisfied because G is increasing function. Employing the linearity hypothesis of G , the last inequality yields, $G(f(r\epsilon x + s\epsilon y)) \leq rG(f(x)) + sG(f(y))$. Thus, $G \circ f$ is a semi (p, ϵ) -convex on A .

3. Applications to Optimization Programming

In this section, a non-linear optimization programming denoted as (NLP) , is defined

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \in A, \end{aligned}$$

where A is (p, ϵ) -convex and f is semi (p, ϵ) -convex function. The following optimality properties hold for the problem (NLP) .

Proposition 3.1: Every local minimum $x^* = \epsilon(x^*) \in \epsilon(A)$ of the problem (NLP) is a global minimum.

Proof: Suppose $x^* = \epsilon(x^*)$ is not a global minimum, then there exists $u \in A$ with $f(u) < f(x^*) = f(\epsilon x^*)$. From the assumptions on f , we have for all $r, s \in [0, 1]$ with $r^p + s^p = 1$ (i.e., $s = (1 - r^p)^{\frac{1}{p}}$),

$$f(r\epsilon u + s\epsilon x^*) \leq rf(u) + sf(x^*) \leq r^p f(u) + s^p f(x^*).$$

Since $f(u) < f(x^*)$ and the fact that $s = (1 - r^p)^{\frac{1}{p}}$, the last inequality yields

$$f\left(r\epsilon u + (1 - r^p)^{\frac{1}{p}}\epsilon x^*\right) < r^p f(x^*) + s^p f(x^*) = (r^p + s^p)f(x^*) = f(x^*) \quad (1)$$

Now, for a sufficiently small $r^p \in (0, 1]$ (hence, $r \in (0, 1]$) then $r\epsilon u + (1 - r^p)^{\frac{1}{p}}\epsilon x^*$ will be close enough to x^* . i.e., there exists $\delta > 0$ such that $r\epsilon u + (1 - r^p)^{\frac{1}{p}}\epsilon x^* \in B_\delta(x^*) \cap A$, where $B_\delta(x^*)$ is an open ball. From the local minimality of x^* , $f(x^*) \leq f(r\epsilon u + (1 - r^p)^{\frac{1}{p}}\epsilon x^*)$ which contradicts (1). Thus, x^* is a global minimum.

Proposition 3.2: Let f be strictly semi (p, ϵ) -convex on A . Then the global minimum of the problem (NLP) is a singleton.

Proof: Let x^*, y^* be two different global minima of (NLP) then $f(x^*) = f(y^*) \leq f(x)$ for any $x \in A$. From the assumptions on f and A , for all $r, s \in (0, 1)$ with $r^p + s^p = 1$, we have $r\epsilon x^* + s\epsilon y^* \in A$ and

$$f(r\epsilon x^* + s\epsilon y^*) < rf(x^*) + sf(y^*) \leq (s^p + t^p)f(x^*) = f(x^*).$$

The above inequality yields that $rex^* + sey^*$ is a global minimum which is a contradiction. Hence, there is a unique global minimum.

Proposition 3.3: The set of global minima of a problem (NLP) is (p, ε) -convex.

Proof: Let $argmin_A f = \{x^* \in A: f(x^*) \leq f(x) \quad \forall x \in A\}$ the set of global minima of the problem (NLP) . Let $x_1^*, x_2^* \in argmin_A f$ we must prove $rex_1^* + sex_2^* \in argmin_A f$. since f is semi (p, ε) -convex on the (p, ε) -convex set A , we have $rex_1^* + sex_2^* \in A$ and for each $x \in A$

$$\begin{aligned} f(rex_1^* + sex_2^*) &\leq rf(x_1^*) + sf(x_2^*) \leq r^p f(x) + s^p f(x) \\ &= (r^p + s^p)f(x) = f(x). \end{aligned}$$

Therefore, $rex_1^* + sex_2^* \in argmin_A f$ as required.

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