

Solution of quintic equation by geometric method and based upon it: A mathematical model

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Abstract

Purpose of writing this paper is to provide a new *dynamic geometric* method and also a mechanical model based upon it, to solve quintic equations. A triangle ABC with base BC , perpendicular AB of length equal to unity, hypotenuse AC , right angle at point B , is constructed. Perpendiculars BD , EF , GH upon AC , perpendiculars DE , FG upon BC , are drawn. Value of angle C is adjusted so that difference in lengths of BD and GH equals constant term of quintic equation transformed to *Bring-Jerrard* normal form, then one real root of given quintic equation equals length of perpendicular BD . Presently, there are two geometric methods available to solve quintic equations, first by paper folding art, called *Origami* and second given by Australian engineer, *Edward Lill*. This paper presents third simple and unique geometric method. Notwithstanding this *dynamic geometric* method, a mathematical model, explaining its fabrication, operation, has also been devised to present a real life picture of solution to the equations.

Subject Classification: 12E12, 65D18, 51M15, 51N15.

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1. Introduction

A general quintic equation with variable x and a, b, c, d, e, f real numbers such that $a \neq 0$, as coefficients of x^5, x^4, x^3, x^2, x and a constant term respectively, is written as

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0, \quad (1.1)$$

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Solving this equation implies finding five values of x that satisfy this equation. *Abel-Ruffini* theorised that there is no solution to quintic equation in terms of its radicals. But there are some quintic equations called reducible equations that are solvable with radicals [6]. To determine solvability of quintic equations, *Evrste Galois* developed a technique called *Galois* theory. Applying this theory, *Arthur Cayley* found the conditions to determine the quintic equation solvability [6]. The method being taken up in this paper, involves transforming the quintic Equation (1.1) to the form $y^5 + py^3 + qy^2 + ry + s = 0$, where $x = y - b/5a$. Also by application of certain transformation, the quintic equation can be reduced to the form $u^5 - u + a = 0$ and this was the required form for solution by *Hermite-Kronecker-Brioschi's* method. This form of quintic equation was also solvable by *Series Representation*.

In the annals of the history of algebra, one finds *two geometric solutions*, first is based upon the Japanese art of paper folding called *Origami* and second given by Australian engineer *Edward Lill*. Therefore, an earnest attempt has been made to solve quintic equations by the use of geometric construction and, then devise a mathematical model thereof. A cubic equation can be solved using *Origami* [3], [4] by folding a sheet of paper along a straight line such that a set of particular incidences are obtained between points and lines and these incidences are determinable by the coefficients of the equation. As a result, use of *Origami* in mathematics has solved a number of geometric construction problems like trisecting an angle [4]. In the above said artful paper folding, single fold is made at each step but if simultaneous folds are performed, higher degree algebraic equations can be solved. Accordingly, *Alpine* and *Lang* theorised, if $n-2$ folds are performed, algebraic equation of degree n can be solved for its real roots [1]. For solution of a quintic equation, the theorem warrants three simultaneous folds, however, solutions by two simultaneous folds have also been put forth [9], [10]. Another geometric method given by Australian engineer *Edward Lill* for solving a polynomial equation of any degree for real roots by the use of magnitude of its coefficients, is fascinating to look at the way, it solves polynomial equations effortlessly [5], [7]. In his later paper, *Edward Lill* gave the visual method of finding complex roots [8]. *Bixby* and *Riaz* also gave methods for solution of algebraic equations but these methods were more or less based upon *Lill's* method [2], [12].

Coming to the geometric method devised in this paper, I refer to Figure 1. A right angled triangle ABC with right angle at point B , base BC , perpendicular AB , hypotenuse AC is constructed, then a perpendicular BD is dropped from point B upon hypotenuse AC meeting it at point D ,

from point D , another perpendicular DE is dropped from point D upon base BC , meeting it at point E and the process of dropping subsequent perpendiculars upon base and hypotenuse is continued ad infinitum, then the ratio of the lengths of two consecutive perpendiculars is constant and equals $\cos C$. This property of similar right angled triangles are utilised in solving quintic equations for real roots in term of value of $\cos C$. According to this method, negative length is not drawable, therefore, negative roots are extracted by substituting the variable with negative of itself, roots with magnitude larger than one are discovered by substituting the variable with reciprocal of itself, however, complex roots are not drawable in real plane, these are not discoverable by this method.

This explanation of extraction of real roots by use of *dynamic geometry*, makes it explicitly clear that if a model replicating this geometric figure, is fabricated with arrangement of adjustment of lengths of the perpendiculars, real roots can be discovered by operating the model thus obviating the need of drawing a geometric figure. Such mechanical model of equation solver has been put forth in this paper. Its design, construction and operation are detailed in Section 4. *This is a step towards use of Mathematical modelling in aid of science and technology. Needless to say, quintic equations appear in the problems relating to subjects 'Gravitation' as well as 'Celestial Mechanics' and a handy tool in the form of a model to solve these, would always be welcome [11].*

Reverting to the solvability by this *dynamic geometric* method, essential condition to be satisfied is that the equation should have at least one real root in the domain of -1 to $+1$. That is, if α (alpha) is one real root of the equation such that $-1 < \alpha < 1$, then quintic equation is solvable for at least one real root. However, if the quintic has a real root with magnitude more than one, then the value of the root can be brought down in the domain of -1 to $+1$ by substituting variable say x with $1/x$ or $-1/x$. For the purpose of uniformity, I will solve quintic equations of the form $u^5 - u + a = 0$. Since much work has been done in providing methods of transformation of general quintic equations to $u^5 - u + a = 0$ called *Bring-Jerrard normal form* and also it is not the purpose of this paper to innovate a method of such transformation, therefore, geometric solution to quintic equation $u^5 - u + a = 0$ coupled with some other quintic equations transformable to this form, will be taken up. To start with, an equation of the form

$$x^5 - Px = -Q, \quad (1.2)$$

when $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, will be completely solved by *dynamic geometric* method. In addition to solution to above equation, following Equations (1.3), (1.4), (1.5) will also be solved for real roots.

$$x^5 - Px = -Q, \quad (1.3)$$

when $\infty > Q / P^{5/4} > 4 / 5^{5/4}$,

$$x^5 + Px = Q, \quad (1.4)$$

when $0 < Q / P^{5/4} < 2$,

$$x^5 + Px = Q, \quad (1.5)$$

when $\infty > Q / P^{5/2} > 2$.

2. Solution To Quintic Equations

For solving quintic Equations (1.2), (1.3), (1.4), (1.5), these will be transformed to the form $y^5 - y = -R$.

2.1. *Transformation Of Quintic Equations* $x^5 - Px = -Q$ To $y^5 - y = -R$

Substituting $y = (x / P^{1/4})$ transforms equations $x^5 - Px = -Q$ and $x^5 + Px = Q$ to

$$y^5 - y = -Q / P^{5/4}, \quad (2.1)$$

$$y^5 + y = Q / P^{5/4}, \quad (2.2)$$

respectively. Assuming y equal to $-Y$, Equations (2.1) and (2.2) transform to equations $Y^5 - Y = Q / P^{5/4}$ and $Y^5 + Y = -Q / P^{5/4}$, respectively.

2.2. *Domain Of Variable y*

Domain of variable y (or denoted by some other alphabet) of quintic equations that will be solved in this paper, is from 0 to +1 and domain 0 to -1 will stand covered by substitution of variable with negative of itself.

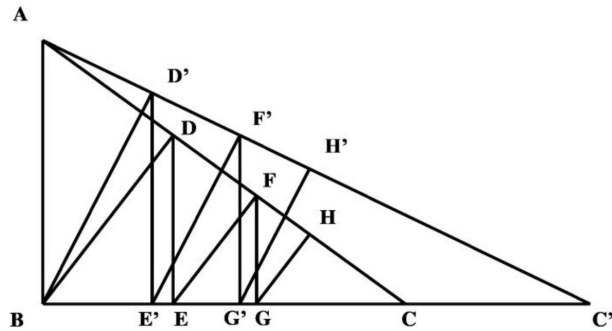


Figure 1

Computation of real roots of quintic equation

2.3. Solution Of Quintic Equation $y^5 - y = -Q / P^{5/4}$, When $-4 / 5^{5/4} \leq Q / P^{5/4} \leq 4 / 5^{5/4}$

2.3.1. Construction

For solution of this quintic equation $y^5 - y = -Q / P^{5/4}$, when $-4 / 5^{5/4} \leq Q / P^{5/4} \leq 4 / 5^{5/4}$ and also for all other equations, Figure 1 will be used invariably. As $y^5 - y = Q / P^{5/4}$ transforms to $Y^5 - Y = -Q / P^{5/4}$ by substituting $y = -Y$, therefore, cases of negative y will be avoided and in coming sections, it may be mentioned, solution to equation $y^5 - y = -Q / P^{5/4}$, when $Q / P^{5/4} \leq 4 / 5^{5/4}$ avoiding mention of solution of $y^5 - y = Q / P^{5/4}$, when $0 > Q / P^{5/4} \geq -4 / 5^{5/4}$. It will be presumed that solution to equation transformed by substitution of $y = -Y$ also stands covered. Construction of Figure 1 has been explained in introduction and is not repeated here.

2.3.2. Theory

Using simple trigonometry, it is found that $BD = AB \cos C$, $DE = AB \cos^2 C$, $EF = AB \cos^3 C$, $FG = AB \cos^4 C$, so on [13]. Thus the lengths of perpendiculars on base BC, have even powers of $\cos C$, whereas lengths of perpendiculars on hypotenuse, have odd powers of $\cos C$. If length of perpendicular AB is assumed unity and $\cos C$ is assumed equal to y , then $BD = y$, $DE = y^2$, $EF = y^3$, $FG = y^4$, $GH = y^5$, so on [13].

In this way, $y - y^5$ is the difference in lengths of perpendiculars BD and GH or

$$y - y^5 = BD - GH = Q / P^{5/4}. \quad (2.3)$$

2.3.3. Maximum value of $y - y^5$

Derivative of $y, -y^5$ when equated with zero, yields maxima at $y = 1/5^{1/4}$ and at this value of y , maximum value of $Q/P^{5/4}$ equals $4/5^{5/4}$. That means, value of $Q/P^{5/4}$, when $1 > y > 1/5^{1/4}$ or when $0 < y < 1/5^{1/4}$ is always less than maximum value $4/5^{5/4}$ and a single value of $Q/P^{5/4}$ between 0 and $4/5^{5/4}$ except $4/5^{5/4}$ corresponds to two values of y , say y_1 and y_2 such that $1 \geq y_2 > 1/5^{1/4}$ and $0 \leq y_1 < 1/5^{1/4}$.

2.3.4. Geometric solution of $y - y^5 = Q/P^{5/4}$ where $-4/5^{5/4} < Q/P^{5/4} \leq 4/5^{5/4}$.

Geometric construction as shown in Figure 1, is drawn taking value of angle C arbitrarily and length AB as unity. Length of perpendicular BD that equals y and length of perpendicular GH that equals y^5 , are measured and their difference is calculated. This difference is, then compared with given value of $Q/P^{5/4}$. If both are found equal, then one root of equation $y - y^5 = Q/P^{5/4}$, when $0 < Q/P^{5/4} \leq 4/5^{5/4}$, equals the length of perpendicular BD. *If both are found unequal, then value of angle C is changed to C' keeping length of perpendicular AB equal to unity and fresh perpendiculars now BD', E'F', G'H' are drawn upon hypotenuse now AC' and fresh perpendiculars D'E', F'G' are drawn upon base BC. Length of BD' - G'H' is compared with given $Q/P^{5/4}$. If now both are found equal, then value of one real root α will equal BD'. If length of BD' - G'H' on comparison is still not found equal to $Q/P^{5/4}$, the process is repeated till the difference say BD'' - G''H'' is found equal to $Q/P^{5/4}$. Then value of one root will be BD''. Other real positive root β can also be determined by following the same procedure as detailed above. For determining other roots (positive and negative) with magnitude more than one, the procedure as detailed below will be adopted.*

2.3.4.1. Determination of remaining roots:

For discovering roots having magnitude more than one, the equation $y - y^5 = Q/P^{5/4}$, by substituting $t = 1/y$, is transformed to

$$Rt^4 - t^5 = R. \quad (2.4)$$

where $R = P^{5/4}/Q$ and its value lies between $5^{5/4}/4$ and ∞ . Since t lies between 0 and 1, therefore, $Rt^4 < R$ and $Rt^4 - t^5$ will further be less than R , therefore, $Rt^4 - t^5$ can not be equal to R . In other words, Equation (2.4) does not have a positive real root between 0 and 1 or equation

$y - y^5 = Q / P^{5/4}$, when $0 < Q / P^{5/4} < 4 / 5^{5/4}$, does not have positive real roots of magnitude more than 1.

For extracting negative roots with magnitude more than one, $-t$ is assumed equal to $1/y$ and that transforms equation $y - y^5 = Q / P^{5/4}$ to

$$Rt^4 + t^5 = R, \quad (2.5)$$

where $R = P^{5/4} / Q$ and it lies between $5^{5/4} / 4$ and ∞ . Since variation of value of t is from 0 to 1, and Rt^4 , is less than R , therefore, $Rt^4 + t^5 = R$ can equal R for certain values of t . Referring to geometric Figure 1, the root is discovered by following the same method as explained for Equation (2.3) except the fact that in this case, comparison of length $R(FG) + GH$ will be made with R and when length $R(FG) + GH$ equals R , then root of Equation (2.5) equals length BD . Accordingly, root $-\gamma$ of the equation (2.4) will equal $-1 / BD$.

2.3.4.2. Determination of remaining roots of Equation $y - y^5 = Q / P^{5/4}$, when $0 \leq Q / P^{5/4} \leq 4 / 5^{5/4}$:

Three real roots $\alpha, \beta, -\gamma$ of equations $y - y^5 = Q / P^{5/4}$, when $0 \leq Q / P^{5/4} \leq 4 / 5^{5/4}$, have already been determined. Since sum of the roots of this equation is zero, therefore, its other two roots can be assumed as $-(1/2)(\alpha + \beta - \gamma) \pm \delta$. Sum of product of roots taken two at a time provides relation $\delta^2 = -(1/4)[2(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta - \gamma)^2]$. Hence δ is imaginary quantity given by relation

$$\delta = \pm i(1/2)\sqrt{2(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta - \gamma)^2}, \quad (2.6)$$

where i is an imaginary quantity equal to $\sqrt{-1}$. Thus quintic equation $y - y^5 = Q / P^{5/4}$, when $0 \leq Q / P^{5/4} \leq 4 / 5^{5/4}$, has three real and two complex roots given by $\alpha, \beta, -\gamma, -(1/2)(\alpha + \beta - \gamma) \pm i(1/2)\sqrt{2(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta - \gamma)^2}$. Roots of equation $x^5 - Px = -Q$, where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, can be determined from substitution, $y = (x / P^{1/4})$. The method detailed above proves Lemmas 2.1 and 2.2.

Lemma 2.1: *If in a right angled triangle ABC with right angle at point B, perpendicular AB equal to unity, base BC, hypotenuse AC and angle C of any arbitrary value, from point B, a perpendicular BD is dropped upon hypotenuse AC meeting it at point D, from point D, a perpendicular DE is dropped upon base BC meeting it at point E, from point E, a perpendicular EF is dropped upon hypotenuse AC meeting it at point F, from point F, a perpendicular FG is dropped*

upon base BC meeting it at point G, from point G, a perpendicular GH is dropped upon hypotenuse AC meeting it at point H and length BD – GH equals $Q / P^{5/4}$, then one root say α of equation $y - y^5 = Q / P^{5/4}$, when $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, will equal BD. If length BD – GH $\neq Q / P^{5/4}$, then value of angle C is further changed, keeping length of AB = 1 till length BD – GH equals $Q / P^{5/4}$, then its one root say α of equation $y - y^5 = Q / P^{5/4}$ where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, will equal BD. For discovering second root, value of angle C is further changed keeping length of AB = 1, till length BD – GH equals $Q / P^{5/4}$, then its second root say β of equation $y - y^5 = Q / P^{5/4}$ where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, will equal BD. For discovering third root, value of angle C is changed keeping length of AB = 1 till $R(FH) + GH = R$, where $R = P^{5/4} / Q$, then its third root say $-\gamma$ of equation $y - y^5 = Q / P^{5/4}$, where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, will equal $-1 / BD$.

Lemma 2.2: If $\alpha, \beta, -\gamma$ are real roots of equation $y - y^5 = Q / P^{5/4}$, where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$, then its two roots are complex and are given by

$$-(1/2)(\alpha + \beta - \gamma) \pm i(1/3)\sqrt{2(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta - \gamma)^2}.$$

Corollary: A quintic equation $y - y^5 = Q / P^{5/4}$, where $0 < Q / P^{5/4} \leq 4 / 5^{5/4}$ has two positive real roots with magnitude less than one, one negative real root with magnitude more than one and two complex roots.

2.4. Solution To Quintic Equation $x^5 - Px = -Q$, When $\infty > Q / P^{5/4} \geq 4 / 5^{5/4}$

By substituting $y = (x / P^{1/4})$, equation $x^5 - Px = -Q$, transforms to $y^5 - y = -Q / P^{5/4}$, where $\infty > Q / P^{5/4} \geq 4 / 5^{5/4}$. For finding root of values less than minus one and more than minus infinity, variable y is substituted with $-1/t$ and that transforms the equation $y^5 - y = -Q / P^{5/4}$ to

$$t^4(R) + t^5 = R, \quad (2.7)$$

where $R = P^{5/4} / Q$ is such that $0 < R < 5^{5/4} / 4$.

2.4.1. Geometric solution to $t^4(R) + t^5 = R$, when $R = P^{5/4} / Q$ is such that $0 < R < 5^{5/4} / 4$

Referring to Figure 1, value of angle C is taken arbitrarily and length of AB is taken as unity. Procedure for discovering real root of the equation is same as described in section 2.3.4 except the fact that length of $R(FG) + GH$ will be compared with the given quantity R. When the length of $R(FG) + GH$ will equal R on varying angle C, length of BD is, then measured, Then value of one of the real roots say $-\alpha$ of the equation

$y^5 - y = -Q/P^{5/4}$, when $\infty > Q/P^{5/4} > 4/5^{5/4}$, will equal $-1/BD$. Since maximum value $y^5 - y$ can achieve is $4/5^{5/4}$, when y varies from 0 to 1 and it is given $y - y^5 > 4/5^{5/4}$, therefore, there is no real root of this equation between 0 and 1. Kindly refer to section 2.3.3, where it is proved that $y - y^5$ can have maximum value $4/5^{5/4}$. Substituting $t = 1/y$, equation transforms to $Rt^4 - t^5 = R$ which again has no root with real magnitude between 0 and 1, since in such a situation $Rt^4 - t^5$ is always less than R . Thus there is no real root other than $-\alpha$ and thus other four roots are complex. That proves Lemma 2.3.

Lemma 2.3: If in a right angled triangle ABC with right angle at point B, perpendicular AB equal to unity, base BC, hypotenuse AC and angle C of any arbitrary value, from point B, a perpendicular BD is dropped upon hypotenuse AC meeting it at point D, from point D, a perpendicular DE is dropped upon base BC meeting it at point E, from point E, a perpendicular EF is dropped upon hypotenuse AC meeting it at point F, from point F, a perpendicular FG is dropped upon base BC meeting it at point G, from point G, a perpendicular GH is dropped upon hypotenuse AC meeting it at point H and length $R(FH) + GH$ is compared with R , where $R = P^{5/4}/Q$ and it is found $R(FH) + GH \neq R$, value of angle C is further changed till $R(FH) + GH = R$, then one real root $-\alpha$ of equation $y^5 - y = -Q/P^{5/4}$, where $\infty > Q/P^{5/4} \geq 4/5^{5/4}$, equals $-1/BD$ and its four other roots are complex.

2.5. Solution To Quintic Equation $x^5 + Px = Q$, When $0 < Q/P^{5/4} \leq 2$

By substituting $y = (x/P^{1/4})$, equation $x^5 + Px = Q$, transforms to $y^5 + y = Q/P^{5/4}$, when $0 < Q/P^{5/4} \leq 2$.

2.5.1. Geometric solution

For solving $y^5 + y = Q/P^{5/4}$, when $0 < Q/P^{5/4} \leq 2$, geometry construction as shown in Figure 1, is drawn taking value of angle C arbitrarily and length AB unity. Procedure for discovering real root of the equation is same as described in section 2.3.4 except the fact that length of $GH + BD$ will be compared with the given $Q/P^{5/4}$. When length of $GH + BD$ will equal $Q/P^{5/4}$ on varying angle C, then value of one of the real roots say α of the equation will equal BD . Since $Q/P^{5/4}$ is a positive quantity, it can not have negative real roots. By substituting y with $1/t$ and the equation $y^5 + y = Q/P^{5/4}$ transforms to obtain equation

$$t^5 - Rt^4 = R,$$

When $R = P^{5/4} / Q$. Since R can achieve maximum value of $1/2$, when t varies from 0 to 1, therefore, $t^5 - Rt^4 = R$ does not have real root between 0 and 1 as given $R > 1/2$ and less than infinity. That proves equation $y^5 + y = P^{5/4} / Q$, when $0 < Q / P^{5/4} < 2$, has only real root and its other four roots are complex. That proves Lemma 2.4.

Lemma 2.4: *If in a right angled triangle ABC with right angle at point B, perpendicular AB equal to unity, base BC, hypotenuse AC and angle C of any arbitrary value, from point B, a perpendicular BD is dropped upon hypotenuse AC meeting it at point D, from point D, a perpendicular DE is dropped upon base BC meeting it at point E, from point E, a perpendicular EF is dropped upon hypotenuse AC meeting it at point F, from point F, a perpendicular FG is dropped upon base BC meeting it at point G, from point G, a perpendicular GH is dropped upon hypotenuse AC meeting it at point H and length $GH + FG \neq R$, then value of angle C is changed till $GH + FG = R$, where $R = P^{5/4} / Q$, then one real root α of equation $y^5 - y = -Q / P^{5/4}$, where $0 > Q / P^{5/4} \leq 2$, equals $1/BD$ and its other four roots are complex.*

2.6. Solution To Quintic Equation $x^5 + Px = Q$, When $\infty > Q / P^{5/4} \geq 2$

By substituting $y = (x / P^{1/4})$, equation $x^5 + Px = Q$, when $\infty > Q / P^{5/4} > 2$ transforms to equation $y^5 + y = Q / P^{5/4}$. Since y varies from 0 to 1 and $\infty > Q / P^{5/4} \geq 2$, the equation does not have a root between 0 and 1. Terms y and y^5 both have odd powers and $Q / P^{5/4}$ is positive, therefore, $y^5 + y = Q / P^{5/4}$ can not have a root with negative value. Extraction of roots of magnitude more than 1, will be made by substitution of $t = 1/y$ and that transforms equation is $t^5 - Rt^4 = R$, when $R = P^{5/4} / Q$ such that $1/2 \geq R > 0$.

2.6.1. Maximum value of $t^5 - Rt^4$

Value of R is given en by relation, $R = t^5 / (1 + t^4)$ and t being equal to $\cos C$, when varies from 0 to 1, value of R steadily rises from 0 to maximum value of $1/2$ and single value of R corresponds to one value of t . Hence equation, $t^5 - Rt^4 = R$, has only one real root and its other four roots are complex.

2.6.2. Geometric solution of $y^5 + y = Q / P^{5/4}$, when $0 > Q / P^{5/4} \geq 2$.

Referring to Figure 1, procedure applied for discovering real root of the the equation is same as described in section 2.3.4 except the fact that

length of $GH - FG(R)$ will be compared with the given $Q/P^{5/4}$. When length of $GH - FG(R)$ will equal $Q/P^{5/4}$ on varying angle C , then value of one of the real roots say α of the equation $y^5 + y = Q/P^{5/4}$, where $0 > Q/P^{5/4} \geq 2$, will equal $1/BD$ and its other four roots are complex. That proves lemma 2.5.

Lemma 2.5: *If in a right angled triangle ABC with right angle at point B , perpendicular AB equal to unity, base BC , hypotenuse AC and angle C of any arbitrary value, from point B , a perpendicular BD is dropped upon hypotenuse AC meeting it at point D , from point D , a perpendicular DE is dropped upon base BC meeting it at point E , from point E , a perpendicular EF is dropped upon hypotenuse AC meeting it at point F , from point F , a perpendicular FG is dropped upon base BC meeting it at point G , from point G , a perpendicular GH is dropped upon hypotenuse AC meeting it at point H and length $GH - R(FG) \neq R$, then value of angle C is further changed till $GH - R(FG) = R$, where $R = P^{5/4}/Q$, then one real root α of equation $y^5 + y = Q/P^{5/4}$, where $0 < Q/P^{5/4} \leq 2$, equals $1/BD$ and its other four roots are complex.*

3. Solution of quintic equation transformable to $x^5 + Px + Q = 0$

Although detailed methods have already been provided by mathematicians for transforming a general quintic equation to *Bring-Jerrard normal form*, some quintic equations are readily transformable to this form without resorting to those methods. For example, quintic equation $x^5 - ax^3 + (a^2/4)x = b$, transforms to $z(1-z^2)^2 = b(2^{5/2}/a^{5/2})$ on substituting $x = z\sqrt{a/2}$ and further transforms to $y^5 - Py = -P$ on substituting $1/y = 1 - z^2$ and $P = \{(a/2)^5/b^2\}$. Also equation $x(a-x)^4 = b$, transforms to $y - Py^5 = 1$ on substituting $1/y = (1-x/a)$ and $P = b/a^5$. These equations are then solvable as explained in section 2.

4. Mechanical Model Of Quintic Equation Solver

It has been the earnest wish of common man in general and mathematics pursuers in particular to know what the mathematics in real world looks like. Mathematics that is taught in classrooms, at times, is not tangible and is difficult to envisage how a mathematical problem looks like. No doubt, examples illustrate the solution but are more of analogy than real picture of the problem and its solution. That needs transformation of the problem and its solution to material objects that could be seen by naked eyes. In that pursuit, this mechanical model is fabricated on the

basis of the problem and its geometric solution. Working of the model not only clarifies the solution but also leaves ever lasting impression in ones mind of the solution of the problem presented by it. Further, *it is a step towards application of mathematics to engineering. For the growth of technology, inherent didactic qualities and simplicity of fabrication, enthused me to come forward with this model of equation solver.*

4.1. Construction/Fabrication of the model

The mathematical model of quintic equation solver is based upon the geometric method given in section 2. It comprises of three main metallic telescopic tubular arms $A'B$, BC' , CA of right angled triangle ABC . Arm AB is perpendicular upon arm BC and arm AC is hypotenuse arm, thus scribing an angle BCA denoted by C . Perpendicular Arms $A'B$, BC' , BD , DE , EF , FG , GH ... are also metallic telescopic tubes, which can be enlarged or shortened giving it a feature of length adjustment. All the tubes are graduated so that their lengths can be displayed and measured. Blackened circles B , C , E , G , I , K , M ... in Figures 2a and 2b are joints slidable on arm BC , whereas D , F , J , L ... are joints slidable on arm $A'C$. Also joint A is slidable on arm $A'C$.

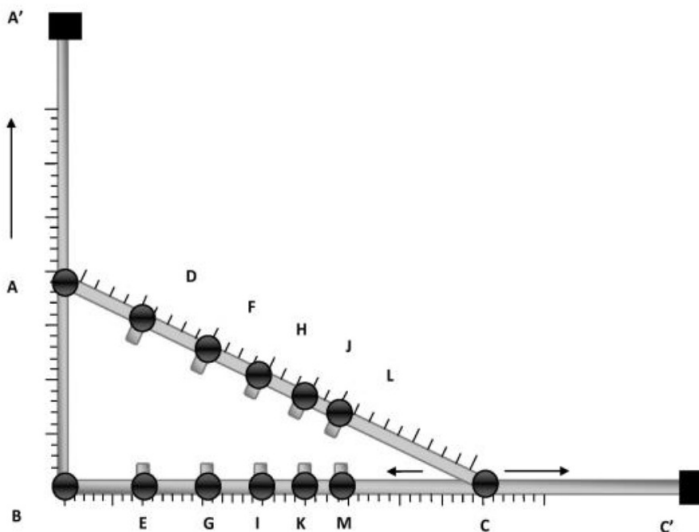


Figure 2a

Mechanical model of equation solver with perpendicular arms collapsed

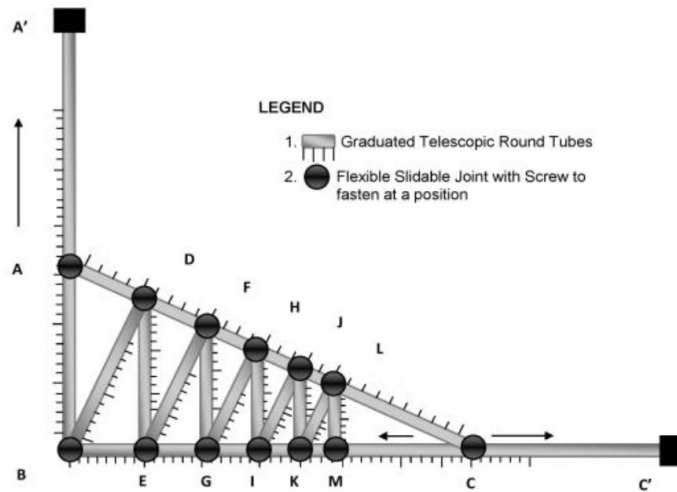


Figure 2b

Mechanical model of equation solver in operating condition

Although only perpendicular arms BD , DE , EF , FG , GH along with arms $A'B$, BC' , AC will be in use for solving quintic equations, more such arms have been provided making the model versatile and useful for solving equations of higher degrees. Above stated joints are also made of metals and have screws to fasten these at points, when their lengths have been finally adjusted according to the quintic equation. The joints also have mechanism so that open ends of the other perpendicular arms can be push-fitted upon their heads. Joints E , G , I , ... have arms ED , FG , HI ,... respectively fixed perpendicularly upon these, whereas joints D , F , H ,... have arms DB , EF , GH ,... respectively fixed perpendicularly upon these.

4.2. Operation

4.2.1. Solution of quintic equation $y - y^5 = Q / P^{5/4}$ when $Q / P^{5/4} \leq 4 / 5^{5/4}$

Referring to model at Figure 2a, joint A is slid upon arm $A'C$ and length of arm AB is adjusted to unity. Open end A' of arm CA' is push fitted upon joint A and its open end C' is push fitted upon joint C thus constructing a right angled triangle ABC with arbitrary value of angle C . It is submitted, wherever an alphabet with apostrophe sign appears in this section, that will mean, the end being denoted is an open end and is yet to be push fitted on the head of the other joint. Joint D with perpendicular arm DB' is slid on arm AC so that its free end B' could be push fitted on

joint B . After open end B' is push fitted, joint E is slid upon arm BC and its open end D' is enlarged and push fitted upon joint D . Similarly joint F is slid upon arm AC and its open end E' is push fitted upon joint E , joint G is slid upon arm BC and its open end F' is push fitted upon joint E and finally joint H is slid upon arm AC and open end I' is push fitted upon joint I . Lengths of perpendicular arms BD and GH are measured and length $BD - GH$ is compared with $Q / P^{5/4}$. If both are unequal, then value of angle C is slightly changed keeping length $AB = 1$ and the whole exercise is repeated till length $BD - GH$ is finally found equal to $Q / P^{5/4}$. One real root of $y - y^5 = Q / P^{5/4}$, when $Q / P^{5/4} \leq 4 / 5^{5/4}$, then equals length of BD . In this way, angle C is scanned from 0 to 90 degrees and second real root is also extracted the way the first real root was found. Similarly, by using transformation as explained in section 2.3 and following the method as described above, its third root is also extracted. Remaining two complex roots can be calculated using Equation 2.6.

4.2.2. Solution of quintic equation $y + y^5 = Q / P^{5/4}$, when $Q / P^{5/4} \leq 2$

Procedure as described in section 4.2.1 is applicable to this case also with a slight change that length $BD + GH$ is compared with $Q / P^{5/4}$.

4.2.3. Solution of quintic equation $y - y^5 = Q / P^{5/4}$, when $Q / P^{5/4} > 4 / 5^{5/4}$

Substituting $y = -1/t$, yields the equation $t^5 + Rt^4 = R$, when $R = P^{5/4} / Q$ and $0 < R < 5^{5/4} / 4$. For solving this equation, procedure as described in section 4.2.1 is applicable with a slight change that length $GH + R(FG)$ is compared with $P^{5/4} / Q$ and when both equal, then root of equation $y - y^5 = Q / P^{5/4}$ is $-1 / BD$.

4.2.4 Solution of quintic equation $y + y^5 = Q / P^{5/4}$, when $Q / P^{5/4} > 2$

Substituting $y = 1/t$, yields the equation $t^5 - Rt^4 = R$, when $R = P^{5/4} / Q$ and $0 < R < 1/2$. For solving this equation, procedure as described in section 4.2.1 is applicable with a slight change that length $GH + R(FG)$ is compared with $P^{5/4} / Q$ and when both equal, then root of equation $y - y^5 = Q / P^{5/4}$ is $1 / BD$.

5. Accuracy and Precision

No doubt method used for solving equations with the aid of geometry is error free but the fact that errors in drawing and measurement of lengths

of geometric figure, might creep in, can not be denied. To obviate such errors, use of precision and drawing equipments are need of the hour. However, inherent errors relating to how precise the measuring or drawing instruments perform, are liable to affect results. It may also be noted, when value of angle C is in the vicinity of zero or ninety degrees, congestion may occur in the figure. This situation can be controlled appreciably if length of perpendicular AB is scaled up by a multiplier say m where m has a large value. In that case, length of AB will be m units and accordingly, lengths of all perpendiculars will also increase by a multiplying factor m and $Q/P^{5/4}$ will also have to be multiplied with m at the time of its comparison with corresponding perpendiculars of the geometric figure. However, the length of BD would be considered BD/m . When geometric figure size gets enlarged, congestion will automatically be taken care of. Larger the length of AB , better would be the results but length of AB can not be increased beyond a certain limit to avoid mammoth unworkable size of the geometric figure.

6. Results, Conclusions and Discussions

Overview of the paper indicates that the characteristic nature of perpendiculars $BD, DE, EF, FG, GH, \dots$ which equal $\cos C, \cos^2 C, \cos^3 C, \cos^4 C, \cos^5 C, \dots$ respectively, when length $AB=1$, paves the way for geometric solution to quintic equation $x^5 - x = R$ [13]. Kindly refer to Figure 1. Roots of a quintic equation, being odd, always facilitates the existence of at least one real root, thus ensuring the applicability of this geometric method to all quintic equations, since complex roots occur in conjugate pairs. Important point that needs discussion is the fact that lengths of perpendiculars $BD, DE, EF, FG, GH, \dots$, represent x, x^2, x^3, x^4, x^5 , since $x = \cos C$. Also $\cos C$ varies from -1 to $+1$, therefore, the essential requirement for solution is that the quintic equation must have, at least, one root of magnitude equal to or less than 1. For this reason, the roots in this paper were considered between 0 and 1 and for roots between 0 to -1 , variable was considered as negative of itself. For roots with magnitude more than 1, variable was considered as reciprocal of itself and for roots with magnitude more than 1 but with negative sign, variable was considered as reciprocal of itself but with negative sign. For solving equation $x - x^5 = R$, where $R \leq 4/5^{5/4}$, length $BD - GH$ was compared with R and angle C was adjusted so that both should match. Length BD is then the value of one of the real root. If quintic equation has multiple real positive roots of magnitude less than unity, all such roots can be extracted

by scanning angle C through 90 degrees. Similarly, all negative roots of magnitude can also be extracted by adopting the same method. However, complex roots can not be discovered but can be calculated, when three real roots have been extracted. *Changes in value of angle C requires reconstruction of the geometric figure. That means, the figure is not fixed according to the parameters of the equation but the figure changes to extract the roots and is thus given the nomenclature of dynamic geometry. Reading the theory of this paper between the lines, reveals that in this case geometric figure is calculating values of x, x^2, x^3, x^4, x^5 , when value of x changes with change in angle C . Recall Lill's method, one will have to change the angle of incident so that the final ray after passing through a number of stages of striking and bending when falling upon consecutive perpendiculars, just manages to strike past the top of the terminal perpendicular. It can not be achieved in one go, one will have to attempt it a few times to hit the top of the finishing perpendicular. It is also a case of dynamic geometry. Here geometry performs algebraic operations, when value of angle C is changed.*

Another important question that comes to one's mind is whether a general quintic equation without transformation to the form $u^5 - u + a = 0$, is solvable by this method. Reply is in affirmative provided magnitude of one of its real root is brought down to the domain of -1 to $+1$. As stated above, a quintic equation always has at least one real root but its magnitude can be more than or equal to 1 or less than or equal to -1 . If it lies between -1 and $+1$, no further treatment to quintic equation is needed but if it is more than 1 or less than -1 , magnitude of the root can be brought down in the domain of -1 to $+1$ by substituting variable with reciprocal or negative reciprocal of itself. Let the general quintic equation be

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

where a, b, c, d, e, f are positive and $a \neq 0$, has at least say one root with magnitude between 0 and -1 , then substituting $x = -y$, the equation transforms to

$$ay^5 - by^4 + cy^3 - dy^2 + ey = f.$$

Substituting $y^5, y^4, y^3, \dots, y$ with corresponding lengths of perpendiculars, the equation

$$a(GH) - b(FG) + c(EF) - d(DE) + e(BD) = f$$

is obtained. Referring to Figure 1 and using *dynamic geometry* by varying the value of angle C , the length $a(GH) - b(FG) + c(EF) - d(DE) + e(BD)$ can be made equal to f , then value of one real root of equation $ax^5 + bx^4 +$

$cx^3 + dx^2 + ex + f = 0$ is $-BD$. This evidences that the method also does not require transformation of quintic equation to a specific form and is equally applicable to higher degree equations, provided the equation has at least one real root.

Coming to the importance of a model in advancement of science and technology, I submit, the object of putting theory into practice coupled with the simplicity of the geometric figure and also significance of solution to quintic equations in subjects *Gravitation and Celestial mechanics* prompted me to present a mathematical model of equation solver [11]. As the length and placement of perpendiculars $BD, DE, EF, \dots$ vary, therefore, model with manoeuvrability of lengths and placements adjustment were the need of hour and these were accomplished by use of graduated telescopic tubes for length adjustment of the perpendicular tubes. Slidable joints helped in placement of perpendiculars at the desired places. The perpendicularity of $BD, DE, EF, \dots$ was ensured by permanent fixation of these at 90 degrees to their respective joints. That completed the task of fabrication of mathematical model. Needless to say, physical models pave way for progress of science and technology but also instil faith of young learners in science.

To exemplify it, I submit, it is very difficult to convince a lay man that light is constituted of seven colours but when Newton's disc marked with seven colours of rainbow Violet, Indigo, Blue, Green, Yellow, Orange, Red is shown rotating with only one white colour appearing, it does not need further explanation to convince him. I further submit, when one pursuing research, encounters problems in his theoretical explanation, a model of the problem comes to his rescue as a guide and mentor. Practical problems which could not be envisaged are brought to limelight and solved by modelling.

Referring to Figures 2a and 2b, model displayed in these, proves beyond doubt, when $x < 1$, then $x > x^2 > x^3 \dots$ the variation in magnitudes of perpendiculars is reflection of magnitude of $x, x^2, x^3, \dots$. Further, this model obviates the necessity of consideration of squares, cubes or multidimensional objects for envisaging $x^2, x^3 \dots x^m$, it gives alternate way of considering these as lengths of perpendiculars $DE, EF, \dots$, in a geometric figure. It further proves, variable x appearing in an equation, is not necessarily an abstract quantity but is tangible as an arm of the model displayed in Figure 2b.

Statement of Conflict of Interests

There are no competing interests or conflict of interest financial or non-financial or in any other way involved in this research.

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